Preface

Welcome to *Calculus Volume 2*, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

About OpenStax

OpenStax is a nonprofit based at Rice University, and it’s our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 20 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

About OpenStax's Resources

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Errata

All OpenStax textbooks undergo a rigorous review process. However, like any professional-grade textbook, errors sometimes occur. Since our books are web based, we can make updates periodically when deemed pedagogically necessary. If you have a correction to suggest, submit it through the link on your book page on openstax.org. Subject matter experts review all errata suggestions. OpenStax is committed to remaining transparent about all updates, so you will also find a list of past errata changes on your book page on openstax.org.

Format

You can access this textbook for free in web view or PDF through openstax.org, and for a low cost in print.

About *Calculus Volume 2*

Calculus is designed for the typical two- or three-semester general calculus course, incorporating innovative features to enhance student learning. The book guides students through the core concepts of calculus and helps them understand how those concepts apply to their lives and the world around them. Due to the comprehensive nature of the material, we are offering the book in three volumes for flexibility and efficiency. Volume 2 covers integration, differential equations, sequences and series, and parametric equations and polar coordinates.

Coverage and Scope

Our *Calculus Volume 2* textbook adheres to the scope and sequence of most general calculus courses nationwide. We have worked to make calculus interesting and accessible to students while maintaining the mathematical rigor inherent in the subject. With this objective in mind, the content of the three volumes of *Calculus* have been developed and arranged to provide a logical progression from fundamental to more advanced concepts, building upon what students have already learned and emphasizing connections between topics and between theory and applications. The goal of each section is to enable students not just to recognize concepts, but work with them in ways that will be useful in later courses and future careers. The organization and pedagogical features were developed and vetted with feedback from mathematics educators dedicated to the project.

Volume 1

Chapter 1: Functions and Graphs
Chapter 2: Limits
Chapter 3: Derivatives
Chapter 4: Applications of Derivatives
Chapter 5: Integration
Chapter 6: Applications of Integration

Volume 2

Chapter 1: Integration
Chapter 2: Applications of Integration
Chapter 3: Techniques of Integration
Chapter 4: Introduction to Differential Equations
Chapter 5: Sequences and Series
Chapter 6: Power Series
Chapter 7: Parametric Equations and Polar Coordinates

Volume 3

Chapter 1: Parametric Equations and Polar Coordinates
Chapter 2: Vectors in Space
Chapter 3: Vector-Valued Functions

Chapter 4: Differentiation of Functions of Several Variables
Chapter 5: Multiple Integration
Chapter 6: Vector Calculus
Chapter 7: Second-Order Differential Equations

Pedagogical Foundation

Throughout *Calculus Volume 2* you will find examples and exercises that present classical ideas and techniques as well as modern applications and methods. Derivations and explanations are based on years of classroom experience on the part of long-time calculus professors, striving for a balance of clarity and rigor that has proven successful with their students. Motivational applications cover important topics in probability, biology, ecology, business, and economics, as well as areas of physics, chemistry, engineering, and computer science. **Student Projects** in each chapter give students opportunities to explore interesting sidelights in pure and applied mathematics, from showing that the number e is irrational, to calculating the center of mass of the Grand Canyon Skywalk or the terminal speed of a skydiver. **Chapter Opening Applications** pose problems that are solved later in the chapter, using the ideas covered in that chapter. Problems include the hydraulic force against the Hoover Dam, and the comparison of the relative intensity of two earthquakes. **Definitions, Rules,** and **Theorems** are highlighted throughout the text, including over 60 **Proofs** of theorems.

Assessments That Reinforce Key Concepts

In-chapter **Examples** walk students through problems by posing a question, stepping out a solution, and then asking students to practice the skill with a “Check Your Learning” component. The book also includes assessments at the end of each chapter so students can apply what they’ve learned through practice problems. Many exercises are marked with a **[T]** to indicate they are suitable for solution by technology, including calculators or Computer Algebra Systems (CAS). Answers for selected exercises are available in the **Answer Key** at the back of the book. The book also includes assessments at

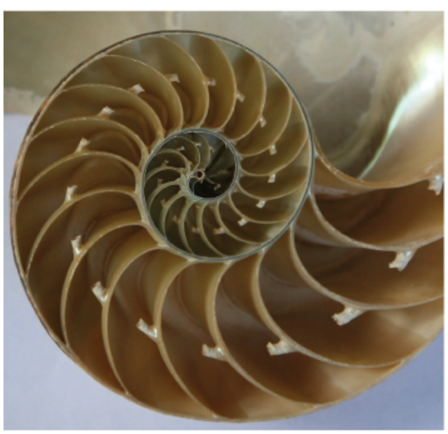
the end of each chapter so students can apply what they’ve learned through practice problems.

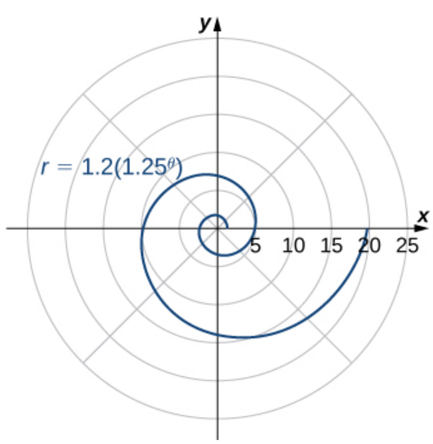
Early or Late Transcendentals

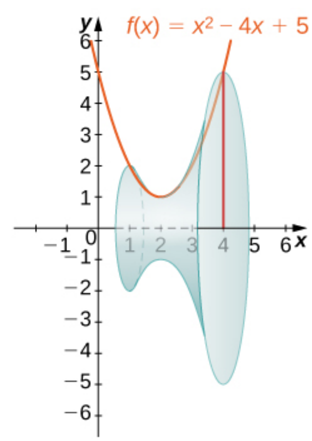
Calculus Volume 2 is designed to accommodate both Early and Late Transcendental approaches to calculus. Exponential and logarithmic functions are presented in Chapter 2. Integration of these functions is covered in Chapters 1 for instructors who want to include them with other types of functions. These discussions, however, are in separate sections that can be skipped for instructors who prefer to wait until the integral definitions are given before teaching the calculus derivations of exponentials and logarithms.

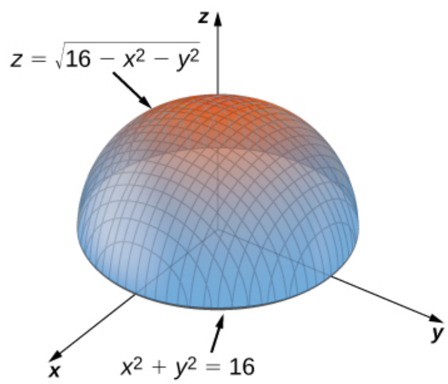
Comprehensive Art Program

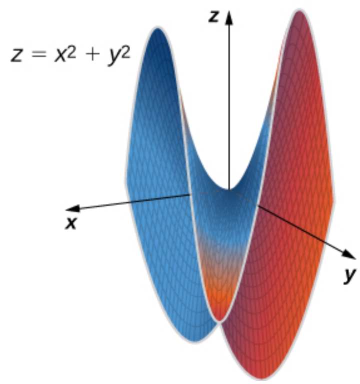
Our art program is designed to enhance students’ understanding of concepts through clear and effective illustrations, diagrams, and photographs.

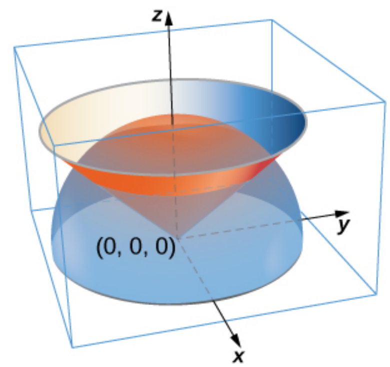












Additional Resources

Student and Instructor Resources

We’ve compiled additional resources for both students and instructors, including Getting Started Guides, an instructor solution manual, and PowerPoint slides. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

Partner Resources

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Derivatives of Inverse Functions

- Calculate the derivative of an inverse function.
- Recognize the derivatives of the standard inverse trigonometric functions.

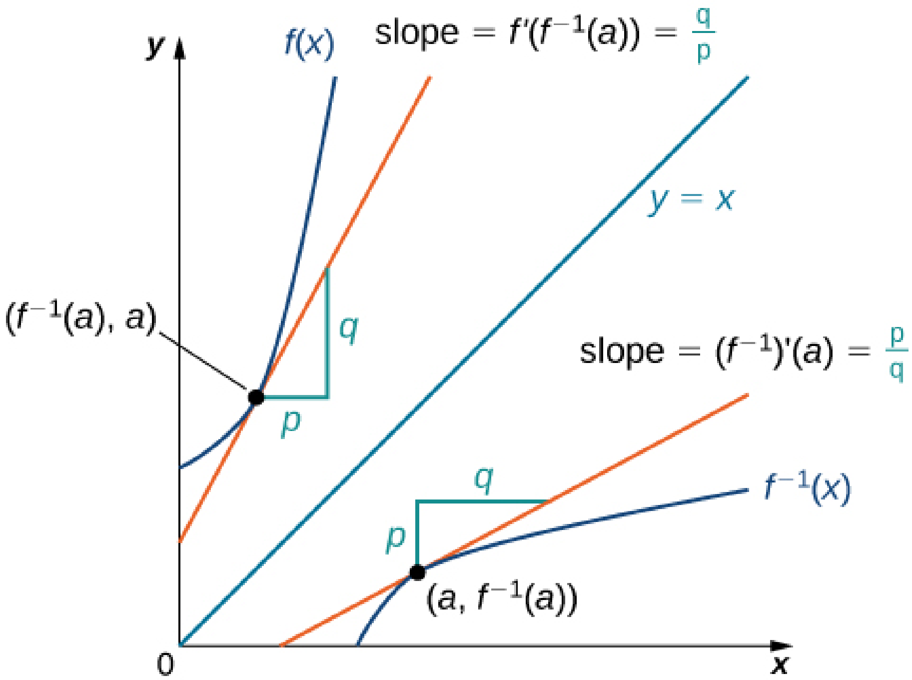
In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. [link] shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $\left(a, f^{-1}(a)\right)$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of $\left(f^{-1}\right)'(a)=\frac{p}{q}$. This point corresponds to a point $\left(f^{-1}(a), a\right)$ on the graph of $f(x)$ having a tangent line with a slope of $f'\left(f^{-1}(a)\right)=\frac{q}{p}$. Thus, if $f^{-1}(x)$ is differentiable at a, then it must be the case that

Equation:

$$\left(f^{-1}\right)'(a)=\frac{1}{f'\left(f^{-1}(a)\right)}.$$



The tangent lines of a function and its inverse are related; so, too, are the derivatives of these functions.

We may also derive the formula for the derivative of the inverse by first recalling that $x = f\left(f^{-1}(x)\right)$. Then by differentiating both sides of this equation (using the chain rule on the right), we obtain

Equation:

$$1 = f'\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x).$$

Solving for $\left(f^{-1}\right)'(x)$, we obtain

Equation:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

We summarize this result in the following theorem.

Note:
Inverse Function Theorem
Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$,
Equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then
Equation:

$$g(x) = \frac{1}{f'(g(x))}.$$

Example:
Exercise:

Problem:
Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution:

The inverse of $g(x) = \frac{x+2}{x}$ is $f(x) = \frac{2}{x-1}$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,
Equation:

$$f'(x) = \frac{-2}{(x-1)^2} \text{ and } f'(g(x)) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\frac{x+2}{x}-1\right)^2} = -\frac{x^2}{2}.$$

Finally,

Equation:

$$g'(x) = \frac{1}{f'(g(x))} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

Equation:

$$g'(x) = -\frac{2}{x^2}.$$

Note:
Exercise:

Problem:

Use the inverse function theorem to find the derivative of $g(x) = \frac{1}{x+2}$. Compare the result obtained by differentiating $g(x)$ directly.

Solution:

$g'(x) = -\frac{1}{(x+2)^2}$

Hint

Use the preceding example as a guide.

Example:
Exercise:

Problem:
Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

Solution:

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

Equation:

$$f'(x) = 3x^3 \text{ and } f'(g(x)) = 3\left(\sqrt[3]{x}\right)^2 = 3x^{2/3}.$$

Finally,

Equation:

$$g'(x) = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

Note:

Exercise:

Problem:

Find the derivative of $g(x) = \sqrt[5]{x}$ by applying the inverse function theorem.

Solution:

$g(x) = \frac{1}{5}x^{-4/5}$

Hint

$g(x)$ is the inverse of $f(x) = x^5$.

From the previous example, we see that we can use the inverse function theorem to extend the power rule to exponents of the form $\frac{1}{n}$, where n is a

positive integer. This extension will ultimately allow us to differentiate x^q, where q is any rational number.

Note:
Extending the Power Rule to Rational Exponents
The power rule may be extended to rational exponents. That is, if n is a positive integer, then
Equation:

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}.$$

Also, if n is a positive integer and m is an arbitrary integer, then
Equation:

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.$$

Proof

The function $g\left(x\right) = x^{1/n}$ is the inverse of the function $f\left(x\right) = x^n$. Since $g'\left(x\right) = \frac{1}{f'(g(x))}$, begin by finding $f'\left(x\right)$. Thus,
Equation:

$$f'\left(x\right) = nx^{n-1} \text{ and } f'\left(g\left(x\right)\right) = n\left(x^{1/n}\right)^{n-1} = nx^{(n-1)/n}.$$

Finally,
Equation:

$$g'\left(x\right) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{(1-n)/n} = \frac{1}{n}x^{(1/n)-1}.$$

To differentiate $x^{m/n}$ we must rewrite it as $\left(x^{1/n}\right)^m$ and apply the chain rule. Thus,

Equation:

$$\frac{d}{dx}\left(x^{m/n}\right)=\frac{d}{dx}\left(\left(x^{1/n}\right)^m\right)=m\left(x^{1/n}\right)^{m-1}\cdot\frac{1}{n}x^{(1/n)-1}=\frac{m}{n}x^{(m/n)-1}.$$

□

Example:

Exercise:

Problem:

Applying the Power Rule to a Rational Power

Find the equation of the line tangent to the graph of $y=x^{2/3}$ at $x=8$.

Solution:

First find $\frac{dy}{dx}$ and evaluate it at $x=8$. Since

Equation:

$$\frac{dy}{dx}=\frac{2}{3}x^{-1/3}\text{ and }\left.\frac{dy}{dx}\right|_{x=8}=\frac{1}{3}$$

the slope of the tangent line to the graph at $x=8$ is $\frac{1}{3}$.

Substituting $x=8$ into the original function, we obtain $y=4$. Thus, the tangent line passes through the point $(8,4)$. Substituting into the point-slope formula for a line, we obtain the tangent line

Equation:

$$y=\frac{1}{3}x+\frac{4}{3}.$$

Note:
Exercise:

Problem: Find the derivative of $s\left(t\right)=\sqrt{2t+1}$.

Solution:

$s'\left(t\right)=\left(2t+1\right)^{-1/2}$

Hint

Use the chain rule.

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Example:
Exercise:

Problem:
Derivative of the Inverse Sine Function

Use the inverse function theorem to find the derivative of $g\left(x\right)=\sin^{-1}x$.

Solution:

Since for x in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $f\left(x\right)=\sin x$ is the inverse of $g\left(x\right)=\sin^{-1}x$, begin by finding $f'(x)$. Since

Equation:

$$f'(x) = \cos x \text{ and } f'(g(x)) = \cos\left(\sin^{-1}x\right) = \sqrt{1-x^2},$$

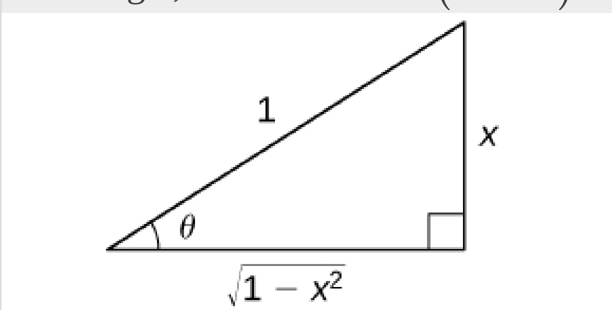
we see that

Equation:

$$g'(x) = \frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{f'(g(x))} = \frac{1}{\sqrt{1-x^2}}.$$

Analysis

To see that $\cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}$, consider the following argument. Set $\sin^{-1}x = \theta$. In this case, $\sin\theta = x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We begin by considering the case where $0 < \theta < \frac{\pi}{2}$. Since θ is an acute angle, we may construct a right triangle having acute angle θ, a hypotenuse of length 1 and the side opposite angle θ having length x. From the Pythagorean theorem, the side adjacent to angle θ has length $\sqrt{1-x^2}$. This triangle is shown in [link]. Using the triangle, we see that $\cos\left(\sin^{-1}x\right) = \cos\theta = \sqrt{1-x^2}$.



Using a right triangle having acute angle θ, a hypotenuse of length 1, and the side opposite angle θ having length x, we can see that $\cos\left(\sin^{-1}x\right) = \cos\theta = \sqrt{1-x^2}$.

In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence

Equation:

$$\cos\left(\sin^{-1}x\right) = \cos\theta = \cos\left(-\theta\right) = \sqrt{1-x^2}.$$

Now if $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$, $x = 1$ or $x = -1$, and since in either case $\cos\theta = 0$ and $\sqrt{1-x^2} = 0$, we have

Equation:

$$\cos\left(\sin^{-1}x\right) = \cos\theta = \sqrt{1-x^2}.$$

Consequently, in all cases, $\cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}$.

Example:

Exercise:

Problem:

Applying the Chain Rule to the Inverse Sine Function

Apply the chain rule to the formula derived in [link] to find the derivative of $h\left(x\right) = \sin^{-1}\left(g\left(x\right)\right)$ and use this result to find the derivative of $h\left(x\right) = \sin^{-1}\left(2x^3\right)$.

Solution:

Applying the chain rule to $h\left(x\right) = \sin^{-1}\left(g\left(x\right)\right)$, we have

Equation:

$$h'\left(x\right) = \frac{1}{\sqrt{1-\left(g\left(x\right)\right)^2}}g'\left(x\right).$$

Now let $g\left(x\right)=2x^3$, so $g'\left(x\right)=6x^2$. Substituting into the previous result, we obtain

Equation:

$$\begin{aligned} h'\left(x\right) &= \frac{1}{\sqrt{1-4x^6}}\cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}}. \end{aligned}$$

Note:

Exercise:

Problem:

Use the inverse function theorem to find the derivative of $g\left(x\right)=\tan^{-1}x$.

Solution:

$g'\left(x\right)=\frac{1}{1+x^2}$

Hint

The inverse of $g(x)$ is $f\left(x\right)=\tan x$. Use [link] as a guide.

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Note:

Derivatives of Inverse Trigonometric Functions

Equation:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-(x)^2}}$$

Equation:

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-(x)^2}}$$

Equation:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+(x)^2}$$

Equation:

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+(x)^2}$$

Equation:

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{(x)^2-1}}$$

Equation:

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{(x)^2-1}}$$

Example:
Exercise:

Problem:
Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f\left(x\right)=\tan^{-1}\left(x^{2}\right)$.

Solution:

Let $g\left(x\right)=x^{2}$, so $g'\left(x\right)=2x$. Substituting into [link], we obtain

Equation:

$$f'\left(x\right)=\frac{1}{1+\left(x^{2}\right)^{2}}\cdot\left(2x\right).$$

Simplifying, we have

Equation:

$$f'\left(x\right)=\frac{2x}{1+x^{4}}.$$

Example:

Exercise:

Problem:

Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h\left(x\right)=x^{2}\sin^{-1}x$.

Solution:

By applying the product rule, we have

Equation:

$$h'\left(x\right)=2x\sin^{-1}x+\frac{1}{\sqrt{1-x^{2}}}\cdot x^{2}.$$

Note:
Exercise:

Problem: Find the derivative of $h\left(x\right)=\cos^{-1}\left(3x-1\right)$.

Solution:

$h'\left(x\right)=\frac{-3}{\sqrt{6x-9x^2}}$

Hint

Use [link]. with $g\left(x\right)=3x-1$

Example:
Exercise:

Problem:
Applying the Inverse Tangent Function

The position of a particle at time t is given by $s\left(t\right)=\tan^{-1}\left(\frac{1}{t}\right)$ for $t\ge\frac{1}{2}$. Find the velocity of the particle at time $t=1$.

Solution:

Begin by differentiating $s\left(t\right)$ in order to find $v\left(t\right)$. Thus,
Equation:

$$v\left(t\right)=s'\left(t\right)=\frac{1}{1+\left(\frac{1}{t}\right)^2}\cdot\frac{-1}{t^2}.$$

Simplifying, we have
Equation:

$$v\left(t\right)=-\frac{1}{t^2+1}.$$

Thus, $v(1) = -\frac{1}{2}$.

Note:
Exercise:

Problem:

Find the equation of the line tangent to the graph of $f(x) = \sin^{-1}x$ at $x = 0$.

Solution:

$y = x$

Hint

$f'(0)$ is the slope of the tangent line.

Key Concepts

- The inverse function theorem allows us to compute derivatives of inverse functions without using the limit definition of the derivative.
- We can use the inverse function theorem to develop differentiation formulas for the inverse trigonometric functions.

Key Equations

- **Inverse function theorem**
  $\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$ whenever $f'\left(f^{-1}(x)\right) \neq 0$ and $f(x)$ is differentiable.
- **Power rule with rational exponents**
  $\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}$.
- **Derivative of inverse sine function**
  $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-(x)^2}}$

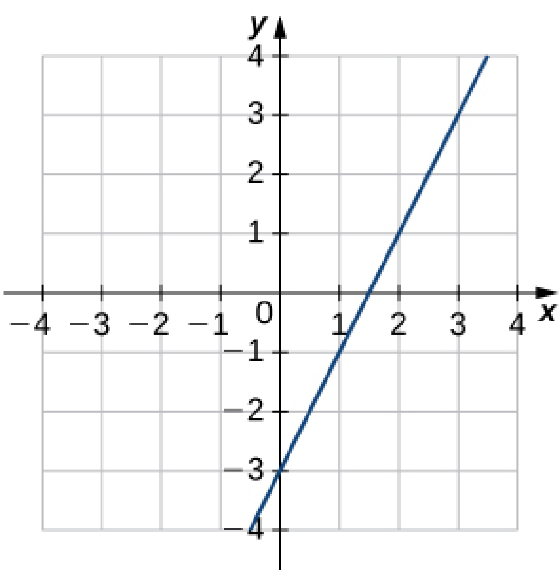
- **Derivative of inverse cosine function**
$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-(x)^2}}$
- **Derivative of inverse tangent function**
$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+(x)^2}$
- **Derivative of inverse cotangent function**
$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+(x)^2}$
- **Derivative of inverse secant function**
$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{(x)^2-1}}$
- **Derivative of inverse cosecant function**
$\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{(x)^2-1}}$

For the following exercises, use the graph of $y = f(x)$ to

a. sketch the graph of $y = f^{-1}(x)$, and
b. use part a. to estimate $\left(f^{-1}\right)'(1)$.

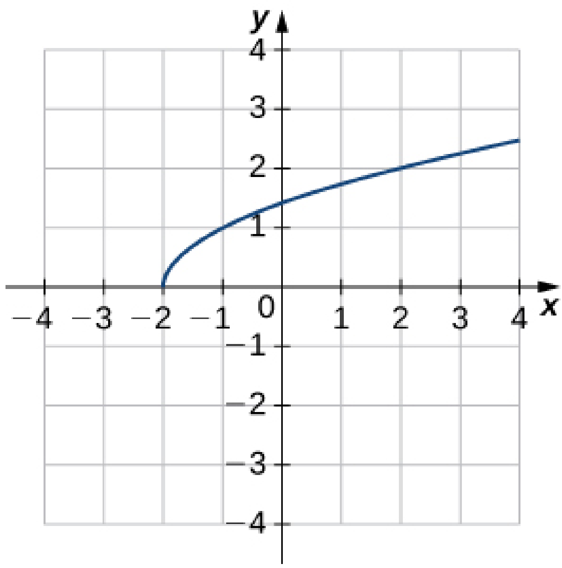
Exercise:

Problem:



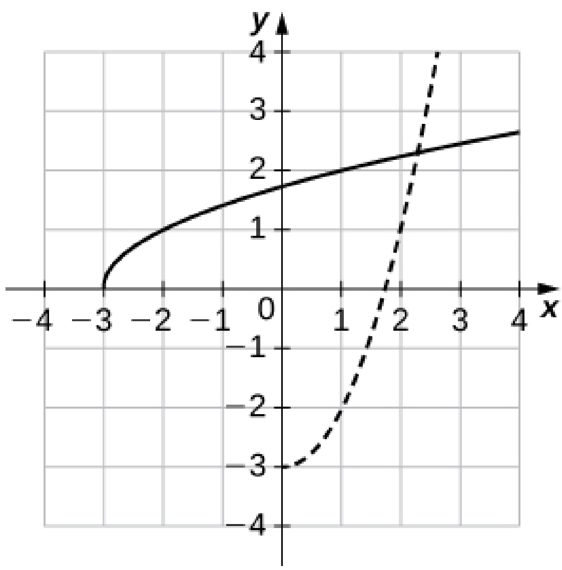
Exercise:

Problem:



Solution:

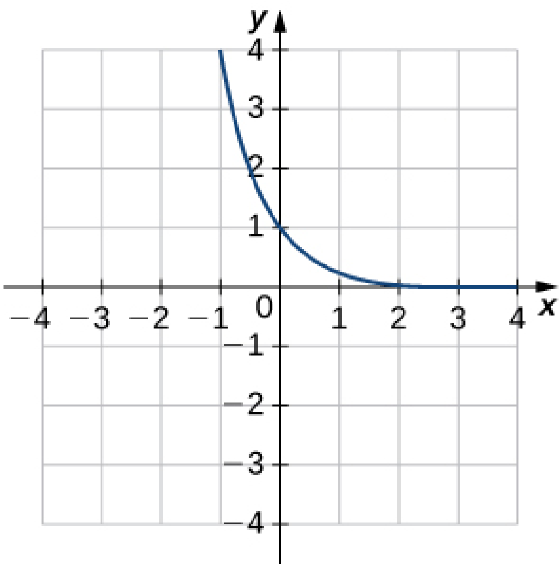
a.



b. $\left(f^{-1}\right)'(1)$~2

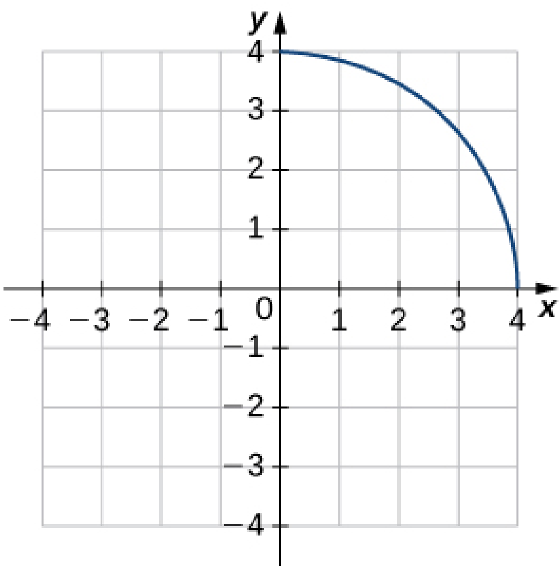
Exercise:

Problem:



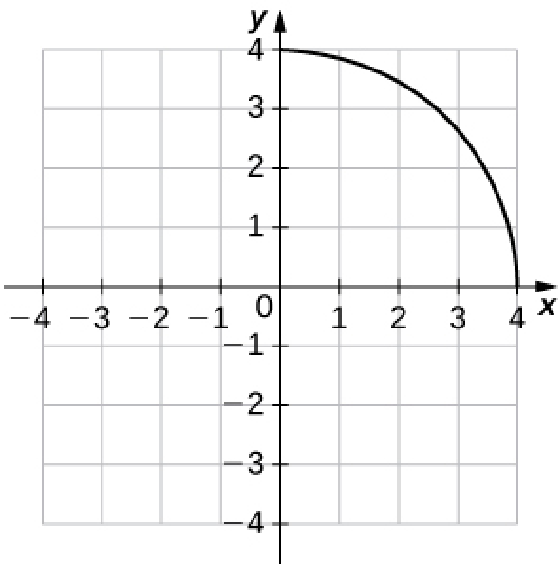
Exercise:

Problem:



Solution:

a.



b. $\left(f^{-1}\right)'(1)\text{~} - 1/\sqrt{3}$

For the following exercises, use the functions $y = f(x)$ to find

a. $\frac{df}{dx}$ at $x = a$ and
b. $x = f^{-1}(y)$.
c. Then use part b. to find $\frac{df^{-1}}{dy}$ at $y = f(a)$.

Exercise:

Problem: $f(x) = 6x - 1, x = -2$

Exercise:

Problem: $f(x) = 2x^3 - 3, x = 1$

Solution:

a. 6, b. $x = f^{-1}(y) = \left(\frac{y+3}{2}\right)^{1/3}$, c. $\frac{1}{6}$

Exercise:

Problem: $f(x) = 9 - x^2, 0 \le x \le 3, x = 2$

Exercise:

Problem: $f(x) = \sin x, x = 0$

Solution:

a. 1, b. $x = f^{-1}(y) = \sin^{-1} y$, c. 1

For each of the following functions, find $\left(f^{-1}\right)'(a)$.

Exercise:

Problem: $f(x) = x^2 + 3x + 2, x \ge -1, a = 2$

Exercise:

Problem: $f(x) = x^3 + 2x + 3, a = 0$

Solution:

$\frac{1}{5}$

Exercise:

Problem: $f(x) = x + \sqrt{x}, a = 2$

Exercise:

Problem: $f(x) = x - \frac{2}{x}, x < 0, a = 1$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $f(x) = x + \sin x, a = 0$

Exercise:

Problem: $f(x) = \tan x + 3x^2, a = 0$

Solution:

1

For each of the given functions $y = f(x)$,

a. find the slope of the tangent line to its inverse function f^{-1} at the indicated point P, and
b. find the equation of the tangent line to the graph of f^{-1} at the indicated point.

Exercise:

Problem: $f(x) = \frac{4}{1+x^2}, P(2, 1)$

Exercise:

Problem: $f(x) = \sqrt{x-4}, P(2, 8)$

Solution:

a. 4, b. $y = 4x$

Exercise:

Problem: $f(x) = \left(x^3 + 1\right)^4, P(16, 1)$

Exercise:

Problem: $f(x) = -x^3 - x + 2, P(-8, 2)$

Solution:

a. $-\frac{1}{96}$, b. $y=-\frac{1}{13}x+\frac{18}{13}$

Exercise:

Problem: $f\left(x\right)=x^5+3x^3-4x-8, P(-8,1)$

For the following exercises, find $\frac{dy}{dx}$ for the given function.

Exercise:

Problem: $y=\sin^{-1}\left(x^2\right)$

Solution:

$\frac{2x}{\sqrt{1-x^4}}$

Exercise:

Problem: $y=\cos^{-1}\left(\sqrt{x}\right)$

Exercise:

Problem: $y=\sec^{-1}\left(\frac{1}{x}\right)$

Solution:

$\frac{-1}{\sqrt{1-x^2}}$

Exercise:

Problem: $y=\sqrt{\csc^{-1}x}$

Exercise:

Problem: $y=\left(1+\tan^{-1}x\right)^3$

Solution:

$$\frac{3\left(1+\tan^{-1}x\right)^2}{1+x^2}$$

Exercise:

Problem: $y = \cos^{-1}(2x) \cdot \sin^{-1}(2x)$

Exercise:

Problem: $y = \frac{1}{\tan^{-1}(x)}$

Solution:

$$\frac{-1}{(1+x^2)\left(\tan^{-1}x\right)^2}$$

Exercise:

Problem: $y = \sec^{-1}(-x)$

Exercise:

Problem: $y = \cot^{-1}\sqrt{4-x^2}$

Solution:

$$\frac{x}{(5-x^2)\sqrt{4-x^2}}$$

Exercise:

Problem: $y = x \cdot \csc^{-1}x$

For the following exercises, use the given values to find $\left(f^{-1}\right)'(a)$.

Exercise:

Problem: $f(\pi) = 0, f\prime(\pi) = -1, a = 0$

Solution:

-1

Exercise:

Problem: $f\left(6\right)=2, f'\left(6\right)=\frac{1}{3}, a=2$

Exercise:

Problem: $f\left(\frac{1}{3}\right)=-8, f\prime\left(\frac{1}{3}\right)=2, a=-8$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $f\left(\sqrt{3}\right)=\frac{1}{2}, f\prime\left(\sqrt{3}\right)=\frac{2}{3}, a=\frac{1}{2}$

Exercise:

Problem: $f\left(1\right)=-3, f\prime\left(1\right)=10, a=-3$

Solution:

$\frac{1}{10}$

Exercise:

Problem: $f\left(1\right)=0, f\prime\left(1\right)=-2, a=0$

Exercise:

Problem:

[T] The position of a moving hockey puck after t seconds is $s\left(t\right)=\tan^{-1}t$ where s is in meters.

a. Find the velocity of the hockey puck at any time t.
b. Find the acceleration of the puck at any time t.
c. Evaluate a. and b. for $t=2, 4,$ and 6 seconds.

d. What conclusion can be drawn from the results in c.?

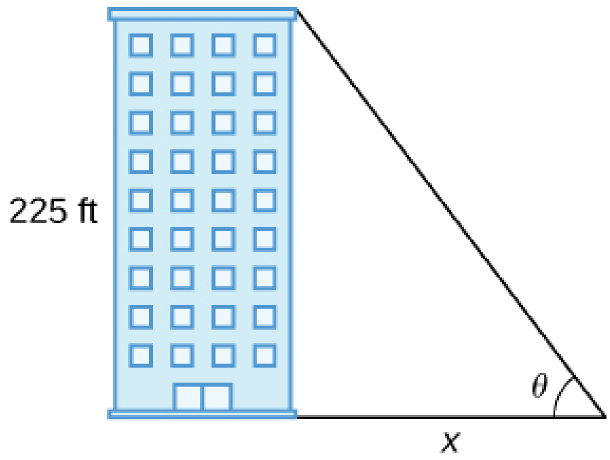
Solution:

a. $v(t) = \frac{1}{1+t^2}$ b. $a(t) = \frac{-2t}{(1+t^2)^2}$ c. $(a)0.2, 0.06, 0.03; (b) - 0.16, -0.028, -0.0088$ d. The hockey puck is decelerating/slowing down at 2, 4, and 6 seconds.

Exercise:

Problem:

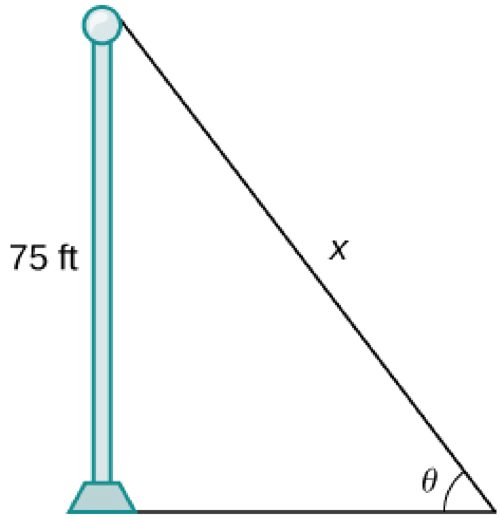
[T] A building that is 225 feet tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 272$ feet.



Exercise:

Problem:

[T] A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



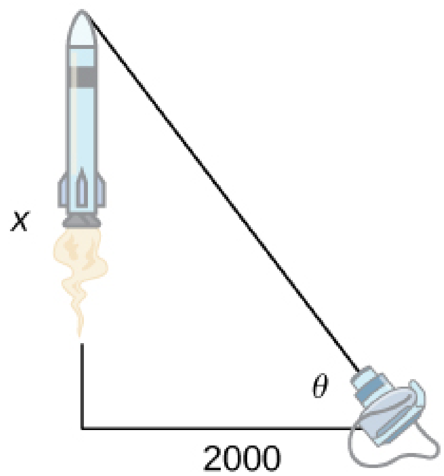
Solution:

-0.0168 radians per foot

Exercise:

Problem:

[T] A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by $\theta = \tan^{-1}\left(\frac{x}{2000}\right)$, where x is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.

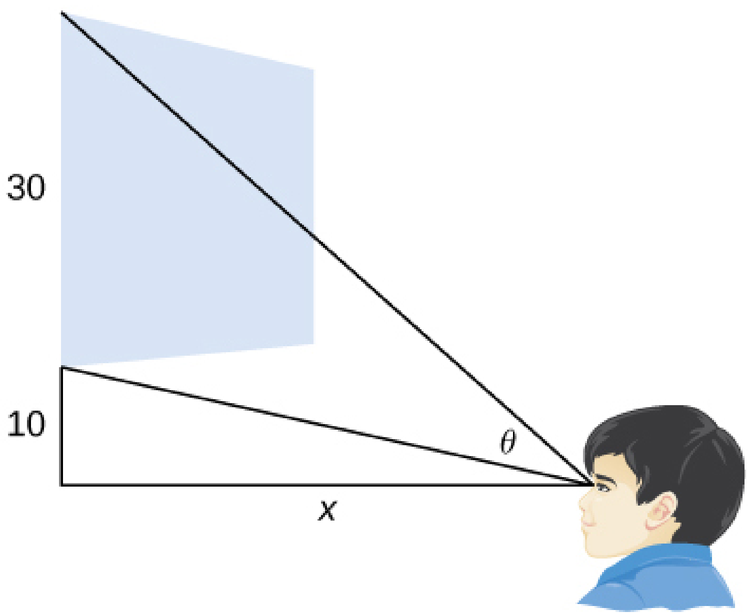


Exercise:

Problem:

[T] A local movie theater with a 30-foot-high screen that is 10 feet above a person's eye level when seated has a viewing angle θ (in radians) given by $\theta = \cot^{-1}\frac{x}{40} - \cot^{-1}\frac{x}{10}$,

where x is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.



a. Find $\frac{d\theta}{dx}$.
b. Evaluate $\frac{d\theta}{dx}$ for $x = 5, 10, 15,$ and 20.
c. Interpret the results in b..
d. Evaluate $\frac{d\theta}{dx}$ for $x = 25, 30, 35,$ and 40
e. Interpret the results in d. At what distance x should the person stand to maximize his or her viewing angle?

Solution:

a. $\frac{d\theta}{dx} = \frac{10}{100+x^2} - \frac{40}{1600+x^2}$ b. $\frac{18}{325}, \frac{9}{340}, \frac{42}{4745}, 0$ c. As a person moves farther away from the screen, the viewing angle is increasing, which implies that as he or she moves farther away, his or her screen vision is

widening. d. $-\frac{54}{12905}, -\frac{3}{500}, -\frac{198}{29945}, -\frac{9}{1360}$ e. As the person moves beyond 20 feet from the screen, the viewing angle is decreasing. The optimal distance the person should stand for maximizing the viewing angle is 20 feet.

Integrals Resulting in Inverse Trigonometric Functions

- Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall from Functions and Graphs that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also in Derivatives, we developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Sine Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Note:
Rule: Integration Formulas Resulting in Inverse Trigonometric Functions
The following integration formulas yield inverse trigonometric functions:

1. **Equation:**

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$$

2. **Equation:**

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

3.

Equation:

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

Proof

Let $y = \sin^{-1}\frac{x}{a}$. Then $a\sin y = x$. Now let's use implicit differentiation. We obtain

Equation:

$$\begin{aligned} \frac{d}{dx}(a\sin y) &= \frac{d}{dx}(x) \\ a\cos y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{a\cos y}. \end{aligned}$$

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$. Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1 - \sin^2 y}$. This gives

Equation:

$$\begin{aligned} \frac{1}{a\cos y} &= \frac{1}{a\sqrt{1-\sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - a^2\sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - x^2}}. \end{aligned}$$

Then for $-a \le x \le a$, we have

Equation:

$$\int \frac{1}{\sqrt{a^2 - u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + C.$$

□

Example:
Exercise:

Problem:
Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

Solution:

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

Equation:

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \Big|_0^1 \\ &= \sin^{-1} 1 - \sin^{-1} 0 \\ &= \tfrac{\pi}{2} - 0 \\ &= \tfrac{\pi}{2}. \end{aligned}$$

Note:
Exercise:

Problem: Find the antiderivative of $\int \frac{dx}{\sqrt{1-16x^2}}$.

Solution:

$\frac{1}{4}\sin^{-1}(4x)+C$

Hint

Substitute $u=4x$

Example:
Exercise:

Problem:
Finding an Antiderivative Involving an Inverse Trigonometric Function

Evaluate the integral $\int \frac{dx}{\sqrt{4-9x^2}}$.

Solution:

Substitute $u=3x$. Then $du=3dx$ and we have
Equation:

$$\int \frac{dx}{\sqrt{4-9x^2}}=\frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}.$$

Applying the formula with $a=2$, we obtain
Equation:

$$\begin{aligned} \int \frac{dx}{\sqrt{4-9x^2}} &= \frac{1}{3} \int \frac{du}{\sqrt{4-u^2}} \\ &= \frac{1}{3} \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= \frac{1}{3} \sin^{-1}\left(\frac{3x}{2}\right) + C. \end{aligned}$$

Note:
Exercise:

Problem:

Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9-x^2}}$.

Solution:

$\sin^{-1}\left(\frac{x}{3}\right) + C$

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.

Example:
Exercise:

Problem:
Evaluating a Definite Integral

Evaluate the definite integral $\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}$.

Solution:

The format of the problem matches the inverse sine formula. Thus,
Equation:

$$\begin{aligned}\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} &= \sin^{-1} u \Big|_0^{\sqrt{3}/2} \\ &= \left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[\sin^{-1}(0)\right] \\ &= \frac{\pi}{3}.\end{aligned}$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out −1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example:
Exercise:

Problem:
Finding an Antiderivative Involving the Inverse Tangent Function

Find an antiderivative of $\int \frac{1}{1+4x^2} dx.$

Solution:

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1}u + C$. So we use substitution, letting $u = 2x$, then $du = 2dx$ and $1/2du = dx$. Then, we have

Equation:

$$\frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(2x) + C.$$

Note:
Exercise:

Problem: Use substitution to find the antiderivative of $\displaystyle\int \frac{dx}{25+4x^2}$.

Solution:

$\frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + C$

Hint

Use the solving strategy from [link] and the rule on integration formulas resulting in inverse trigonometric functions.

Example:
Exercise:

Problem:
Applying the Integration Formulas

Find the antiderivative of $\int \frac{1}{9+x^2}dx$.

Solution:

Apply the formula with $a=3$. Then,
Equation:

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Note:
Exercise:

Problem: Find the antiderivative of $\int \frac{dx}{16+x^2}$.

Solution:

$\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$

Hint

Follow the steps in [link].

Example:
Exercise:

Problem:
Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution:

Use the formula for the inverse tangent. We have
Equation:

$$\begin{aligned}\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_{\sqrt{3}/3}^{\sqrt{3}} \\ &= \left[\tan^{-1}\left(\sqrt{3}\right)\right] - \left[\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right] \\ &= \frac{\pi}{6}.\end{aligned}$$

Note:
Exercise:

Problem: Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

Solution:

$\frac{\pi}{8}$

Hint

Follow the procedures from [link] to solve the problem.

Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in Derivatives of Exponential and Logarithmic Functions lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**
  $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$
  $$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
  $$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$

In the following exercises, evaluate each integral in terms of an inverse trigonometric function.

Exercise:

Problem: $\displaystyle\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Solution:

$\sin^{-1}x\Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$

Exercise:

Problem: $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

Exercise:

Problem: $\displaystyle\int_{\sqrt{3}}^{1} \frac{dx}{\sqrt{1+x^2}}$

Solution:

$\tan^{-1}x\Big|_{\sqrt{3}}^{1} = -\frac{\pi}{12}$

Exercise:

Problem: $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

Exercise:

Problem: $\displaystyle\int_{1}^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

Solution:

$\sec^{-1}x\Big|_{1}^{\sqrt{2}} = \frac{\pi}{4}$

Exercise:

Problem: $\displaystyle\int_{1}^{2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

In the following exercises, find each indefinite integral, using appropriate substitutions.

Exercise:

Problem: $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

Solution:

$\sin^{-1}\left(\frac{x}{3}\right)+C$

Exercise:

Problem: $\displaystyle\int \frac{dx}{\sqrt{1-16x^2}}$

Exercise:

Problem: $\displaystyle\int \frac{dx}{9+x^2}$

Solution:

$\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$

Exercise:

Problem: $\displaystyle\int \frac{dx}{25+16x^2}$

Exercise:

Problem: $\displaystyle\int \frac{dx}{|x|\sqrt{x^2-9}}$

Solution:

$\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right)+C$

Exercise:

Problem: $\displaystyle\int \frac{dx}{|x|\sqrt{4x^2-16}}$

Exercise:

Problem:

Explain the relationship $-\cos^{-1}t + C = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}t + C$. Is it true, in general, that $\cos^{-1}t = -\sin^{-1}t$?

Solution:

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$. So, $\sin^{-1}t = \frac{\pi}{2} - \cos^{-1}t$. They differ by a constant.

Exercise:

Problem:

Explain the relationship $\sec^{-1}t + C = \int \frac{dt}{|t|\sqrt{t^2-1}} = -\csc^{-1}t + C$. Is it true, in general, that $\sec^{-1}t = -\csc^{-1}t$?

Exercise:

Problem:

Explain what is wrong with the following integral: $\int_1^2 \frac{dt}{\sqrt{1-t^2}}$.

Solution:

$\sqrt{1-t^2}$ is not defined as a real number when $t > 1$.

Exercise:

Problem:

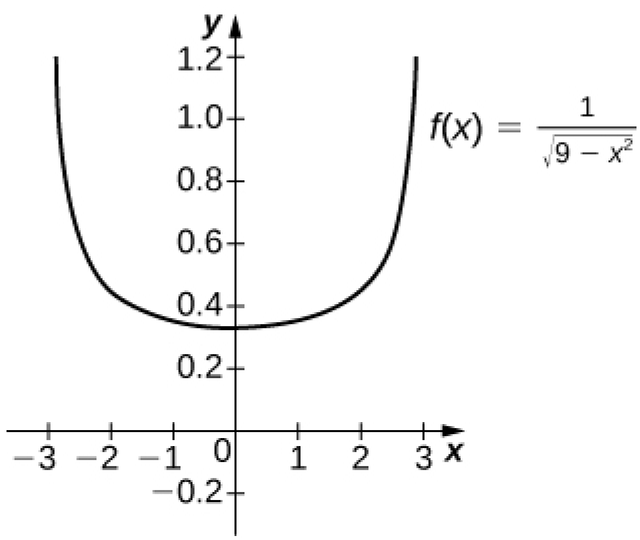
Explain what is wrong with the following integral: $\int_{-1}^1 \frac{dt}{|t|\sqrt{t^2-1}}$.

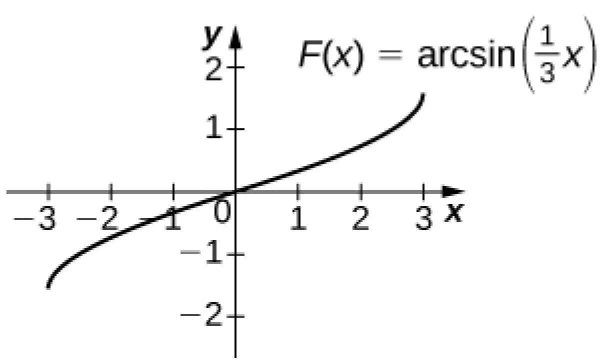
In the following exercises, solve for the antiderivative $\int f$ of f with $C = 0$, then use a calculator to graph f and the antiderivative over the given interval $[a, b]$. Identify a value of C such that adding C to the antiderivative recovers the definite integral $F(x) = \int_a^x f(t)dt$.

Exercise:

Problem: **[T]** $\int \frac{1}{\sqrt{9 - x^2}} dx$ over $[-3, 3]$

Solution:





The antiderivative is $\sin^{-1}\left(\frac{x}{3}\right) + C$. Taking $C = \frac{\pi}{2}$ recovers the definite integral.

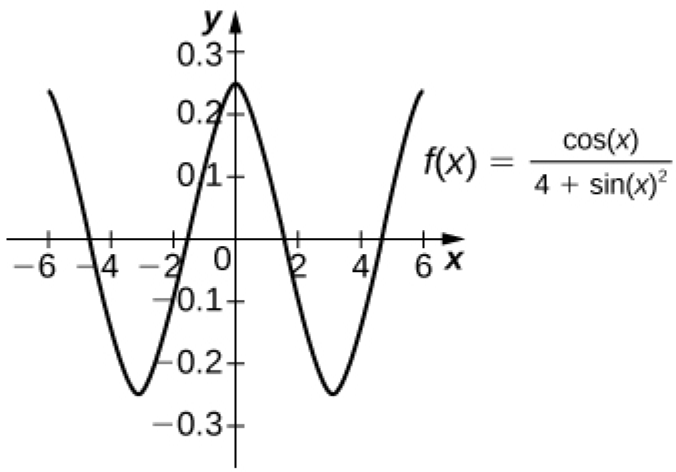
Exercise:

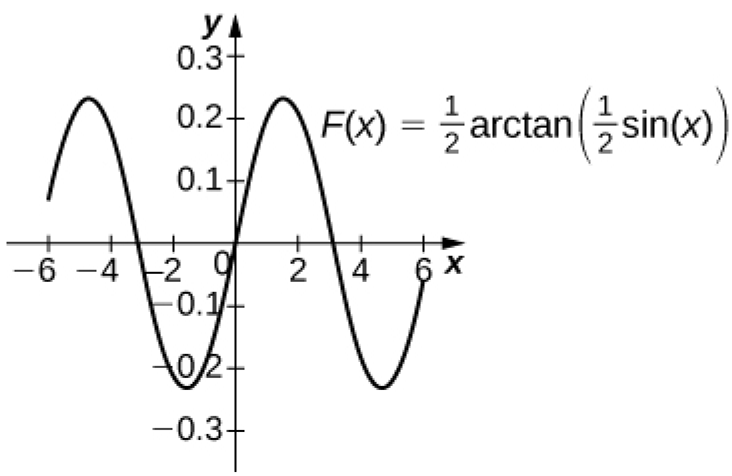
Problem: **[T]** $\displaystyle\int \frac{9}{9 + x^2} dx$ over $[-6, 6]$

Exercise:

Problem: **[T]** $\displaystyle\int \frac{\cos x}{4 + \sin^2 x} dx$ over $[-6, 6]$

Solution:





The antiderivative is $\frac{1}{2}\tan^{-1}\left(\frac{\sin x}{2}\right) + C$. Taking $C = \frac{1}{2}\tan^{-1}\left(\frac{\sin(6)}{2}\right)$ recovers the definite integral.

Exercise:

Problem: **[T]** $\int \frac{e^x}{1 + e^{2x}} dx$ over $[-6, 6]$

In the following exercises, compute the antiderivative using appropriate substitutions.

Exercise:

Problem: $\displaystyle\int \frac{\sin^{-1}t dt}{\sqrt{1-t^2}}$

Solution:

$\frac{1}{2}\left(\sin^{-1}t\right)^2 + C$

Exercise:

Problem: $\displaystyle\int \frac{dt}{\sin^{-1}t\sqrt{1-t^2}}$

Exercise:

Problem: $\displaystyle\int \frac{\tan^{-1}(2t)}{1+4t^2}dt$

Solution:

$\frac{1}{4}\left(\tan^{-1}(2t)\right)^2$

Exercise:

Problem: $\displaystyle\int \frac{t\tan^{-1}\left(t^2\right)}{1+t^4}dt$

Exercise:

Problem: $\displaystyle\int \frac{\sec^{-1}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}dt$

Solution:

$\frac{1}{4}\left(\sec^{-1}\left(\frac{t}{2}\right)^2\right) + C$

Exercise:

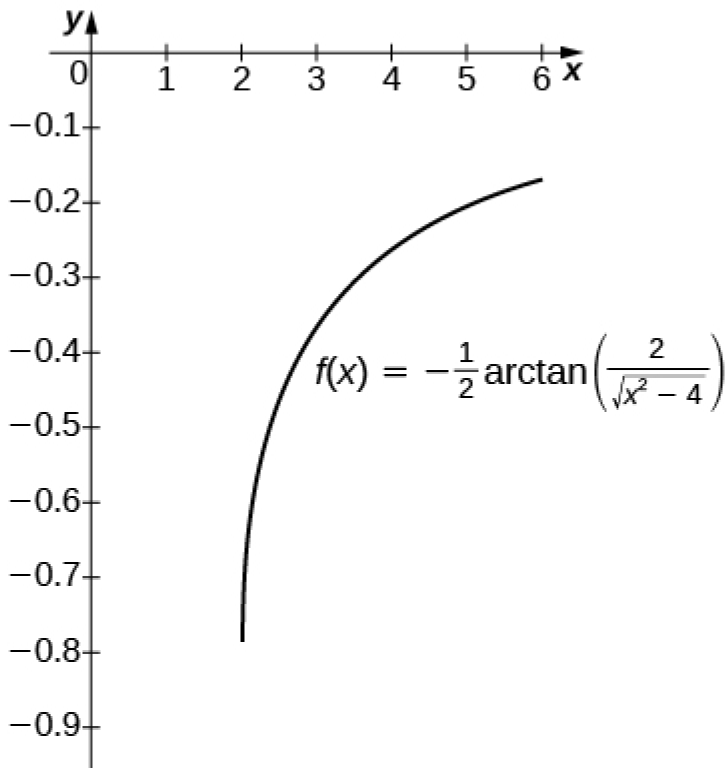
Problem: $\int \frac{t\sec^{-1}\left(t^2\right)}{t^2\sqrt{t^4-1}}dt$

In the following exercises, use a calculator to graph the antiderivative $\int f$ with $C=0$ over the given interval $\left[a,b\right]$. Approximate a value of C, if possible, such that adding C to the antiderivative gives the same value as the definite integral $F\left(x\right)=\int_a^x f\left(t\right)dt$.

Exercise:

Problem: **[T]** $\int \frac{1}{x\sqrt{x^2-4}}dx$ over $\left[2,6\right]$

Solution:



The antiderivative is $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C$. Taking $C = 0$ recovers the definite integral over $[2, 6]$.

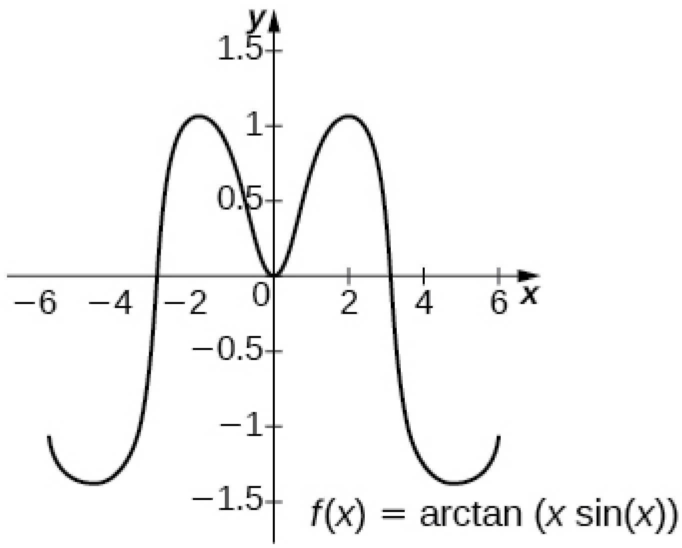
Exercise:

Problem: **[T]** $\int \frac{1}{(2x+2)\sqrt{x}}dx$ over $[0, 6]$

Exercise:

Problem: **[T]** $\int \frac{(\sin x + x\cos x)}{1 + x^2\sin^2 x}dx$ over $[-6, 6]$

Solution:



The general antiderivative is $\tan^{-1}(x\sin x)+C$. Taking $C=-\tan^{-1}(6\sin(6))$ recovers the definite integral.

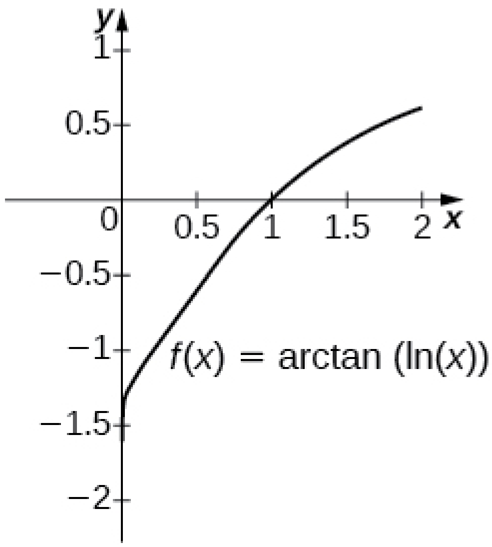
Exercise:

Problem: [T] $\int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}dx$ over $[0,2]$

Exercise:

Problem: [T] $\int \frac{1}{x+x\ln^2 x}$ over $[0,2]$

Solution:



The general antiderivative is $\tan^{-1}(\ln x) + C$. Taking $C = \frac{\pi}{2} = \tan^{-1}\infty$ recovers the definite integral.

Exercise:

Problem: [T] $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ over $[-1, 1]$

In the following exercises, compute each integral using appropriate substitutions.

Exercise:

Problem: $\int \frac{e^x}{\sqrt{1-e^{2t}}} dt$

Solution:

$\sin^{-1}\left(e^t\right) + C$

Exercise:

Problem: $\int \frac{e^t}{1+e^{2t}} dt$

Exercise:

Problem: $\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

Solution:

$\sin^{-1}(\ln t) + C$

Exercise:

Problem: $\int \frac{dt}{t\left(1+\ln^2 t\right)}$

Exercise:

Problem: $\int \frac{\cos^{-1}(2t)}{\sqrt{1-4t^2}} dt$

Solution:

$-\frac{1}{2}\left(\cos^{-1}(2t)\right)^2 + C$

Exercise:

Problem: $\int \frac{e^t \cos^{-1}\left(e^t\right)}{\sqrt{1-e^{2t}}} dt$

In the following exercises, compute each definite integral.

Exercise:

Problem: $\displaystyle\int_0^{1/2} \frac{\tan\left(\sin^{-1} t\right)}{\sqrt{1-t^2}}\,dt$

Solution:

$\frac{1}{2}\ln\left(\frac{4}{3}\right)$

Exercise:

Problem: $\displaystyle\int_{1/4}^{1/2} \frac{\tan\left(\cos^{-1} t\right)}{\sqrt{1-t^2}}\,dt$

Exercise:

Problem: $\displaystyle\int_0^{1/2} \frac{\sin\left(\tan^{-1} t\right)}{1+t^2}\,dt$

Solution:

$1-\frac{2}{\sqrt{5}}$

Exercise:

Problem: $\displaystyle\int_0^{1/2} \frac{\cos\left(\tan^{-1} t\right)}{1+t^2}\,dt$

Exercise:

Problem:

For $A>0$, compute $I(A)=\displaystyle\int_{-A}^{A} \frac{dt}{1+t^2}$ and evaluate $\lim\limits_{a\to\infty} I(A)$, the area under the graph of $\frac{1}{1+t^2}$ on $[-\infty,\infty]$.

Solution:

$2\tan^{-1}(A) \to \pi$ as $A \to \infty$

Exercise:

Problem:

For $1 < B < \infty$, compute $I(B) = \int_1^B \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim_{B\to\infty} I(B)$, the area under the graph of $\frac{1}{t\sqrt{t^2-1}}$ over $[1, \infty)$.

Exercise:

Problem:

Use the substitution $u = \sqrt{2}\cot x$ and the identity $1 + \cot^2 x = \csc^2 x$ to evaluate $\int \frac{dx}{1+\cos^2 x}$. (*Hint:* Multiply the top and bottom of the integrand by $\csc^2 x$.)

Solution:

Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x} dx = \int \frac{\csc^2 x}{1 + 2\cot^2 x} dx.$ Set $u = \sqrt{2}\cot x$. Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}} \int \frac{du}{1+u^2} = -\frac{1}{\sqrt{2}}\tan^{-1} u + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\cot x\right) + C.$ If one uses the identity $\tan^{-1} s + \tan^{-1}\left(\frac{1}{s}\right) = \frac{\pi}{2}$, then this can also be written $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C.$

Exercise:

Problem:

[T] Approximate the points at which the graphs of $f(x) = 2x^2 - 1$ and $g(x) = \left(1 + 4x^2\right)^{-3/2}$ intersect, and approximate the area between their graphs accurate to three decimal places.

Exercise:

Problem:

47. **[T]** Approximate the points at which the graphs of $f(x) = x^2 - 1$ and $f(x) = x^2 - 1$ intersect, and approximate the area between their graphs accurate to three decimal places.

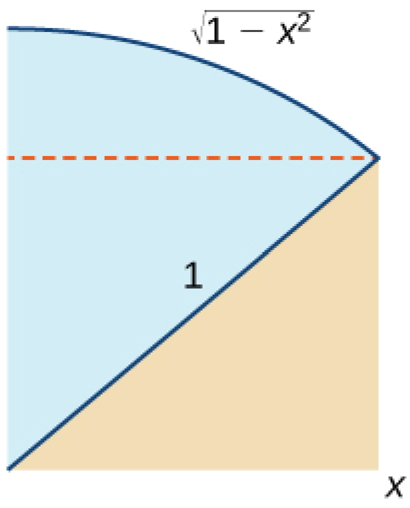
Solution:

$x \approx \pm 1.13525.$ The left endpoint estimate with $N = 100$ is 2.796 and these decimals persist for $N = 500.$

Exercise:

Problem:

Use the following graph to prove that $\int_0^x \sqrt{1-t^2}dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x.$



Chapter Review Exercises

True or False. Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

Exercise:

Problem:

If $f(x) > 0, f'(x) > 0$ for all x, then the right-hand rule underestimates the integral $\int_a^b f(x)$. Use a graph to justify your answer.

Solution:

False

Exercise:

Problem: $\int_a^b f(x)^2 dx = \int_a^b f(x)dx \int_a^b f(x)dx$

Exercise:

Problem:

If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \leq \int_a^b g(x)$.

Solution:

True

Exercise:

Problem: All continuous functions have an antiderivative.

Evaluate the Riemann sums L_4 and R_4 for the following functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

Exercise:

Problem: $y = 3x^2 - 2x + 1$ over $[-1, 1]$

Solution:

$L_4 = 5.25, R_4 = 3.25,$ exact answer: 4

Exercise:

Problem: $y = \ln\left(x^2 + 1\right)$ over $[0, e]$

Exercise:

Problem: $y = x^2 \sin x$ over $[0, \pi]$

Solution:

$L_4 = 5.364, R_4 = 5.364,$ exact answer: 5.870

Exercise:

Problem: $y = \sqrt{x} + \frac{1}{x}$ over $[1, 4]$

Evaluate the following integrals.

Exercise:

Problem: $\int_{-1}^{1} \left(x^3 - 2x^2 + 4x\right) dx$

Solution:

$-\frac{4}{3}$

Exercise:

Problem: $\int_0^4 \frac{3t}{\sqrt{1+6t^2}}\,dt$

Exercise:

Problem: $\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)\,d\theta$

Solution:

1

Exercise:

Problem: $\int_0^{\pi/4} e^{\cos^2 x}\sin x\cos dx$

Find the antiderivative.

Exercise:

Problem: $\int \frac{dx}{(x+4)^3}$

Solution:

$-\frac{1}{2(x+4)^2}+C$

Exercise:

Problem: $\int x\ln\left(x^2\right)dx$

Exercise:

Problem: $\int \frac{4x^2}{\sqrt{1-x^6}} dx$

Solution:

$\frac{4}{3} \sin^{-1}\left(x^3\right) + C$

Exercise:

Problem: $\int \frac{e^{2x}}{1+e^{4x}} dx$

Find the derivative.

Exercise:

Problem: $\frac{d}{dt} \int_0^t \frac{\sin x}{\sqrt{1+x^2}} dx$

Solution:

$\frac{\sin t}{\sqrt{1+t^2}}$

Exercise:

Problem: $\frac{d}{dx} \int_1^{x^3} \sqrt{4-t^2} dt$

Exercise:

Problem: $\frac{d}{dx} \int_1^{\ln(x)} \left(4t+e^t\right) dt$

Solution:

$4\frac{\ln x}{x} + 1$

Exercise:

Problem: $\frac{d}{dx} \int_0^{\cos x} e^{t^2} dt$

The following problems consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change ($)
1980	0
1985	−5,468,750
1990	−755,495
1995	−73,005
2000	−29,768
2005	−918
2010	−177

Exercise:

Problem:

If the average cost per gigabyte of RAM in 2010 is $12, find the average cost per gigabyte of RAM in 1980.

Solution:

$6,328,113

Exercise:

Problem:

The average cost per gigabyte of RAM can be approximated by the function $C\left(t\right)=8,500,000(0.65)^{t}$, where t is measured in years since 1980, and C is cost in US$. Find the average cost per gigabyte of RAM for 1980 to 2010.

Exercise:

Problem: Find the average cost of 1GB RAM for 2005 to 2010.

Solution:

$73.36

Exercise:

Problem:

The velocity of a bullet from a rifle can be approximated by $v\left(t\right)=6400t^{2}-6505t+2686$, where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0\le t\le 0.5$. What is the total distance the bullet travels in 0.5 sec?

Exercise:

Problem:

What is the average velocity of the bullet for the first half-second?

Solution:

$\frac{19117}{12}\,\text{ft/sec}$, or $1593\,\text{ft/sec}$

Derivatives of Exponential and Logarithmic Functions

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in Introduction to Functions and Graphs, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B\left(x\right)=b^{x}, b>0$, is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^{n}, where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^{0}=1, b^{-n}=\frac{1}{b^{n}}$, for a positive integer n, and $b^{s/t}=(\sqrt[t]{b})^{s}$ for positive integers s and t. These definitions leave open the question of the value of b^{r} where r is an arbitrary real number. By assuming the *continuity* of $B\left(x\right)=b^{x}, b>0$, we may interpret b^{r} as $\lim\limits_{x\to r}b^{x}$ where the values of x as we take the limit are rational. For example, we may view 4^{π} as the number satisfying
Equation:

$$4^{3}<4^{\pi}<4^{4}, 4^{3.1}<4^{\pi}<4^{3.2}, 4^{3.14}<4^{\pi}<4^{3.15},$$
$$4^{3.141}<4^{\pi}<4^{3.142}, 4^{3.1415}<4^{\pi}<4^{3.1416}, \ldots.$$

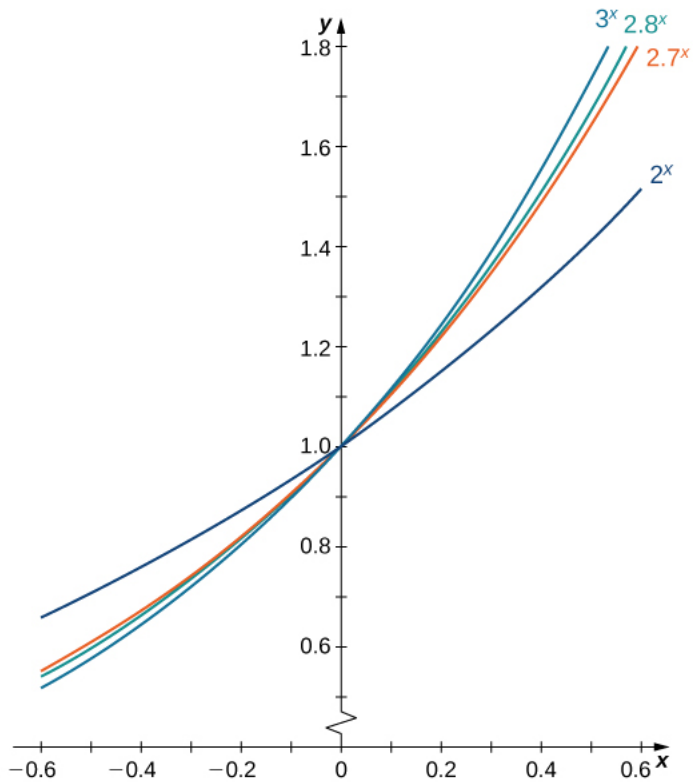
As we see in the following table, $4^{\pi}\approx 77.88$.

x	4^{x}	x	4^{x}
4^{3}	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^{4}	256

Approximating a Value of 4^{π}

We also assume that for $B(x) = b^x, b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in Introduction to Functions and Graphs. [link] provides graphs of the functions $y = 2^x, y = 3^x, y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.



The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

Equation:

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$ to obtain the estimate

Equation:

$$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}.$$

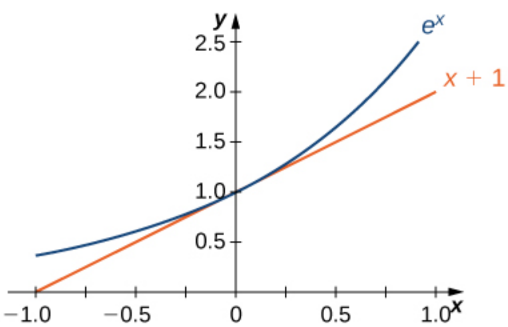
See the following table.

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

Estimating a Value of e

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in [link]. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.



The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x, b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

Equation:

$$B'(0) = \lim_{h\to 0}\frac{b^{0+h}-b^0}{h} = \lim_{h\to 0}\frac{b^h-1}{h}.$$

Turning to $B'(x)$, we obtain the following.
Equation:

$$\begin{aligned} B'(x) &= \lim_{h \to 0} \frac{b^{x+h}-b^x}{h} && \text{Apply the limit definition of the derivative.} \\ &= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\ &= \lim_{h \to 0} \frac{b^x(b^h-1)}{h} && \text{Factor out } b^x. \\ &= b^x \lim_{h \to 0} \frac{b^h-1}{h} && \text{Apply a property of limits.} \\ &= b^x B'(0) && \text{Use } B'(0) = \lim_{h \to 0} \frac{b^{0+h}-b^0}{h} = \lim_{h \to 0} \frac{b^h-1}{h}. \end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is
Equation:

$$B'(x) = b^x B'(0).$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

Note:
Derivative of the Natural Exponential Function
Let $E(x) = e^x$ be the natural exponential function. Then
Equation:

$$E'(x) = e^x.$$

In general,
Equation:

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x).$$

Example:
Exercise:

Problem:
Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution:

Using the derivative formula and the chain rule,
Equation:

$$\begin{aligned} f'(x) &= e^{\tan(2x)} \frac{d}{dx}(\tan(2x)) \\ &= e^{\tan(2x)} \sec^2(2x) \cdot 2. \end{aligned}$$

Example:
Exercise:

Problem:
Combining Differentiation Rules

Find the derivative of $y = \frac{e^{x^2}}{x}$.

Solution:

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.
Equation:

$$\begin{aligned} y' &= \frac{\left(e^{x^2}\cdot 2\right)x\cdot x - 1\cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\ &= \frac{e^{x^2}\left(2x^2-1\right)}{x^2} && \text{Simplify.} \end{aligned}$$

Note:
Exercise:

Problem: Find the derivative of $h\left(x\right) = xe^{2x}$.

Solution:

$h'\left(x\right) = e^{2x} + 2xe^{2x}$

Hint

Don't forget to use the product rule.

Example:
Exercise:

Problem:
Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A\left(t\right) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'\left(t\right)$, to the population, $A\left(t\right)$ is constant.

Solution:

First find $A'\left(t\right)$. By using the chain rule, we have $A'\left(t\right) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by
Equation:

$$A'\left(t\right) = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Note:
Exercise:

Problem:

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Solution:

996

Hint

Find $A'(4)$.

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Note:
The Derivative of the Natural Logarithmic Function
If $x > 0$ and $y = \ln x$, then
Equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by
Equation:

$$h'(x) = \frac{1}{g(x)} g'(x).$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation
Equation:

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

Equation:

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain
Equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have
Equation:

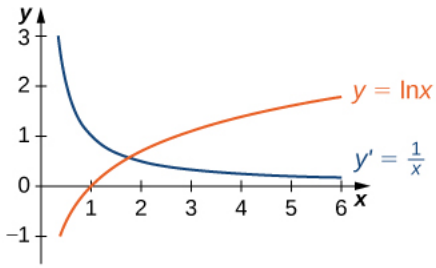
$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields
Equation:

$$h'(x) = \frac{1}{g(x)} g'(x).$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in [link].



The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$.

Example:
Exercise:

Problem:
Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln\left(x^3 + 3x - 4\right)$.

Solution:

Use [link] directly.

Equation:

$$\begin{aligned} f'(x) &= \frac{1}{x^3+3x-4} \cdot \left(3x^2+3\right) && \text{Use } g(x) = x^3+3x-4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x). \\ &= \frac{3x^2+3}{x^3+3x-4} && \text{Rewrite.} \end{aligned}$$

Example:

Exercise:

Problem:

Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right)$.

Solution:

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

Equation:

$$\begin{aligned} f(x) &= \ln\left(\frac{x^2 \sin x}{2x+1}\right) = 2\ln x + \ln(\sin x) - \ln(2x+1) && \text{Apply properties of logarithms.} \\ f'(x) &= \frac{2}{x} + \cot x - \frac{2}{2x+1} && \text{Apply sum rule and } h'(x) = \frac{1}{g(x)} g'(x). \end{aligned}$$

Note:

Exercise:

Problem: Differentiate: $f(x) = \ln(3x+2)^5$.

Solution:

$f'(x) = \frac{15}{3x+2}$

Hint

Use a property of logarithms to simplify before taking the derivative.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = log_b x$ and $y = b^x$ for $b > 0, b \neq 1$.

Note:
Derivatives of General Exponential and Logarithmic Functions
Let $b > 0, b \neq 1$, and let $g(x)$ be a differentiable function.

i. If, $y = \log_b x$, then
Equation:

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,
Equation:

$$h'(x) = \frac{g'(x)}{g(x) \ln b}.$$

ii. If $y = b^x$, then
Equation:

$$\frac{dy}{dx} = b^x \ln b.$$

More generally, if $h(x) = b^{g(x)}$, then
Equation:

$$h'(x) = b^{g(x)} g\prime\prime(x) \ln b.$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that
Equation:

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in [link] now follows from the chain rule.

If $y = b^x$, then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y} \frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that
Equation:

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative ([link]) follows from the chain rule.

□

Example:
Exercise:

Problem:
Applying Derivative Formulas

Find the derivative of $h\left(x\right)=\frac{3^x}{3^x+2}$.

Solution:

Use the quotient rule and [link].
Equation:

$$\begin{array}{ccccc}h'\left(x\right)\hfill & =\hfill & \frac{3^x \ln 3(3^x+2)-3^x \ln 3(3^x)}{\left(3^x+2\right)^2}\hfill & & \text{Apply the quotient rule.}\hfill \\ & =\hfill & \frac{2\cdot 3^x \ln 3}{\left(3^x+2\right)^2}\hfill & & \text{Simplify.}\hfill \end{array}$$

Example:
Exercise:

Problem:
Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y=\log_2\left(3x+1\right)$ at $x=1$.

Solution:

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x=1$. Using [link], we see that
Equation:

$$\frac{dy}{dx}=\frac{3}{\ln 2\left(3x+1\right)}.$$

By evaluating the derivative at $x=1$, we see that the tangent line has slope
Equation:

$$\frac{dy}{dx}\Big|_{x=1}=\frac{3}{4\ln 2}=\frac{3}{\ln 16}.$$

Note:
Exercise:

Problem: Find the slope for the line tangent to $y=3^x$ at $x=2$.

Solution:

$9\ln\left(3\right)$

Hint

Evaluate the derivative at $x = 2$.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Note:
Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example:
Exercise:

Problem:
Using Logarithmic Differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}$.

Solution:

Use logarithmic differentiation to find this derivative.
Equation:

$$\begin{aligned}
\ln y &= \ln\left(2x^4+1\right)^{\tan x} && \text{Step 1. Take the natural logarithm o} \\
\ln y &= \tan x \ln\left(2x^4+1\right) && \text{Step 2. Expand using properties of lo} \\
\frac{1}{y}\frac{dy}{dx} &= \sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x && \text{Step 3. Differentiate both sides. Use product rule on the right.} \\
\frac{dy}{dx} &= y\cdot\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x\right) && \text{Step 4. Multiply by } y \text{ on both sides.} \\
\frac{dy}{dx} &= \left(2x^4+1\right)^{\tan x}\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x\right) && \text{Step 5. Substitute } y = \left(2x^4+1\right)^{\tan x}.
\end{aligned}$$

Example:

Exercise:

Problem:
Using Logarithmic Differentiation

Find the derivative of $y=\frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution:

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

Equation:

$$\begin{array}{rcll} \ln y & = & \ln \frac{x\sqrt{2x+1}}{e^x \sin^3 x} & \text{Step 1. Take the natural logarithm of both side} \\ \ln y & = & \ln x+\frac{1}{2}\ln(2x+1)-x\ln e-3\ln \sin x & \text{Step 2. Expand using properties of logarithms.} \\ \frac{1}{y}\frac{dy}{dx} & = & \frac{1}{x}+\frac{1}{2x+1}-1-3\frac{\cos x}{\sin x} & \text{Step 3. Differentiate both sides.} \\ \frac{dy}{dx} & = & y\left(\frac{1}{x}+\frac{1}{2x+1}-1-3\cot x\right) & \text{Step 4. Multiply by } y \text{ on both sides.} \\ \frac{dy}{dx} & = & \frac{x\sqrt{2x+1}}{e^x \sin^3 x}\left(\frac{1}{x}+\frac{1}{2x+1}-1-3\cot x\right) & \text{Step 5. Substitute } y=\frac{x\sqrt{2x+1}}{e^x \sin^3 x}. \end{array}$$

Example:
Exercise:

Problem:
Extending the Power Rule

Find the derivative of $y=x^r$ where r is an arbitrary real number.

Solution:

The process is the same as in [link], though with fewer complications.

Equation:

$$\begin{array}{rcll} \ln y & = & \ln x^r & \text{Step 1. Take the natural logarithm of both sides.} \\ \ln y & = & r\ln x & \text{Step 2. Expand using properties of logarithms.} \\ \frac{1}{y}\frac{dy}{dx} & = & r\frac{1}{x} & \text{Step 3. Differentiate both sides.} \\ \frac{dy}{dx} & = & y\frac{r}{x} & \text{Step 4. Multiply by } y \text{ on both sides.} \\ \frac{dy}{dx} & = & x^r\frac{r}{x} & \text{Step 5. Substitute } y=x^r. \\ \frac{dy}{dx} & = & rx^{r-1} & \text{Simplify.} \end{array}$$

Note:
Exercise:

Problem: Use logarithmic differentiation to find the derivative of $y=x^x$.

Solution:

$\frac{dy}{dx} = x^x(1 + \ln x)$

Hint

Follow the problem solving strategy.

Note:
Exercise:

Problem: Find the derivative of $y = (\tan x)^\pi$.

Solution:

$y' = \pi(\tan x)^{\pi - 1}\sec^2 x$

Hint

Use the result from [link].

Key Concepts

- On the basis of the assumption that the exponential function $y = b^x, b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \frac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.
- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

Key Equations

- **Derivative of the natural exponential function**
  $\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g'(x)$
- **Derivative of the natural logarithmic function**
  $\frac{d}{dx}(\ln g(x)) = \frac{1}{g(x)}g'(x)$
- **Derivative of the general exponential function**
  $\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}g'(x)\ln b$
- **Derivative of the general logarithmic function**
  $\frac{d}{dx}(\log_b g(x)) = \frac{g'(x)}{g(x)\ln b}$

For the following exercises, find $f'(x)$ for each function.
Exercise:

Problem: $f(x) = x^2 e^x$

Solution:

$2xe^x + x^2 e^x$

Exercise:

Problem: $f\left(x\right) = \frac{e^{-x}}{x}$

Exercise:

Problem: $f\left(x\right) = e^{x^3 \ln x}$

Solution:

$e^{x^3 \ln x}\left(3x^2 \ln x + x^2\right)$

Exercise:

Problem: $f\left(x\right) = \sqrt{e^{2x} + 2x}$

Exercise:

Problem: $f\left(x\right) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution:

$\frac{4}{\left(e^x + e^{-x}\right)^2}$

Exercise:

Problem: $f\left(x\right) = \frac{10^x}{\ln 10}$

Exercise:

Problem: $f\left(x\right) = 2^{4x} + 4x^2$

Solution:

$2^{4x+2} \cdot \ln 2 + 8x$

Exercise:

Problem: $f\left(x\right) = 3^{\sin 3\mathrm{x}}$

Exercise:

Problem: $f\left(x\right) = x^\pi \cdot \pi^x$

Solution:

$\pi x^{\pi - 1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$

Exercise:

Problem: $f\left(x\right) = \ln\left(4x^3 + x\right)$

Exercise:

Problem: $f\left(x\right) = \ln \sqrt{5x - 7}$

Solution:

$\frac{5}{2(5x-7)}$

Exercise:

Problem: $f\left(x\right)=x^2\ln 9x$

Exercise:

Problem: $f\left(x\right)=\log\left(\sec x\right)$

Solution:

$\frac{\tan x}{\ln 10}$

Exercise:

Problem: $f\left(x\right)=\log_7\left(6x^4+3\right)^5$

Exercise:

Problem: $f\left(x\right)=2^x\cdot\log_3 7^{x^2-4}$

Solution:

$2^x\cdot\ln 2\cdot\log_3 7^{x^2-4}+2^x\cdot\frac{2x\ln 7}{\ln 3}$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

Exercise:

Problem: $y=x^{\sqrt{x}}$

Exercise:

Problem: $y=\left(\sin 2x\right)^{4x}$

Solution:

$\left(\sin 2x\right)^{4x}\left[4\cdot\ln\left(\sin 2x\right)+8x\cdot\cot 2x\right]$

Exercise:

Problem: $y=\left(\ln x\right)^{\ln x}$

Exercise:

Problem: $y=x^{\log_2 x}$

Solution:

$x^{\log_2 x}\cdot\frac{2\ln x}{x\ln 2}$

Exercise:

Problem: $y=\left(x^2-1\right)^{\ln x}$

Exercise:

Problem: $y=x^{\cot x}$

Solution:

$x^{\cot x}\cdot\left[-\csc^2 x\cdot\ln x+\frac{\cot x}{x}\right]$

Exercise:

Problem: $y=\frac{x+11}{\sqrt[3]{x^2-4}}$

Exercise:

Problem: $y=x^{-1/2}\left(x^2+3\right)^{2/3}(3x-4)^4$

Solution:

$x^{-1/2}\left(x^2+3\right)^{2/3}(3x-4)^4\cdot\left[\frac{-1}{2x}+\frac{4x}{3(x^2+3)}+\frac{12}{3x-4}\right]$

Exercise:

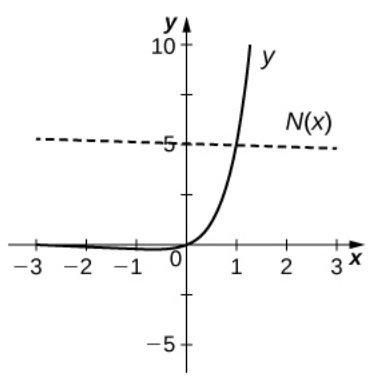
Problem: **[T]** Find an equation of the tangent line to the graph of $f(x)=4xe^{(x^2-1)}$ at the point where $x=-1$. Graph both the function and the tangent line.

Exercise:

Problem:

[T] Find the equation of the line that is normal to the graph of $f(x)=x\cdot 5^x$ at the point where $x=1$. Graph both the function and the normal line.

Solution:



$y=\frac{-1}{5+5\ln 5}x+\left(5+\frac{1}{5+5\ln 5}\right)$

Exercise:

Problem:

[T] Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (*Hint*: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

Exercise:

Problem: Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.
b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Solution:

a. $x = e$~2.718 b. $(e, \infty), (0, e)$

Exercise:

Problem: The formula $I\left(t\right) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\frac{\pi}{2}$	(ii)
π	(iii)
$\frac{3\pi}{2}$	(iv)
2π	(v)
2π	(vi)
3π	(vii)
$\frac{7\pi}{2}$	(viii)
4π	(ix)

b. Using only the values in the table, determine where the tangent line to the graph of $I\left(t\right)$ is horizontal.

Exercise:

Problem:

[T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.
b. Use a. to determine the rate at which the population is increasing in t years.
c. Use b. to determine the rate at which the population is increasing in 10 years.

Solution:

a. $P = 500{,}000(1.05)^t$ individuals b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year c. $39{,}737$ individuals per year

Exercise:

Problem:

[T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.
b. Use a. to determine the rate at which the substance is decaying in t hours.
c. Use b. to determine the rate of decay at $t = 4$ hours.

Exercise:

Problem:

[T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function

$N(t) = 5.3e^{0.093t^2 - 0.87t}, (0 \leq t \leq 4),$

where $N(t)$ gives the number of cases (in thousands) and *t* is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in the United States.

Solution:

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States. b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

Exercise:

Problem:

[T] The *relative rate of change* of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a, b, and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.
b. Use a. to find the relative rate of change of a population in $x = 20$ months when $a = 204, b = 0.0198$, and $c = 0.15$.
c. Briefly interpret what the result of b. means.

For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

Years since 1790	**Population**
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over TimeSource: http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade.

Exercise:

Problem: **[T]** Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Solution:

$p = 35741(1.045)^t$

Exercise:

Problem:

[T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

Exercise:

Problem:

[T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Solution:

Years since 1790	P''
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

Exercise:

Problem: **[T]** Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?
b. Estimate the population in 2010. Was the prediction correct from a.?

Chapter Review Exercises

True or False? Justify the answer with a proof or a counterexample.

Exercise:

Problem: Every function has a derivative.

Solution:

False.

Exercise:

Problem: A continuous function has a continuous derivative.

Exercise:

Problem: A continuous function has a derivative.

Solution:

False

Exercise:

Problem: If a function is differentiable, it is continuous.

Use the limit definition of the derivative to exactly evaluate the derivative.

Exercise:

Problem: $f\left(x\right)=\sqrt{x+4}$

Solution:

$\frac{1}{2\sqrt{x+4}}$

Exercise:

Problem: $f\left(x\right)=\frac{3}{x}$

Find the derivatives of the following functions.

Exercise:

Problem: $f\left(x\right)=3x^{3}-\frac{4}{x^{2}}$

Solution:

$9x^{2}+\frac{8}{x^{3}}$

Exercise:

Problem: $f\left(x\right)=\left(4-x^{2}\right)^{3}$

Exercise:

Problem: $f\left(x\right)=e^{\sin x}$

Solution:

$e^{\sin x}\cos x$

Exercise:

Problem: $f\left(x\right)=\ln\left(x+2\right)$

Exercise:

Problem: $f\left(x\right)=x^{2}\cos x+x\tan\left(x\right)$

Solution:

$x\sec^{2}\left(x\right)+2x\cos\left(x\right)+\tan\left(x\right)-x^{2}\sin\left(x\right)$

Exercise:

Problem: $f\left(x\right)=\sqrt{3x^{2}+2}$

Exercise:

Problem: $f\left(x\right)=\frac{x}{4}\sin^{-1}\left(x\right)$

Solution:

$\frac{1}{4}\left(\frac{x}{\sqrt{1-x^2}}+\sin^{-1}(x)\right)$

Exercise:

Problem: $x^2y=(y+2)+xy\sin(x)$

Find the following derivatives of various orders.

Exercise:

Problem: First derivative of $y=x\ln(x)\cos x$

Solution:

$\cos x\cdot(\ln x+1)-x\ln(x)\sin x$

Exercise:

Problem: Third derivative of $y=(3x+2)^2$

Exercise:

Problem: Second derivative of $y=4^x+x^2\sin(x)$

Solution:

$4^x(\ln 4)^2+2\sin x+4x\cos x-x^2\sin x$

Find the equation of the tangent line to the following equations at the specified point.

Exercise:

Problem: $y=\cos^{-1}(x)+x$ at $x=0$

Exercise:

Problem: $y=x+e^x-\frac{1}{x}$ at $x=1$

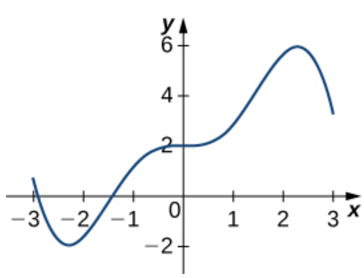
Solution:

$T=(2+e)x-2$

Draw the derivative for the following graphs.

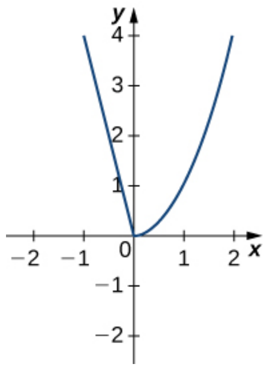
Exercise:

Problem:

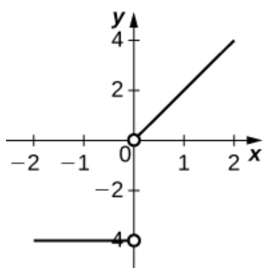


Exercise:

Problem:



Solution:



The following questions concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9\cos\left(\frac{\pi}{6}t\right)$, where t is measured in hours after midnight, and the height is measured in feet.

Exercise:

Problem: Find and graph the derivative. What is the physical meaning?

Exercise:

Problem: Find $w'(3)$. What is the physical meaning of this value?

Solution:

$w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

The following questions consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource: http://news.nationalgeographic.com/news/2005/09/0914_050914_katrina_timeline.html.

Exercise:

Problem:

Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

Exercise:

Problem: Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Solution:

-7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

Glossary

logarithmic differentiation
: is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

Integrals Involving Exponential and Logarithmic Functions

- Integrate functions involving exponential functions.
- Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

Note:
Rule: Integrals of Exponential Functions
Exponential functions can be integrated using the following formulas.
Equation:

$$\begin{aligned} \int e^x dx &= e^x + C \\ \int a^x dx &= \frac{a^x}{\ln a} + C \end{aligned}$$

Example:
Exercise:

Problem:
Finding an Antiderivative of an Exponential Function

Find the antiderivative of the exponential function e^{-x}.

Solution:

Use substitution, setting $u = -x$, and then $du = -1dx$. Multiply the du equation by -1, so you now have $-du = dx$. Then,

Equation:

$$\begin{aligned}\int e^{-x} dx &= -\int e^{u} du \\ &= -e^{u} + C \\ &= -e^{-x} + C.\end{aligned}$$

Note:

Exercise:

Problem:

Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

Solution:

$$\int x^2 e^{-2x^3} dx = -\frac{1}{6} e^{-2x^3} + C$$

Hint

Let u equal the exponent on e.

A common mistake when dealing with exponential expressions is treating the exponent on e the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on e. This can

be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

Example:
Exercise:

Problem:
Square Root of an Exponential Function

Find the antiderivative of the exponential function $e^x\sqrt{1+e^x}$.

Solution:

First rewrite the problem using a rational exponent:
Equation:

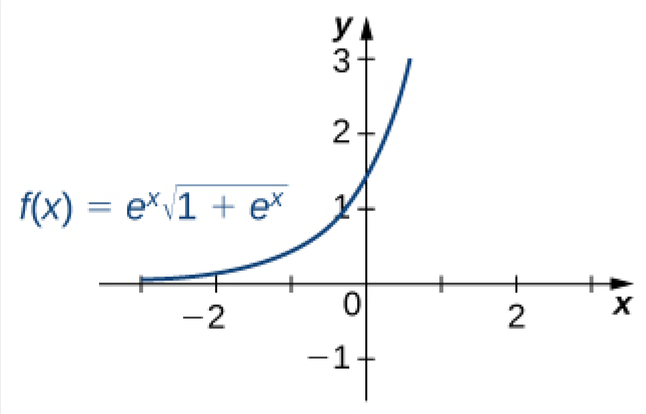
$$\int e^x\sqrt{1+e^x}dx=\int e^x(1+e^x)^{1/2}dx.$$

Using substitution, choose $u=1+e^x.u=1+e^x$. Then, $du=e^x dx$. We have ([link])
Equation:

$$\int e^x(1+e^x)^{1/2}dx=\int u^{1/2}du.$$

Then
Equation:

$$\int u^{1/2}du=\frac{u^{3/2}}{3/2}+C=\frac{2}{3}u^{3/2}+C=\frac{2}{3}(1+e^x)^{3/2}+C.$$



The graph shows an exponential function times the square root of an exponential function.

Note:
Exercise:

Problem: Find the antiderivative of $e^x(3e^x - 2)^2$.

Solution:

$$\int e^x(3e^x - 2)^2 dx = \frac{1}{9}(3e^x - 2)^3$$

Hint

Let $u = 3e^x - 2$ $u = 3e^x - 2$.

Example:
Exercise:

Problem:
Using Substitution with an Exponential Function

Use substitution to evaluate the indefinite integral $\int 3x^2 e^{2x^3} dx$.

Solution:

Here we choose to let *u* equal the expression in the exponent on *e*. Let $u = 2x^3$ and $du = 6x^2 dx.$. Again, *du* is off by a constant multiplier; the original function contains a factor of $3x^2$, not $6x^2$. Multiply both sides of the equation by $\frac{1}{2}$ so that the integrand in *u* equals the integrand in *x*. Thus,

Equation:

$$\int 3x^2 e^{2x^3} dx = \frac{1}{2} \int e^u du.$$

Integrate the expression in *u* and then substitute the original expression in *x* back into the *u* integral:

Equation:

$$\frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{2x^3} + C.$$

Note:
Exercise:

Problem: Evaluate the indefinite integral $\int 2x^3 e^{x^4} dx$.

Solution:

$$\int 2x^3 e^{x^4} dx = \frac{1}{2} e^{x^4}$$

Hint

Let $u = x^4$.

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number e is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

Example:
Exercise:

Problem:
Finding a Price–Demand Equation

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at $2.35 per tube, given that the marginal price—demand function, $p'(x)$, for x number of tubes per week, is given as

Equation:

$$p\prime(x) = -0.015e^{-0.01x}.$$

If the supermarket chain sells 100 tubes per week, what price should it set?

Solution:

To find the price–demand equation, integrate the marginal price–demand function. First find the antiderivative, then look at the particulars. Thus,

Equation:

$$\begin{aligned} p\left(x\right) &= \int -0.015e^{-0.01x}dx \\ &= -0.015\int e^{-0.01x}dx. \end{aligned}$$

Using substitution, let $u = -0.01x$ and $du = -0.01dx$. Then, divide both sides of the du equation by −0.01. This gives

Equation:

$$\begin{aligned} \frac{-0.015}{-0.01}\int e^{u}du &= 1.5\int e^{u}du \\ &= 1.5e^{u} + C \\ &= 1.5e^{-0.01x} + C. \end{aligned}$$

The next step is to solve for C. We know that when the price is \$2.35 per tube, the demand is 50 tubes per week. This means

Equation:

$$\begin{aligned} p\left(50\right) &= 1.5e^{-0.01(50)} + C \\ &= 2.35. \end{aligned}$$

Now, just solve for C:
Equation:

$$\begin{aligned} C &= 2.35 - 1.5e^{-0.5} \\ &= 2.35 - 0.91 \\ &= 1.44. \end{aligned}$$

Thus,
Equation:

$$p\left(x\right) = 1.5e^{-0.01x} + 1.44.$$

If the supermarket sells 100 tubes of toothpaste per week, the price would be
Equation:

$$p(100) = 1.5e^{-0.01(100)} + 1.44 = 1.5e^{-1} + 1.44 \approx 1.99.$$

The supermarket should charge $1.99 per tube if it is selling 100 tubes per week.

Example:
Exercise:

Problem:
Evaluating a Definite Integral Involving an Exponential Function

Evaluate the definite integral $\int_1^2 e^{1-x} dx.$

Solution:

Again, substitution is the method to use. Let $u = 1 - x$, so $du = -1dx$ or $-du = dx$. Then $\int e^{1-x}dx = -\int e^{u}du$. Next, change the limits of integration. Using the equation $u = 1 - x$, we have

Equation:

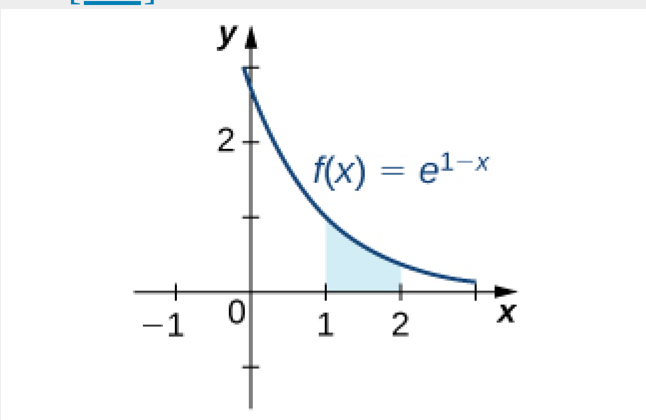
$$\begin{aligned} u &= 1 - (1) = 0 \\ u &= 1 - (2) = -1. \end{aligned}$$

The integral then becomes

Equation:

$$\begin{aligned} \int_1^2 e^{1-x}dx &= -\int_0^{-1} e^{u}du \\ &= \int_{-1}^{0} e^{u}du \\ &= e^{u}\big|_{-1}^{0} \\ &= e^{0} - \left(e^{-1}\right) \\ &= -e^{-1} + 1. \end{aligned}$$

See [link].



The indicated area can be calculated by evaluating a definite integral using substitution.

Note:
Exercise:

Problem: Evaluate $\int_0^2 e^{2x} dx.$

Solution:

$\frac{1}{2}\int_0^4 e^u du = \frac{1}{2}\left(e^4 - 1\right)$

Hint

Let $u = 2x.$

Example:
Exercise:

Problem:
Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with 10,000 bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t. How many bacteria are in the dish after 2 hours?

Solution:

We have
Equation:

$$Q(t) = \int 3^t dt = \frac{3^t}{\ln 3} + C.$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C$, so $C \approx 9.090$ and we get
Equation:

$$Q(t) = \frac{3^t}{\ln 3} + 9.090.$$

At time $t = 2$, we have
Equation:

$$Q(2) = \frac{3^2}{\ln 3} + 9.090$$

Equation:

$$= 17.282.$$

After 2 hours, there are 17,282 bacteria in the dish.

Note:
Exercise:

Problem:

From [link], suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with 10,000 bacteria. Find $Q(t)$. How many bacteria are in the dish after 3 hours?

Solution:

$Q(t) = \frac{2^t}{\ln 2} + 8.557$. There are 20,099 bacteria in the dish after 3 hours.

Hint

Use the procedure from [link] to solve the problem.

Example:
Exercise:

Problem:
Fruit Fly Population Growth

Suppose a population of fruit flies increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

Solution:

Let $G\left(t\right)$ represent the number of flies in the population at time t. Applying the net change theorem, we have
Equation:

$$
\begin{aligned}
G(10) &= G(0) + \int_0^{10} 2e^{0.02t} dt \\
&= 100 + \left[\frac{2}{0.02} e^{0.02t}\right]\Big|_0^{10} \\
&= 100 + \left[100e^{0.02t}\right]\Big|_0^{10} \\
&= 100 + 100e^{0.2} - 100 \\
&\approx 122.
\end{aligned}
$$

There are 122 flies in the population after 10 days.

Note:
Exercise:

Problem:

Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t}$, and the initial fly population is 100 flies. How many flies are in the population after 15 days?

Solution:

There are 116 flies.

Hint

Use the process from [link] to solve the problem.

Example:
Exercise:

Problem:

Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution: $\int_1^2 \frac{e^{1/x}}{x^2}dx.$

Solution:

This problem requires some rewriting to simplify applying the properties. First, rewrite the exponent on *e* as a power of *x*, then bring the x^2 in the denominator up to the numerator using a negative exponent. We have

Equation:

$$\int_1^2 \frac{e^{1/x}}{x^2}dx = \int_1^2 e^{x^{-1}}x^{-2}dx.$$

Let $u = x^{-1}$, the exponent on *e*. Then

Equation:

$$\begin{aligned} du &= -x^{-2}dx \\ -du &= x^{-2}dx. \end{aligned}$$

Bringing the negative sign outside the integral sign, the problem now reads

Equation:

$$-\int e^u du.$$

Next, change the limits of integration:

Equation:

$$u = (1)^{-1} = 1$$
$$u = (2)^{-1} = \frac{1}{2}.$$

Notice that now the limits begin with the larger number, meaning we must multiply by −1 and interchange the limits. Thus,

Equation:

$$\begin{aligned} -\int_1^{1/2} e^u du &= \int_{1/2}^{1} e^u du \\ &= e^u \Big|_{1/2}^{1} \\ &= e - e^{1/2} \\ &= e - \sqrt{e}. \end{aligned}$$

Note:
Exercise:

Problem:

Evaluate the definite integral using substitution: $\int_1^2 \frac{1}{x^3} e^{4x^{-2}} dx.$

Solution:

$$\int_1^2 \frac{1}{x^3} e^{4x^{-2}} dx = \frac{1}{8}\left[e^4 - e\right]$$

Hint

Let $u = 4x^{-2}$.

Integrals Involving Logarithmic Functions

Integrating functions of the form $f\,(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule. Integral formulas for other logarithmic functions, such as $f(x) = \ln x$ and $f(x) = \log_a x$, are also included in the rule.

Note:
Rule: Integration Formulas Involving Logarithmic Functions
The following formulas can be used to evaluate integrals involving logarithmic functions.
Equation:

$$\begin{aligned} \int x^{-1} dx &= \ln|x| + C \\ \int \ln x \, dx &= x \ln x - x + C = x\,(\ln x - 1) + C \\ \int \log_a x \, dx &= \tfrac{x}{\ln a}(\ln x - 1) + C \end{aligned}$$

Example:
Exercise:

Problem:
Finding an Antiderivative Involving $\ln x$

Find the antiderivative of the function $\frac{3}{x-10}$.

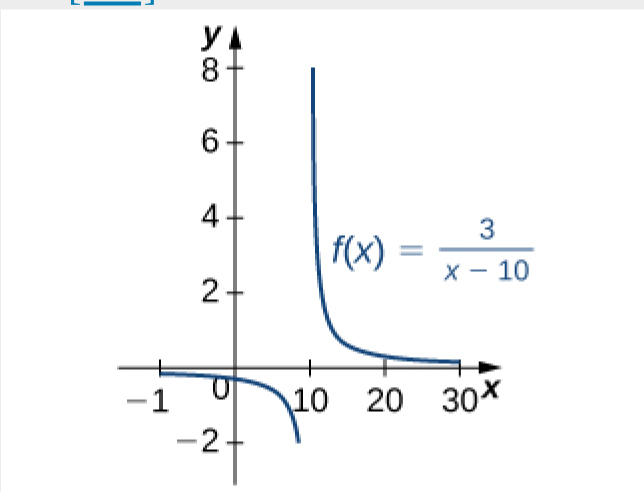
Solution:

First factor the 3 outside the integral symbol. Then use the u^{-1} rule. Thus,

Equation:

$$\begin{aligned}\int \frac{3}{x-10}dx &= 3\int \frac{1}{x-10}dx \\ &= 3\int \frac{du}{u} \\ &= 3\ln|u| + C \\ &= 3\ln|x-10| + C, x \neq 10.\end{aligned}$$

See [link].



The domain of this function is $x \neq 10$.

Note:
Exercise:

Problem: Find the antiderivative of $\frac{1}{x+2}$.

Solution:

$\ln |x+2| + C$

Hint

Follow the pattern from [link] to solve the problem.

Example:
Exercise:

Problem:
Finding an Antiderivative of a Rational Function

Find the antiderivative of $\frac{2x^3+3x}{x^4+3x^2}$.

Solution:

This can be rewritten as $\int (2x^3+3x)(x^4+3x^2)^{-1}dx$. Use substitution. Let $u = x^4 + 3x^2$, then $du = 4x^3 + 6x$. Alter du by factoring out the 2. Thus,

Equation:

$$\begin{aligned} du &= (4x^3+6x)dx \\ &= 2(2x^3+3x)dx \\ \frac{1}{2}du &= (2x^3+3x)dx. \end{aligned}$$

Rewrite the integrand in u:

Equation:

$$\int (2x^3+3x)(x^4+3x^2)^{-1}dx = \frac{1}{2}\int u^{-1}du.$$

Then we have
Equation:

$$\begin{aligned}\frac{1}{2}\int u^{-1}du &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln\left|x^4 + 3x^2\right| + C.\end{aligned}$$

Example:
Exercise:

Problem:
Finding an Antiderivative of a Logarithmic Function

Find the antiderivative of the log function $\log_2 x$.

Solution:

Follow the format in the formula listed in the rule on integration formulas involving logarithmic functions. Based on this format, we have
Equation:

$$\int \log_2 x dx = \frac{x}{\ln 2}(\ln x - 1) + C.$$

Note:
Exercise:

Problem: Find the antiderivative of $\log_3 x$.

Solution:

$\frac{x}{\ln 3}(\ln x - 1) + C$

Hint

Follow [link] and refer to the rule on integration formulas involving logarithmic functions.

[link] is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

Example:
Exercise:

Problem:
Evaluating a Definite Integral

Find the definite integral of $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx.$

Solution:

We need substitution to evaluate this problem. Let $u = 1 + \cos x,$, so $du = -\sin x\,dx.$ Rewrite the integral in terms of u, changing the limits of integration as well. Thus,

Equation:

$$u = 1 + \cos(0) = 2$$
$$u = 1 + \cos\left(\frac{\pi}{2}\right) = 1.$$

Then
Equation:

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin x}{1+\cos x} &= -\int_2^1 u^{-1} du \\ &= \int_1^2 u^{-1} du \\ &= \ln |u| |_1^2 \\ &= [\ln 2 - \ln 1] \\ &= \ln 2.\end{aligned}$$

Key Concepts

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

Key Equations

- **Integrals of Exponential Functions**
  $$\int e^x dx = e^x + C$$
  $$\int a^x dx = \frac{a^x}{\ln a} + C$$
- **Integration Formulas Involving Logarithmic Functions**
  $$\int x^{-1} dx = \ln |x| + C$$

$$\int \ln x \, dx = x \ln x - x + C = x (\ln x - 1) + C$$
$$\int \log_a x \, dx = \frac{x}{\ln a} (\ln x - 1) + C$$

In the following exercises, compute each indefinite integral.

Exercise:

Problem: $\int e^{2x} dx$

Exercise:

Problem: $\int e^{-3x} dx$

Solution:

$\frac{-1}{3} e^{-3x} + C$

Exercise:

Problem: $\int 2^x dx$

Exercise:

Problem: $\int 3^{-x} dx$

Solution:

$-\frac{3^{-x}}{\ln 3} + C$

Exercise:

Problem: $\int \frac{1}{2x} dx$

Exercise:

Problem: $\int \frac{2}{x} dx$

Solution:

$\ln\left(x^2\right) + C$

Exercise:

Problem: $\int \frac{1}{x^2} dx$

Exercise:

Problem: $\int \frac{1}{\sqrt{x}} dx$

Solution:

$2\sqrt{x} + C$

In the following exercises, find each indefinite integral by using appropriate substitutions.

Exercise:

Problem: $\int \frac{\ln x}{x} dx$

Exercise:

Problem: $\int \frac{dx}{x(\ln x)^2}$

Solution:

$-\frac{1}{\ln x} + C$

Exercise:

Problem: $\int \frac{dx}{x \ln x} \; (x > 1)$

Exercise:

Problem: $\int \frac{dx}{x \ln x \ln(\ln x)}$

Solution:

$\ln(\ln(\ln x)) + C$

Exercise:

Problem: $\int \tan\theta \, d\theta$

Exercise:

Problem: $\int \frac{\cos x - x \sin x}{x \cos x} dx$

Solution:

$\ln(x \cos x) + C$

Exercise:

Problem: $\int \frac{\ln(\sin x)}{\tan x} dx$

Exercise:

Problem: $\int \ln(\cos x) \tan x dx$

Solution:

$-\frac{1}{2}(\ln(\cos(x)))^2 + C$

Exercise:

Problem: $\int xe^{-x^2} dx$

Exercise:

Problem: $\int x^2 e^{-x^3} dx$

Solution:

$\frac{-e^{-x^3}}{3} + C$

Exercise:

Problem: $\int e^{\sin x} \cos x dx$

Exercise:

Problem: $\int e^{\tan x} \sec^2 x dx$

Solution:

$e^{\tan x} + C$

Exercise:

Problem: $\int e^{\ln x} \frac{dx}{x}$

Exercise:

Problem: $\int \frac{e^{\ln(1-t)}}{1-t} dt$

Solution:

$t + C$

In the following exercises, verify by differentiation that $\int \ln x \, dx = x (\ln x - 1) + C$, then use appropriate changes of variables to compute the integral.

Exercise:

Problem: $\int \ln x dx$ $(Hint: \int \ln x dx = \frac{1}{2} \int x \ln \left(x^2\right) dx)$

Exercise:

Problem: $\int x^2 \ln^2 x \, dx$

Solution:

$\frac{1}{9} x^3 \left(\ln \left(x^3\right) - 1\right) + C$

Exercise:

Problem: $\int \frac{\ln x}{x^2} dx$ $(Hint: \text{Set } u = \frac{1}{x}.)$

Exercise:

Problem: $\int \frac{\ln x}{\sqrt{x}} dx$ $(Hint: \text{Set } u = \sqrt{x}.)$

Solution:

$2\sqrt{x}\,(\ln x - 2) + C$

Exercise:

Problem:

Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to e^x and evaluate the integral.

Exercise:

Problem:

Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$, and evaluate the integral.

Solution:

$$\int_0^{\ln x} e^t dt = e^t \Big|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$$

In the following exercises, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

Exercise:

Problem: $\int \tan(2x) dx$

Exercise:

Problem: $\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)} dx$

Solution:

$-\frac{1}{3}\ln(\sin(3x) + \cos(3x))$

Exercise:

Problem: $\int \frac{x\sin\left(x^2\right)}{\cos\left(x^2\right)} dx$

Exercise:

Problem: $\int x\csc\left(x^2\right)dx$

Solution:

$-\frac{1}{2}\ln\left|\csc\left(x^2\right) + \cot\left(x^2\right)\right| + C$

Exercise:

Problem: $\int \ln(\cos x)\tan x\ dx$

Exercise:

Problem: $\int \ln(\csc x)\cot x dx$

Solution:

$-\frac{1}{2}(\ln(\csc x))^2 + C$

Exercise:

Problem: $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

In the following exercises, evaluate the definite integral.

Exercise:

Problem: $\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx$

Solution:

$\frac{1}{3} \ln\left(\frac{26}{7}\right)$

Exercise:

Problem: $\int_0^{\pi/4} \tan x \, dx$

Exercise:

Problem: $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Solution:

$\ln\left(\sqrt{3} - 1\right)$

Exercise:

Problem: $\int_{\pi/6}^{\pi/2} \csc x dx$

Exercise:

Problem: $\int_{\pi/4}^{\pi/3} \cot x dx$

Solution:

$\frac{1}{2}\ln \frac{3}{2}$

In the following exercises, integrate using the indicated substitution.
Exercise:

Problem: $\int \frac{x}{x-100}dx; u = x - 100$

Exercise:

Problem: $\int \frac{y-1}{y+1}dy; u = y + 1$

Solution:

$y - 2\ln|y+1| + C$

Exercise:

Problem: $\int \frac{1-x^2}{3x-x^3}dx; u = 3x - x^3$

Exercise:

Problem: $\int \frac{\sin x + \cos x}{\sin x - \cos x}dx; u = \sin x - \cos x$

Solution:

$\ln|\sin x - \cos x| + C$

Exercise:

Problem: $\int e^{2x}\sqrt{1-e^{2x}}dx; u=e^{2x}$

Exercise:

Problem: $\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}dx; u=\ln x$

Solution:

$-\frac{1}{3}\left(1-\left(\ln x^2\right)\right)^{3/2}+C$

In the following exercises, does the right-endpoint approximation overestimate or underestimate the exact area? Calculate the right endpoint estimate R_{50} and solve for the exact area.

Exercise:

Problem: **[T]** $y=e^x$ over $[0,1]$

Exercise:

Problem: **[T]** $y=e^{-x}$ over $[0,1]$

Solution:

Exact solution: $\frac{e-1}{e}$, $R_{50}=0.6258$. Since f is decreasing, the right endpoint estimate underestimates the area.

Exercise:

Problem: **[T]** $y=\ln(x)$ over $[1,2]$

Exercise:

Problem: **[T]** $y = \frac{x+1}{x^2+2x+6}$ over $[0, 1]$

Solution:

Exact solution: $\frac{2\ln(3)-\ln(6)}{2}$, $R_{50} = 0.2033$. Since *f* is increasing, the right endpoint estimate overestimates the area.

Exercise:

Problem: **[T]** $y = 2^x$ over $[-1, 0]$

Exercise:

Problem: **[T]** $y = -2^{-x}$ over $[0, 1]$

Solution:

Exact solution: $-\frac{1}{\ln(4)}$, $R_{50} = -0.7164$. Since *f* is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).

In the following exercises, $f(x) \geq 0$ for $a \leq x \leq b$. Find the area under the graph of $f(x)$ between the given values a and b by integrating.

Exercise:

Problem: $f(x) = \frac{\log_{10}(x)}{x}$; $a = 10, b = 100$

Exercise:

Problem: $f(x) = \frac{\log_2(x)}{x}$; $a = 32, b = 64$

Solution:

$\frac{11}{2}\ln 2$

Exercise:

Problem: $f(x) = 2^{-x}; a = 1, b = 2$

Exercise:

Problem: $f(x) = 2^{-x}; a = 3, b = 4$

Solution:

$\frac{1}{\ln(65{,}536)}$

Exercise:

Problem:

Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

Exercise:

Problem:

Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of N such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 10$ is, at most, 0.01.

Solution:

$\int_N^{N+1} xe^{-x^2} dx = \frac{1}{2}\left(e^{-N^2} - e^{-(N+1)^2}\right)$. The quantity is less than 0.01 when $N = 2$.

Exercise:

Problem:

Find the limit, as N tends to infinity, of the area under the graph of $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

Exercise:

Problem: Show that $\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}$ when $0 < a \leq b$.

Solution:

$$\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln\left(\frac{1}{a}\right) - \ln\left(\frac{1}{b}\right) = \int_{1/b}^{1/a} \frac{dx}{x}$$

Exercise:

Problem:

Suppose that $f(x) > 0$ for all *x* and that *f* and *g* are differentiable. Use the identity $f^g = e^{g \ln f}$ and the chain rule to find the derivative of f^g.

Exercise:

Problem:

Use the previous exercise to find the antiderivative of $h(x) = x^x (1 + \ln x)$ and evaluate $\int_2^3 x^x (1 + \ln x) dx$.

Solution:

23

Exercise:

Problem:

Show that if $c > 0$, then the integral of $1/x$ from *ac* to *bc* $(0 < a < b)$ is the same as the integral of $1/x$ from *a* to *b*.

The following exercises are intended to derive the fundamental properties of the natural log starting from the *definition* $\ln(x) = \int_1^x \frac{dt}{t}$, using

properties of the definite integral and making no further assumptions.

Exercise:

Problem:

Use the identity $\ln(x) = \int_1^x \frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right) = -\ln x$.

Solution:

We may assume that $x > 1$, so $\frac{1}{x} < 1$. Then, $\int_1^{1/x} \frac{dt}{t}$. Now make the substitution $u = \frac{1}{t}$, so $du = -\frac{dt}{t^2}$ and $\frac{du}{u} = -\frac{dt}{t}$, and change endpoints: $\int_1^{1/x} \frac{dt}{t} = -\int_1^x \frac{du}{u} = -\ln x$.

Exercise:

Problem:

Use a change of variable in the integral $\int_1^{xy} \frac{1}{t}dt$ to show that $\ln xy = \ln x + \ln y$ for $x, y > 0$.

Exercise:

Problem:

Use the identity $\ln x = \int_1^x \frac{dt}{x}$ to show that $\ln(x)$ is an increasing function of *x* on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

Exercise:

Problem:

Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a+b) = E(a)E(b)$ for any real numbers *a*, *b*.

Exercise:

Problem:

Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Show that $E\prime(t) = E(t)$.

Solution:

$x = E(\ln(x))$. Then, $1 = \frac{E\prime(\ln x)}{x}$ or $x = E\prime(\ln x)$. Since any number *t* can be written $t = \ln x$ for some *x*, and for such *t* we have $x = E(t)$, it follows that for any t, $E\prime(t) = E(t)$.

Exercise:

Problem:

The sine integral, defined as $S(x) = \displaystyle\int_0^x \frac{\sin t}{t} dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large *x*. Show that for $k \geq 1, |S(2\pi k) - S(2\pi(k+1))| \leq \frac{1}{k(2k+1)\pi}$.
$(Hint: \sin(t+\pi) = -\sin t)$

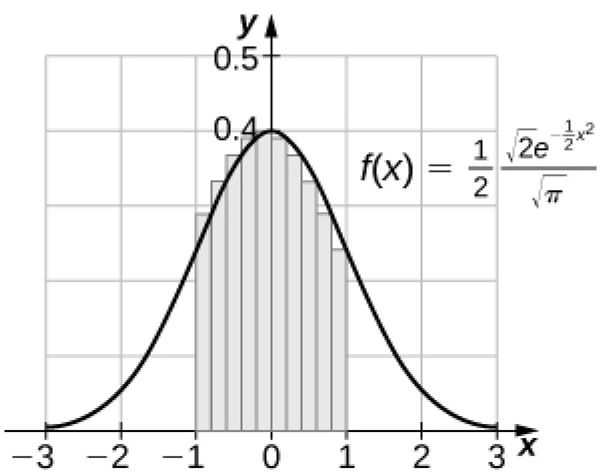
Exercise:

Problem:

[T] The normal distribution in probability is given by $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where σ is the standard deviation and μ is the average. The *standard normal distribution* in probability, p_s, corresponds to $\mu = 0$ and $\sigma = 1$. Compute the left endpoint estimates R_{10} and R_{100} of $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Solution:

$R_{10} = 0.6811, R_{100} = 0.6827$



Exercise:

Problem:

[T] Compute the right endpoint estimates R_{50} and R_{100} of $\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8}$.

Integrals, Exponential Functions, and Logarithms

- Write the definition of the natural logarithm as an integral.
- Recognize the derivative of the natural logarithm.
- Integrate functions involving the natural logarithmic function.
- Define the number e through an integral.
- Recognize the derivative and integral of the exponential function.
- Prove properties of logarithms and exponential functions using integrals.
- Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number e is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number e, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier).

We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number e, and expand these concepts to logarithms and exponential functions of any base.

The Natural Logarithm as an Integral

Recall the power rule for integrals:
Equation:

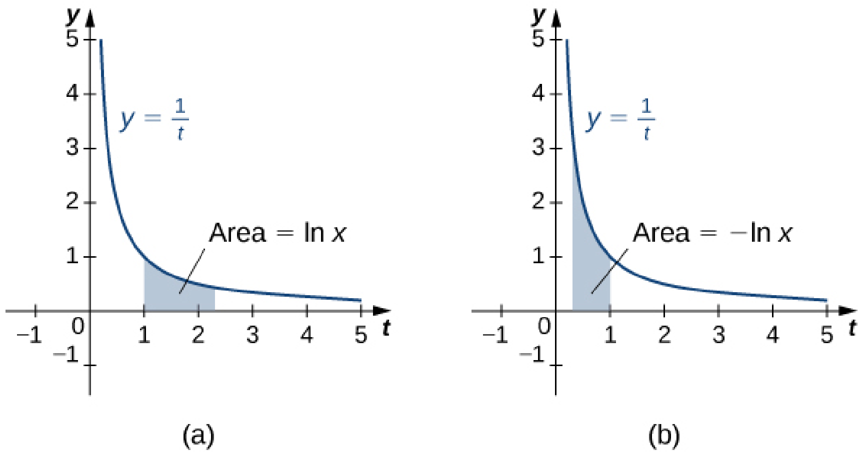
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$$

Clearly, this does not work when $n = -1$, as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x} dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t} dt$ is an antiderivative of $1/x$. Therefore, we can make the following definition.

Note:
Definition
For $x > 0$, define the natural logarithm function by
Equation:

$$\ln x = \int_1^x \frac{1}{t} dt.$$

For $x > 1$, this is just the area under the curve $y = 1/t$ from 1 to x. For $x < 1$, we have $\int_1^x \frac{1}{t} dt = -\int_x^1 \frac{1}{t} dt$, so in this case it is the negative of the area under the curve from x to 1 (see the following figure).



(a) When $x > 1$, the natural logarithm is the area under the curve $y = 1/t$ from 1 to x. (b) When $x < 1$, the natural logarithm is the negative of the area under the curve from x to 1.

Notice that $\ln 1 = 0$. Furthermore, the function $y = 1/t > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

Properties of the Natural Logarithm

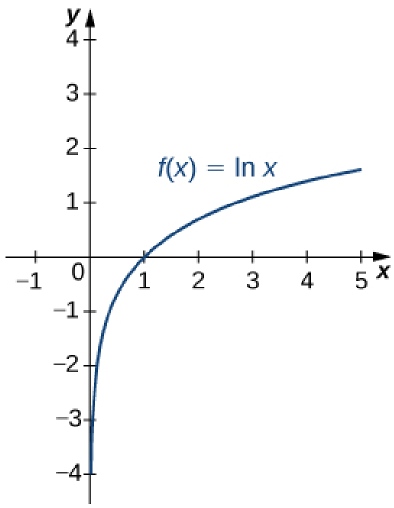
Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of to the Fundamental Theorem of Calculus.

Note:
Derivative of the Natural Logarithm
For $x > 0$, the derivative of the natural logarithm is given by
Equation:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Note:
Corollary to the Derivative of the Natural Logarithm
The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in [link]. Notice that it is continuous throughout its domain of $(0, \infty)$.



The graph of $f(x) = \ln x$ shows that it is a continuous function.

Example:
Exercise:

Problem:
Calculating Derivatives of Natural Logarithms

Calculate the following derivatives:

a. $\frac{d}{dx} \ln\left(5x^3 - 2\right)$
b. $\frac{d}{dx} (\ln(3x))^2$

Solution:

We need to apply the chain rule in both cases.

a. $\frac{d}{dx} \ln\left(5x^3 - 2\right) = \frac{15x^2}{5x^3 - 2}$
b. $\frac{d}{dx} (\ln(3x))^2 = \frac{2(\ln(3x)) \cdot 3}{3x} = \frac{2(\ln(3x))}{x}$

Note:

Exercise:

Problem: Calculate the following derivatives:

a. $\frac{d}{dx}\ln\left(2x^2+x\right)$
b. $\frac{d}{dx}\left(\ln\left(x^3\right)\right)^2$

Solution:

a. $\frac{d}{dx}\ln\left(2x^2+x\right)=\frac{4x+1}{2x^2+x}$
b. $\frac{d}{dx}\left(\ln\left(x^3\right)\right)^2=\frac{6\ln\left(x^3\right)}{x}$

Hint

Apply the differentiation formula just provided and use the chain rule as necessary.

Note that if we use the absolute value function and create a new function $\ln\ |x|$, we can extend the domain of the natural logarithm to include $x<0$. Then $(d/\left(dx\right))\ln\ |x|=1/x$. This gives rise to the familiar integration formula.

Note:
Integral of (1/*u*) *du*
The natural logarithm is the antiderivative of the function $f\left(u\right)=1/u$:
Equation:

$$\int\frac{1}{u}du=\ln\ |u|+C.$$

Example:
Exercise:

Problem:
Calculating Integrals Involving Natural Logarithms

Calculate the integral $\int\frac{x}{x^2+4}dx.$

Solution:

Using u-substitution, let $u=x^2+4$. Then $du=2x\ dx$ and we have
Equation:

$$\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du \frac{1}{2}\ln |u| + C = \frac{1}{2}\ln \left|x^2+4\right| + C = \frac{1}{2}\ln \left(x^2+4\right) + C.$$

Note:
Exercise:

Problem: Calculate the integral $\int \frac{x^2}{x^3+6}dx$.

Solution:

$$\int \frac{x^2}{x^3+6}dx = \tfrac{1}{3}\ln \left|x^3+6\right| + C$$

Hint

Apply the integration formula provided earlier and use u-substitution as necessary.

Although we have called our function a "logarithm," we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

Note:
Properties of the Natural Logarithm
If $a, b > 0$ and r is a rational number, then

i. $\ln 1 = 0$
ii. $\ln (ab) = \ln a + \ln b$
iii. $\ln \left(\frac{a}{b}\right) = \ln a - \ln b$
iv. $\ln (a^r) = r \ln a$

Proof

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t}dt = 0.$
ii. We have
Equation:

$$\ln (ab) = \int_1^{ab} \frac{1}{t}dt = \int_1^{a} \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt.$$

Use u-substitution on the last integral in this expression. Let $u = t/a$. Then $du = (1/a)dt$.

Furthermore, when $t = a, u = 1$, and when $t = ab, u = b$. So we get
Equation:

$$\ln(ab) = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_1^{ab} \frac{a}{t} \cdot \frac{1}{a}dt = \int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{u}du = \ln a + \ln b.$$

iii. Note that
Equation:

$$\frac{d}{dx}\ln(x^r) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}.$$

Furthermore,
Equation:

$$\frac{d}{dx}(r\ln x) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have
Equation:

$$\ln(x^r) = r\ln x + C$$

for some constant C. Taking $x = 1$, we get
Equation:

$$\begin{aligned} \ln(1^r) &= r\ln(1) + C \\ 0 &= r(0) + C \\ C &= 0. \end{aligned}$$

Thus $\ln(x^r) = r\ln x$ and the proof is complete. Note that we can extend this property to irrational values of r later in this section.
Part iii. follows from parts ii. and iv. and the proof is left to you.

□

Example:
Exercise:

Problem:
Using Properties of Logarithms

Use properties of logarithms to simplify the following expression into a single logarithm:
Equation:

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right).$$

Solution:

We have
Equation:

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right) = \ln\left(3^2\right) - 2\ln 3 + \ln\left(3^{-1}\right) = 2\ln 3 - 2\ln 3 - \ln 3 = -\ln 3.$$

Note:
Exercise:

Problem: Use properties of logarithms to simplify the following expression into a single logarithm:
Equation:

$$\ln 8 - \ln 2 - \ln\left(\frac{1}{4}\right).$$

Solution:

$4\ln 2$

Hint

Apply the properties of logarithms.

Defining the Number *e*

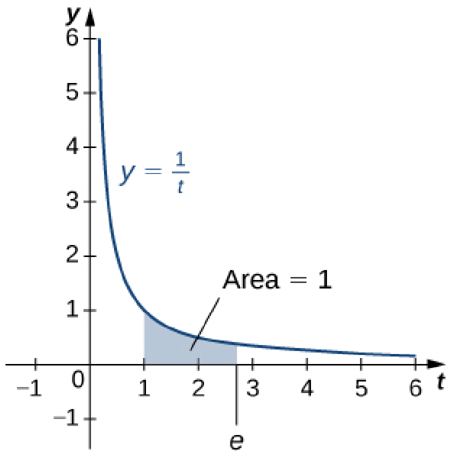
Now that we have the natural logarithm defined, we can use that function to define the number e.

Note:
Definition
The number e is defined to be the real number such that
Equation:

$$\ln e = 1.$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 ([link]). The proof that such a number exists and is unique is left to you. (*Hint*: Use the Intermediate Value Theorem to prove

existence and the fact that $\ln x$ is increasing to prove uniqueness.)



The area under the curve from 1 to e is equal to one.

The number e can be shown to be irrational, although we won't do so here (see the Student Project in Taylor and Maclaurin Series). Its approximate value is given by
Equation:

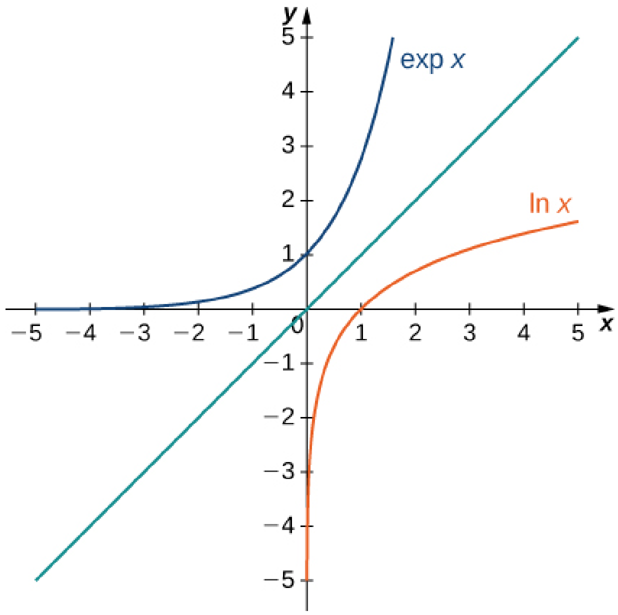
$$e \approx 2.71828182846.$$

The Exponential Function

We now turn our attention to the function e^x. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,
Equation:

$$\exp(\ln x) = x \text{ for } x > 0 \text{ and } \ln(\exp x) = x \text{ for all } x.$$

The following figure shows the graphs of $\exp x$ and $\ln x$.



The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of x, this is easy to show. If x is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when x is rational, $e^x = \exp x$. For irrational values of x, we simply define e^x as the inverse function of $\ln x$.

Note:
Definition
For any real number x, define $y = e^x$ to be the number for which
Equation:

$$\ln y = \ln(e^x) = x.$$

Then we have $e^x = \exp(x)$ for all x, and thus
Equation:

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln(e^x) = x$$

for all x.

Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of e, we must verify that the usual laws of exponents hold for the function e^x.

Note:
Properties of the Exponential Function
If p and q are any real numbers and r is a rational number, then

i. $e^p e^q = e^{p+q}$
ii. $\frac{e^p}{e^q} = e^{p-q}$
iii. $\left(e^p\right)^r = e^{pr}$

Proof

Note that if p and q are rational, the properties hold. However, if p or q are irrational, we must apply the inverse function definition of e^x and verify the properties. Only the first property is verified here; the other two are left to you. We have
Equation:

$$\ln\left(e^p e^q\right) = \ln\left(e^p\right) + \ln\left(e^q\right) = p + q = \ln\left(e^{p+q}\right).$$

Since $\ln x$ is one-to-one, then
Equation:

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of r, and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then
Equation:

$$\begin{aligned} \ln y &= x \\ \frac{d}{dx}\ln y &= \frac{d}{dx}x \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y. \end{aligned}$$

Thus, we see
Equation:

$$\frac{d}{dx}e^x = e^x$$

as desired, which leads immediately to the integration formula
Equation:

$$\int e^x dx = e^x + C.$$

We apply these formulas in the following examples.

Example:
Exercise:

Problem:
Using Properties of Exponential Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}e^{3t}e^{t^2}$
b. $\frac{d}{dx}e^{3x^2}$

Solution:

We apply the chain rule as necessary.

a. $\frac{d}{dt}e^{3t}e^{t^2} = \frac{d}{dt}e^{3t+t^2} = e^{3t+t^2}(3+2t)$
b. $\frac{d}{dx}e^{3x^2} = e^{3x^2}6x$

Note:
Exercise:

Problem: Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right)$
b. $\frac{d}{dt}\left(e^{2t}\right)^3$

Solution:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right) = e^{x^2-5x}(2x-5)$
b. $\frac{d}{dt}\left(e^{2t}\right)^3 = 6e^{6t}$

Hint

Use the properties of exponential functions and the chain rule as necessary.

Example:
Exercise:

Problem:
Using Properties of Exponential Functions

Evaluate the following integral: $\int 2xe^{-x^2}dx$.

Solution:

Using u-substitution, let $u=-x^2$. Then $du=-2x\,dx$, and we have
Equation:

$$\int 2xe^{-x^2}dx=-\int e^u du=-e^u+C=-e^{-x^2}+C.$$

Note:
Exercise:

Problem: Evaluate the following integral: $\int \frac{4}{e^{3x}}dx$.

Solution:

$\int \frac{4}{e^{3x}}dx=-\frac{4}{3}e^{-3x}+C$

Hint

Use the properties of exponential functions and $u\text{-substitution}$ as necessary.

General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than e. Exponential functions are functions of the form $f(x)=a^x$. Note that unless $a=e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x)=a^x$ in terms of the exponential function e^x. We then examine logarithms with bases other than e as inverse functions of exponential functions.

Note:
Definition
For any $a>0$, and for any real number x, define $y=a^x$ as follows:
Equation:

$$y=a^x=e^{x\ln a}.$$

Now a^x is defined rigorously for all values of x. This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of r. It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for a^x. We have
Equation:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\ln a = a^x\ln a.$$

The corresponding integration formula follows immediately.

Note:
Derivatives and Integrals Involving General Exponential Functions
Let $a > 0$. Then,
Equation:

$$\frac{d}{dx}a^x = a^x\ln a$$

and
Equation:

$$\int a^x dx = \frac{1}{\ln a}a^x + C.$$

If $a \neq 1$, then the function a^x is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,
Equation:

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get
Equation:

$$\begin{aligned}\ln x &= \ln\left(a^y\right)\\ \ln x &= y\ln a\\ y &= \tfrac{\ln x}{\ln a}\\ \log^a x &= \tfrac{\ln x}{\ln a}.\end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base a. Again, let $y = \log_a x$. Then,

Equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\log_a x) \\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \left(\frac{1}{\ln a}\right)\frac{d}{dx}(\ln x) \\ &= \frac{1}{\ln a}\cdot\frac{1}{x} \\ &= \frac{1}{x\ln a}. \end{aligned}$$

Note:

Derivatives of General Logarithm Functions

Let $a > 0$. Then,

Equation:

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

Example:

Exercise:

Problem:

Calculating Derivatives of General Exponential and Logarithm Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right)$

b. $\frac{d}{dx}\log_8\left(7x^2+4\right)$

Solution:

We need to apply the chain rule as necessary.

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t}\cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t+t^2}\right) = 2^{2t+t^2}\ln(2)\,(2+2t)$

b. $\frac{d}{dx}\log_8\left(7x^2+4\right) = \frac{1}{(7x^2+4)(\ln 8)}(14x)$

Note:

Exercise:

Problem: Evaluate the following derivatives:

a. $\frac{d}{dt}\, 4^{t^4}$
b. $\frac{d}{dx}\log_3\left(\sqrt{x^2+1}\right)$

Solution:

a. $\frac{d}{dt} 4^{t^4} = 4^{t^4}\,(\ln 4)\,(4t^3)$
b. $\frac{d}{dx}\log_3\left(\sqrt{x^2+1}\right) = \frac{x}{(\ln 3)(x^2+1)}$

Hint

Use the formulas and apply the chain rule as necessary.

Example:
Exercise:

Problem:
Integrating General Exponential Functions

Evaluate the following integral: $\displaystyle\int \frac{3}{2^{3x}}\,dx.$

Solution:

Use u-substitution and let $u = -3x$. Then $du = -3dx$ and we have
Equation:

$$\int \frac{3}{2^{3x}}dx = \int 3\cdot 2^{-3x}dx = -\int 2^u du = -\frac{1}{\ln 2}2^u + C = -\frac{1}{\ln 2}2^{-3x} + C.$$

Note:
Exercise:

Problem: Evaluate the following integral: $\displaystyle\int x^2 2^{x^3}\,dx.$

Solution:

$\displaystyle\int x^2 2^{x^3}\,dx = \frac{1}{3\ln 2}2^{x^3} + C$

Hint

Use the properties of exponential functions and u-substitution as necessary.

Key Concepts

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function e^x is then defined as the inverse of the natural logarithm.
- General exponential functions are defined in terms of e^x, and the corresponding inverse functions are general logarithms.
- Familiar properties of logarithms and exponents still hold in this more rigorous context.

Key Equations

- **Natural logarithm function**
- $\ln x = \int_1^x \frac{1}{t} dt$ Z
- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$ Z

For the following exercises, find the derivative $\frac{dy}{dx}$.

Exercise:

Problem: $y = \ln(2x)$

Solution:

$\frac{1}{x}$

Exercise:

Problem: $y = \ln(2x+1)$

Exercise:

Problem: $y = \frac{1}{\ln x}$

Solution:

$-\frac{1}{x(\ln x)^2}$

For the following exercises, find the indefinite integral.

Exercise:

Problem: $\int \frac{dt}{3t}$

Exercise:

Problem: $\int \frac{dx}{1+x}$

Solution:

$\ln (x+1) + C$

For the following exercises, find the derivative dy/dx. (You can use a calculator to plot the function and the derivative to confirm that it is correct.)

Exercise:

Problem: [T] $y = \frac{\ln(x)}{x}$

Exercise:

Problem: [T] $y = x \ln (x)$

Solution:

$\ln (x) + 1$

Exercise:

Problem: [T] $y = \log_{10} x$

Exercise:

Problem: [T] $y = \ln (\sin x)$

Solution:

$\cot (x)$

Exercise:

Problem: [T] $y = \ln (\ln x)$

Exercise:

Problem: [T] $y = 7 \ln (4x)$

Solution:

$\frac{7}{x}$

Exercise:

Problem: [T] $y = \ln \left((4x)^7 \right)$

Exercise:

Problem: **[T]** $y = \ln(\tan x)$

Solution:

$\csc(x)\sec x$

Exercise:

Problem: **[T]** $y = \ln(\tan(3x))$

Exercise:

Problem: **[T]** $y = \ln\left(\cos^2 x\right)$

Solution:

$-2\tan x$

For the following exercises, find the definite or indefinite integral.

Exercise:

Problem: $\int_0^1 \frac{dx}{3+x}$

Exercise:

Problem: $\int_0^1 \frac{dt}{3+2t}$

Solution:

$\frac{1}{2}\ln\left(\frac{5}{3}\right)$

Exercise:

Problem: $\int_0^2 \frac{x\,dx}{x^2+1}$

Exercise:

Problem: $\int_0^2 \frac{x^3 dx}{x^2+1}$

Solution:

$2 - \frac{1}{2}\ln(5)$

Exercise:

Problem: $\int_2^e \frac{dx}{x \ln x}$

Exercise:

Problem: $\int_2^e \frac{dx}{(x \ln(x))^2}$

Solution:

$\frac{1}{\ln(2)} - 1$

Exercise:

Problem: $\int \frac{\cos x \, dx}{\sin x}$

Exercise:

Problem: $\int_0^{\pi/4} \tan x \, dx$

Solution:

$\frac{1}{2} \ln(2)$

Exercise:

Problem: $\int \cot(3x) dx$

Exercise:

Problem: $\int \frac{(\ln x)^2 dx}{x}$

Solution:

$\frac{1}{3} (\ln x)^3$

For the following exercises, compute dy/dx by differentiating $\ln y$.

Exercise:

Problem: $y = \sqrt{x^2 + 1}$

Exercise:

Problem: $y = \sqrt{x^2 + 1}\sqrt{x^2 - 1}$

Solution:

$\frac{2x^3}{\sqrt{x^2+1}\sqrt{x^2-1}}$

Exercise:

Problem: $y = e^{\sin x}$

Exercise:

Problem: $y = x^{-1/x}$

Solution:

$x^{-2-(1/x)}(\ln x - 1)$

Exercise:

Problem: $y = e^{(ex)}$

Exercise:

Problem: $y = x^e$

Solution:

ex^{e-1}

Exercise:

Problem: $y = x^{(ex)}$

Exercise:

Problem: $y = \sqrt{x}\,\sqrt[3]{x}\,\sqrt[6]{x}$

Solution:

1

Exercise:

Problem: $y = x^{-1/\ln x}$

Exercise:

Problem: $y = e^{-\ln x}$

Solution:

$-\frac{1}{x^2}$

For the following exercises, evaluate by any method.
Exercise:

Problem: $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

Exercise:

Problem: $\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

Solution:

$\pi - \ln(2)$

Exercise:

Problem: $\frac{d}{dx} \int_x^1 \frac{dt}{t}$

Exercise:

Problem: $\frac{d}{dx} \int_x^{x^2} \frac{dt}{t}$

Solution:

$\frac{1}{x}$

Exercise:

Problem: $\frac{d}{dx} \ln(\sec x + \tan x)$

For the following exercises, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.
Exercise:

Problem: Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

Solution:

$e^5 - 6 \text{ units}^2$

Exercise:

Problem: **[T]** Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

Exercise:

Problem: Find the area between $\ln x$ and the *x*-axis from $x = 1$ to $x = 2$.

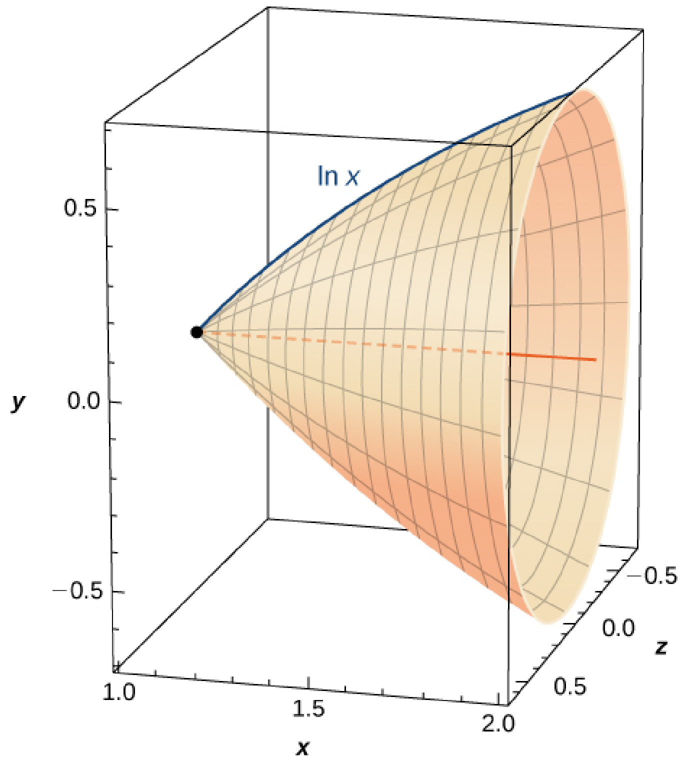
Solution:

$\ln(4) - 1 \text{ units}^2$

Exercise:

Problem:

Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the *x*-axis, as pictured here.



Exercise:

Problem:

[T] Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the *x*-axis.

Solution:

2.8656

If you are unable to find intersection points analytically in the following exercises, use a calculator.

Exercise:

Problem:

Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

Exercise:

Problem: **[T]** Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

Solution:

3.1502

Exercise:

Problem: Find the area under $y = 1/x$ and above the *x*-axis from $x = 1$ to $x = 4$.

For the following exercises, verify the derivatives and antiderivatives.

Exercise:

Problem: $\frac{d}{dx}\ln\left(x + \sqrt{x^2+1}\right) = \frac{1}{\sqrt{1+x^2}}$

Exercise:

Problem: $\frac{d}{dx}\ln\left(\frac{x-a}{x+a}\right) = \frac{2a}{(x^2-a^2)}$

Exercise:

Problem: $\frac{d}{dx}\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = -\frac{1}{x\sqrt{1-x^2}}$

Exercise:

Problem: $\frac{d}{dx}\ln\left(x + \sqrt{x^2-a^2}\right) = \frac{1}{\sqrt{x^2-a^2}}$

Exercise:

Problem: $\displaystyle\int \frac{dx}{x\ln(x)\ln(\ln x)} = \ln(\ln(\ln x)) + C$

Exponential Growth and Decay

- Use the exponential growth model in applications, including population growth and compound interest.
- Explain the concept of doubling time.
- Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
- Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where y_0 represents the initial state of the system and k is a positive constant, called the *growth constant*. Notice that in an exponential growth model, we have
Equation:

$$y' = ky_0 e^{kt} = ky.$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. [link] involves derivatives and is called a *differential equation*. We learn more about differential equations in Introduction to Differential Equations.

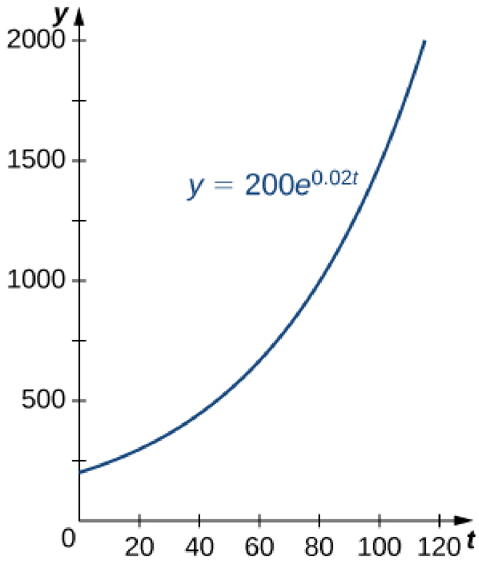
Note:
Rule: Exponential Growth Model
Systems that exhibit **exponential growth** increase according to the mathematical model
Equation:

$$y = y_0 e^{kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *growth constant*.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. [link] and [link] represent the growth of a population of

bacteria with an initial population of 200 bacteria and a growth constant of 0.02. Notice that after only 2 hours (120 minutes), the population is 10 times its original size!



An example of exponential growth for bacteria.

Time (min)	**Population Size (no. of bacteria)**
10	244
20	298
30	364
40	445
50	544
60	664
70	811

Time (min)	Population Size (no. of bacteria)
80	991
90	1210
100	1478
110	1805
120	2205

Exponential Growth of a Bacterial Population

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

Example:
Exercise:

Problem:
Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours $(300$ minutes)? When does the population reach $100{,}000$ bacteria?

Solution:

We have $f(t) = 200e^{0.02t}$. Then
Equation:

$$f(300) = 200e^{0.02(300)} \approx 80{,}686.$$

There are $80{,}686$ bacteria in the population after 5 hours.

To find when the population reaches $100{,}000$ bacteria, we solve the equation
Equation:

$$\begin{aligned} 100{,}000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100{,}000$ bacteria after 310.73 minutes.

Note:
Exercise:

Problem:

Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Solution:

There are $81{,}377{,}396$ bacteria in the population after 4 hours. The population reaches 100 million bacteria after 244.12 minutes.

Hint

Use the process from the previous example.

Let's now turn our attention to a financial application: compound interest. Interest that is not compounded is called *simple interest*. Simple interest is paid once, at the end of the specified time period (usually 1 year). So, if we put $\$1000$ in a savings account earning 2% simple interest per year, then at the end of the year we have

Equation:

$$1000\,(1 + 0.02) = \$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

Equation:

$$1000\left(1+\frac{0.02}{2}\right)^2 = \$1020.10.$$

Similarly, if the interest is compounded every 4 months, we have
Equation:

$$1000\left(1+\frac{0.02}{3}\right)^3 = \$1020.13,$$

and if the interest is compounded daily $(365$ times per year), we have $\$1020.20$. If we extend this concept, so that the interest is compounded continuously, after t years we have
Equation:

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:
Equation:

$$e=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}.$$

Based on this, we want the expression inside the parentheses to have the form $(1+1/m)$. Let $n=0.02m$. Note that as $n\to\infty$, $m\to\infty$ as well. Then we get
Equation:

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt} = 1000\lim_{m\to\infty}\left(1+\frac{0.02}{0.02m}\right)^{0.02mt} = 1000\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e. So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of $r\%$, compounded continuously, then the balance of the account after t years is
Equation:

$$\text{Balance} = Pe^{rt}.$$

Example:

Exercise:

Problem:
Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5% annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age 65? What if she could earn 6% annual interest compounded continuously instead?

Solution:

We have
Equation:

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.05(40)} \\ P &= 135{,}335.28. \end{aligned}$$

She must invest $\$135{,}335.28$ at 5% interest.

If, instead, she is able to earn 6%, then the equation becomes
Equation:

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.06(40)} \\ P &= 90{,}717.95. \end{aligned}$$

In this case, she needs to invest only $\$90{,}717.95$. This is roughly two-thirds the amount she needs to invest at 5%. The fact that the interest is compounded continuously greatly magnifies the effect of the 1% increase in interest rate.

Note:
Exercise:

Problem:

Suppose instead of investing at age $25\sqrt{b^2 - 4ac}$, the student waits until age 35. How much would she have to invest at 5%? At 6%?

Solution:

At 5% interest, she must invest $\$223{,}130.16$. At 6% interest, she must invest $\$165{,}298.89$.

Hint

Use the process from the previous example.

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from 10,000 to 20,000 bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

Equation:

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \tfrac{\ln 2}{k}. \end{aligned}$$

Note:

Definition

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

Equation:

$$\text{Doubling time } = \frac{\ln 2}{k}.$$

Example:

Exercise:

Problem:

Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches 10,000. When will the owner's friends be allowed to fish?

Solution:

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k$. Then,

$k = (\ln 2)/6$. Thus, the population is given by $y = 500e^{((\ln 2)/6)t}$. To figure out when the population reaches 10,000 fish, we must solve the following equation:
Equation:

$$\begin{aligned} 10{,}000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\ln 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right)t \\ t &= \frac{6(\ln 20)}{\ln 2} \approx 25.93. \end{aligned}$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

Note:
Exercise:

Problem:

Suppose it takes 9 months for the fish population in [link] to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

Solution:

38.90 months

Hint

Use the process from the previous example.

Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant k, we have $y = y_0e^{-kt}$.

As with exponential growth, there is a differential equation associated with exponential decay. We have
Equation:

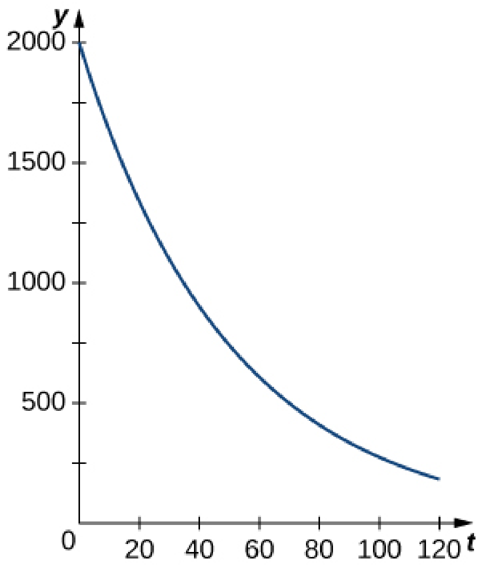
$$y' = -ky_0e^{-kt} = -ky.$$

Note:
Rule: Exponential Decay Model
Systems that exhibit **exponential decay** behave according to the model
Equation:

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *decay constant*.

The following figure shows a graph of a representative exponential decay function.



An example of exponential decay.

Let's look at a physical application of exponential decay. Newton's law of cooling says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then
Equation:

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

Equation:

$$y' = -ky.$$

From our previous work, we know this relationship between *y* and its derivative leads to exponential decay. Thus,

Equation:

$$y = y_0 e^{-kt},$$

and we see that

Equation:

$$\begin{aligned} T - T_a &= (T_0 - T_a)e^{-kt} \\ T &= (T_0 - T_a)e^{-kt} + T_a \end{aligned}$$

where T_0 represents the initial temperature. Let's apply this formula in the following example.

Example:

Exercise:

Problem:

Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between 155°F and 175°F. Suppose coffee is poured at a temperature of 200°F, and after 2 minutes in a 70°F room it has cooled to 180°F. When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

Solution:

We have

Equation:

$$\begin{aligned} T &= (T_0 - T_a)e^{-kt} + T_a \\ 180 &= (200 - 70)e^{-k(2)} + 70 \\ 110 &= 130e^{-2k} \\ \tfrac{11}{13} &= e^{-2k} \\ \ln \tfrac{11}{13} &= -2k \\ \ln 11 - \ln 13 &= -2k \\ k &= \tfrac{\ln 13 - \ln 11}{2}. \end{aligned}$$

Then, the model is
Equation:

$$T = 130e^{(\ln 11 - \ln 13/2)t} + 70.$$

The coffee reaches $175^\circ\mathrm{F}$ when
Equation:

$$\begin{aligned} 175 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 105 &= 130e^{(\ln 11 - \ln 13/2)t} \\ \tfrac{21}{26} &= e^{(\ln 11 - \ln 13/2)t} \\ \ln \tfrac{21}{26} &= \tfrac{\ln 11 - \ln 13}{2}t \\ \ln 21 - \ln 26 &= \tfrac{\ln 11 - \ln 13}{2}t \\ t &= \tfrac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \approx 2.56. \end{aligned}$$

The coffee can be served about 2.5 minutes after it is poured. The coffee reaches $155^\circ\mathrm{F}$ at
Equation:

$$\begin{aligned} 155 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 85 &= 130e^{(\ln 11 - \ln 13)t} \\ \tfrac{17}{26} &= e^{(\ln 11 - \ln 13)t} \\ \ln 17 - \ln 26 &= \left(\tfrac{\ln 11 - \ln 13}{2}\right)t \\ t &= \tfrac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \approx 5.09. \end{aligned}$$

The coffee is too cold to be served about 5 minutes after it is poured.

Note:
Exercise:

Problem:

Suppose the room is warmer $(75\,^{\circ}\mathrm{F})$ and, after 2 minutes, the coffee has cooled only to $185\,^{\circ}\mathrm{F}$. When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Solution:

The coffee is first cool enough to serve about 3.5 minutes after it is poured. The coffee is too cold to serve about 7 minutes after it is poured.

Hint

Use the process from the previous example.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

Equation:

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Note: This is the same expression we came up with for doubling time.

Note:
Definition
If a quantity decays exponentially, the **half-life** is the amount of time it takes the quantity to be reduced by half. It is given by
Equation:

$$\text{Half-life} = \frac{\ln 2}{k}.$$

Example:

Exercise:

Problem:
Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon was originally present in an object and how much carbon remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon now contains 10 g of carbon, how old is it? Round the answer to the nearest hundred years.

Solution:

We have
Equation:

$$\begin{aligned} 5730 &= \frac{\ln 2}{k} \\ k &= \frac{\ln 2}{5730}. \end{aligned}$$

So, the model says
Equation:

$$y = 100e^{-(\ln 2/5730)t}.$$

In 50 years, we have
Equation:

$$\begin{aligned} y &= 100e^{-(\ln 2/5730)(50)} \\ &\approx 99.40. \end{aligned}$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

To determine the age of the artifact, we must solve
Equation:

$$\begin{aligned} 10 &= 100e^{-(\ln 2/5730)t} \\ \frac{1}{10} &= e^{-(\ln 2/5730)t} \\ t &\approx 19035. \end{aligned}$$

The artifact is about 19,000 years old.

Note:
Exercise:

Problem:

If we have 100 g of carbon-14, how much is left after. years? If an artifact that originally contained 100 g of carbon now contains $20g$ of carbon, how old is it? Round the answer to the nearest hundred years.

Solution:

A total of 94.13 g of carbon remains. The artifact is approximately 13,300 years old.

Hint

Use the process from the previous example.

Key Concepts

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

True or False? If true, prove it. If false, find the true answer.

Exercise:

Problem: The doubling time for $y = e^{ct}$ is $(\ln\ (2))/\ (\ln\ (c))$.

Exercise:

Problem:

If you invest $\$500$, an annual rate of interest of 3% yields more money in the first year than a 2.5% continuous rate of interest.

Solution:

True

Exercise:

Problem:

If you leave a $100^\circ\mathrm{C}$ pot of tea at room temperature $(25^\circ\mathrm{C})$ and an identical pot in the refrigerator $(5^\circ\mathrm{C})$, with $k = 0.02$, the tea in the refrigerator reaches a drinkable temperature $(70^\circ\mathrm{C})$ more than 5 minutes before the tea at room temperature.

Exercise:

Problem:

If given a half-life of *t* years, the constant k for $y = e^{kt}$ is calculated by $k = \ln\ (1/2)/t$.

Solution:

False; $k = \frac{\ln(2)}{t}$

For the following exercises, use $y = y_0e^{kt}$.

Exercise:

Problem:

If a culture of bacteria doubles in 3 hours, how many hours does it take to multiply by 10?

Exercise:

Problem:

If bacteria increase by a factor of 10 in 10 hours, how many hours does it take to increase by 100?

Solution:

20 hours

Exercise:

Problem:

How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is 5730 years.

Exercise:

Problem:

If a relic contains 90% as much radiocarbon as new material, can it have come from the time of Christ (approximately 2000 years ago)? Note that the half-life of radiocarbon is 5730 years.

Solution:

No. The relic is approximately 871 years old.

Exercise:

Problem:

The population of Cairo grew from 5 million to 10 million in 20 years. Use an exponential model to find when the population was 8 million.

Exercise:

Problem:

The populations of New York and Los Angeles are growing at 1% and 1.4% a year, respectively. Starting from 8 million (New York) and 6 million (Los Angeles), when are the populations equal?

Solution:

71.92 years

Exercise:

Problem:

Suppose the value of $\$1$ in Japanese yen decreases at 2% per year. Starting from $\$1 = \quad 250$, when will $\$1 = \quad 1$?

Exercise:

Problem:

The effect of advertising decays exponentially. If 40% of the population remembers a new product after 3 days, how long will 20% remember it?

Solution:

5 days 6 hours 27 minutes

Exercise:

Problem: If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$, what was y_0 at $t = 0$?

Exercise:

Problem: If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$, when does $y = 1$?

Solution:

12

Exercise:

Problem:

If a bank offers annual interest of 7.5% or continuous interest of 7.25%, which has a better annual yield?

Exercise:

Problem: What continuous interest rate has the same yield as an annual rate of 9%?

Solution:

8.618%

Exercise:

Problem:

If you deposit $\$5000$ at 8% annual interest, how many years can you withdraw $\$500$ (starting after the first year) without running out of money?

Exercise:

Problem:

You are trying to save $\$50,000$ in 20 years for college tuition for your child. If interest is a continuous 10%, how much do you need to invest initially?

Solution:

$\$6766.76$

Exercise:

Problem:

You are cooling a turkey that was taken out of the oven with an internal temperature of $165^\circ\mathrm{F}$. After 10 minutes of resting the turkey in a $70^\circ\mathrm{F}$ apartment, the temperature has reached $155^\circ\mathrm{F}$. What is the temperature of the turkey 20 minutes after taking it out of the oven?

Exercise:

Problem:

You are trying to thaw some vegetables that are at a temperature of $1^\circ\mathrm{F}$. To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of $44^\circ\mathrm{F}$. You check on your vegetables 2 hours after putting them in the refrigerator to find that they are now $12^\circ\mathrm{F}$. Plot the resulting temperature curve and use it to determine when the vegetables reach $33^\circ\mathrm{F}$.

Solution:

9 hours 13 minutes

Exercise:

Problem:

You are an archaeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era $(146$ million years to 65 million years ago), and you find by radiocarbon dating that there is 0.000001% the amount of radiocarbon. Is this bone from the Cretaceous?

Exercise:

Problem:

The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of $24,000$ years. If 1 barrel containing 10 kg of plutonium-239 is sealed, how many years must pass until only $10g$ of plutonium-239 is left?

Solution:

$239,179$ years

For the next set of exercises, use the following table, which features the world population by decade.

Years since 1950	**Population (millions)**
0	2,556
10	3,039
20	3,706
30	4,453
40	5,279
50	6,083
60	6,849

Source: http://www.factmonster.com/ipka/A0762181.html.

Exercise:

Problem:

[T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

Exercise:

Problem:

[T] Find and graph the derivative y' of your equation. Where is it increasing and what is the meaning of this increase?

Solution:

$P\prime(t) = 43e^{0.01604t}$. The population is always increasing.

Exercise:

Problem:

[T] Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

Exercise:

Problem:

[T] Find the predicted date when the population reaches 10 billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

Solution:

The population reaches 10 billion people in 2027.

For the next set of exercises, use the following table, which shows the population of San Francisco during the 19th century.

Years since 1850	Population (thousands)
0	21.00

Years since 1850	Population (thousands)
10	56.80
20	149.5
30	234.0

Source: http://www.sfgenealogy.com/sf/history/hgpop.htm.

Exercise:

Problem:

[T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

Exercise:

Problem:

[T] Find and graph the derivative y' of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

Solution:

$P\prime(t) = 2.259e^{0.06407t}$. The population is always increasing.

Exercise:

Problem:

[T] Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

Glossary

doubling time
: if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

exponential decay
: systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$

exponential growth
: systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$

half-life

if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

L’Hôpital’s Rule

- Recognize when to apply L’Hôpital’s rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L’Hôpital’s rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L’Hôpital’s rule**, uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L’Hôpital’s Rule

L’Hôpital’s rule can be used to evaluate limits involving the quotient of two functions. Consider
Equation:

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2 \neq 0$, then
Equation:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$? We call this one of the **indeterminate forms**, of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider
Equation:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \text{ and } \lim_{x \to 0} \frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing
Equation:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4.$$

For $\lim\limits_{x \to 0} \frac{\sin x}{x}$ we were able to show, using a geometric argument, that
Equation:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more important, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim\limits_{x \to a} f(x) = 0 = \lim\limits_{x \to a} g(x)$ and such that $g'(a) \neq 0$ For x near a, we can write
Equation:

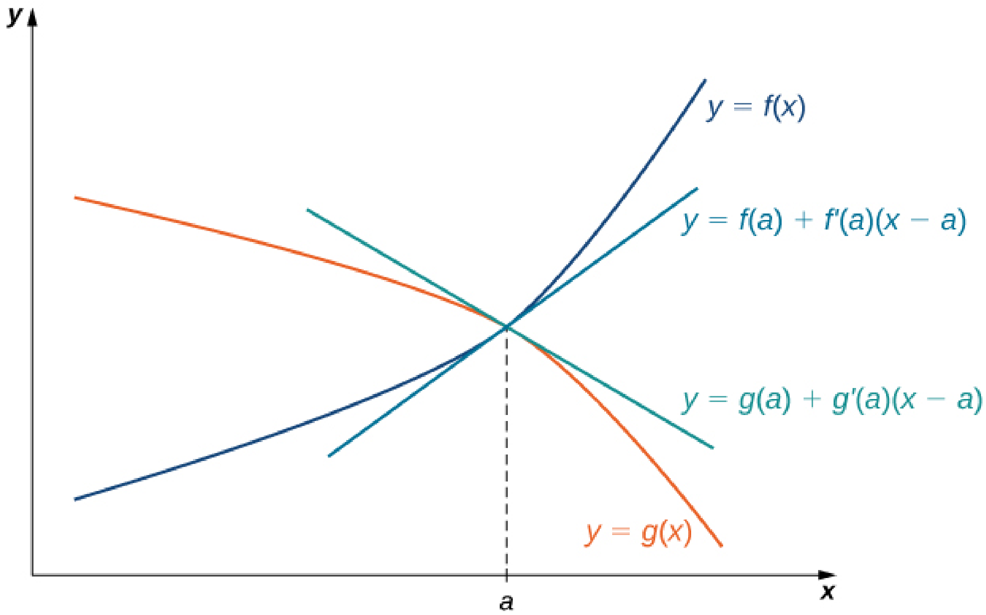
$$f(x) \approx f(a) + f'(a)(x - a)$$

and
Equation:

$$g(x) \approx g(a) + g'(a)(x - a).$$

Therefore,
Equation:

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}.$$



If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x \to a} f(x) = 0$. Similarly, $g(a) = \lim_{x \to a} g(x) = 0$. If we also assume that f' and g' are continuous at $x = a$, then $f'(a) = \lim_{x \to a} f'(x)$ and $g'(a) = \lim_{x \to a} g'(x)$. Using these ideas, we conclude that

Equation:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

Note:
L'Hôpital's Rule (0/0 Case)
Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

Equation:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ and $-\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

Equation:

$$\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} && \text{since } f(a) = 0 = g(a) \\
&= \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} && \text{algebra} \\
&= \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} && \text{limit of a quotient} \\
&= \frac{f'(a)}{g'(a)} && \text{definition of the derivative} \\
&= \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} && \text{continuity of } f' \text{ and } g' \\
&= \lim_{x \to a} \frac{f'(x)}{g'(x)}. && \text{limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example:
Exercise:

Problem:
Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x \to 0} \frac{1-\cos x}{x}$

b. $\lim\limits_{x \to 1} \frac{\sin(\pi x)}{\ln x}$
c. $\lim\limits_{x \to \infty} \frac{e^{1/x}-1}{1/x}$
d. $\lim\limits_{x \to 0} \frac{\sin x - x}{x^2}$

Solution:

a. Since the numerator $1 - \cos x \to 0$ and the denominator $x \to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have
Equation:

$$\begin{aligned} \lim_{x \to 0} \frac{1-\cos x}{x} &= \lim_{x \to 0} \frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)} \\ &= \lim_{x \to 0} \frac{\sin x}{1} \\ &= \frac{\lim\limits_{x \to 0}(\sin x)}{\lim\limits_{x \to 0}(1)} \\ &= \frac{0}{1} = 0. \end{aligned}$$

b. As $x \to 1$, the numerator $\sin(\pi x) \to 0$ and the denominator $\ln(x) \to 0$. Therefore, we can apply L'Hôpital's rule. We obtain
Equation:

$$\begin{aligned} \lim_{x \to 1} \frac{\sin(\pi x)}{\ln x} &= \lim_{x \to 1} \frac{\pi \cos(\pi x)}{1/x} \\ &= \lim_{x \to 1} (\pi x) \cos(\pi x) \\ &= (\pi \cdot 1)(-1) = -\pi. \end{aligned}$$

c. As $x \to \infty$, the numerator $e^{1/x} - 1 \to 0$ and the denominator $\left(\frac{1}{x}\right) \to 0$. Therefore, we can apply L'Hôpital's rule. We obtain
Equation:

$$\lim_{x \to \infty} \frac{e^{1/x} - 1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x \to \infty} e^{1/x} = e^0 = 1.$$

d. As $x \to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain
Equation:

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = \lim_{x \to 0} \frac{\cos x - 1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as

$x \to 0$, we apply L'Hôpital's rule again. In doing so, we see that
Equation:

$$\lim_{x\to0}\frac{\cos x-1}{2x}=\lim_{x\to0}\frac{-\sin x}{2}=0.$$

Therefore, we conclude that
Equation:

$$\lim_{x\to0}\frac{\sin x-x}{x^2}=0.$$

Note:
Exercise:

Problem: Evaluate $\lim\limits_{x\to0}\frac{x}{\tan x}$.

Solution:

1

Hint

$\frac{d}{dx}\tan x=\sec^2 x$

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$. Limits of this form are classified as *indeterminate forms of type* ∞/∞. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞. is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

Note:
L'Hôpital's Rule (∞/∞ Case)
Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim\limits_{x\to a}f(x)=\infty$ (or $-\infty$) and $\lim\limits_{x\to a}g(x)=\infty$ (or $-\infty$). Then,
Equation:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $-\infty$, or the limit is one-sided.

Example:
Exercise:

Problem:
Applying L'Hôpital's Rule $(\infty/\infty$ Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim\limits_{x\to\infty} \frac{3x+5}{2x+1}$
b. $\lim\limits_{x\to 0^+} \frac{\ln x}{\cot x}$

Solution:

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim\limits_{x\to\infty}(3x+5) = \infty$ and $\lim\limits_{x\to\infty}(2x+1) = \infty$. Therefore, we apply L'Hôpital's rule and obtain
Equation:

$$\lim_{x\to\infty} \frac{3x+5}{2x+1} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that
Equation:

$$\lim_{x\to\infty} \frac{3x+5}{2x+1} = \lim_{x\to\infty} \frac{3+5/x}{2x+1/x} = \frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim\limits_{x\to 0^+} \ln x = -\infty$ and $\lim\limits_{x\to 0^+} \cot x = \infty$. Therefore, we can apply L'Hôpital's rule and obtain
Equation:

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{1}{-x\csc^2 x}.$$

Now as $x \to 0^+$, $\csc^2 x \to \infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

Equation:

$$\lim_{x\to 0^+} \frac{1}{-x\csc^2 x} = \lim_{x\to 0^+} \frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to 0^+} \sin^2 x = 0$ and $\lim_{x\to 0^+} x = 0$, so we apply L'Hôpital's rule again. We find

Equation:

$$\lim_{x\to 0^+} \frac{\sin^2 x}{-x} = \lim_{x\to 0^+} \frac{2\sin x\cos x}{-1} = \frac{0}{-1} = 0.$$

We conclude that

Equation:

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = 0.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x\to\infty} \frac{\ln x}{5x}$.

Solution:

0

Hint

$\frac{d}{dx} \ln x = \frac{1}{x}$

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

Example:
Exercise:

Problem:
When L’Hôpital’s Rule Does Not Apply

Consider $\lim_{x\to 1}\frac{x^2+5}{3x+4}$. Show that the limit cannot be evaluated by applying L’Hôpital’s rule.

Solution:

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L’Hôpital’s rule. If we try to do so, we get
Equation:

$$\frac{d}{dx}\left(x^2+5\right)=2x$$

and
Equation:

$$\frac{d}{dx}(3x+4)=3.$$

At which point we would conclude erroneously that
Equation:

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\lim_{x\to 1}\frac{2x}{3}=\frac{2}{3}.$$

However, since $\lim_{x\to 1}\left(x^2+5\right)=6$ and $\lim_{x\to 1}(3x+4)=7$, we actually have
Equation:

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}.$$

We can conclude that
Equation:

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\frac{d}{dx}\left(x^2+5\right)}{\frac{d}{dx}(3x+4)}.$$

Note:
Exercise:

Problem:

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x \to 0^+} \frac{\cos x}{x}$. Evaluate $\lim_{x \to 0^+} \frac{\cos x}{x}$ by other means.

Solution:

$\lim_{x \to 0^+} \cos x = 1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞

Hint

Determine the limits of the numerator and denominator separately.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0 \cdot \infty$, $\infty - \infty$, 1^∞, ∞^0, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim_{x \to a} \left(f(x) \cdot g(x)\right)$, where $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form that arises in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x \to a$. For example, let n be a positive integer and consider

Equation:

$$f(x) = \frac{1}{(x^n + 1)} \text{ and } g(x) = 3x^2.$$

As $x \to \infty$, $f(x) \to 0$ and $g(x) \to \infty$. However, the limit as $x \to \infty$ of $f(x)g(x) = \frac{3x^2}{(x^n+1)}$ varies, depending on n. If $n = 2$, then $\lim_{x \to \infty} f(x)g(x) = 3$. If $n = 1$, then $\lim_{x \to \infty} f(x)g(x) = \infty$. If $n = 3$, then $\lim_{x \to \infty} f(x)g(x) = 0$. Here we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example:
Exercise:

Problem:
Indeterminate Form of Type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution:

First, rewrite the function $x \ln x$ as a quotient to apply L'Hôpital's rule. If we write
Equation:

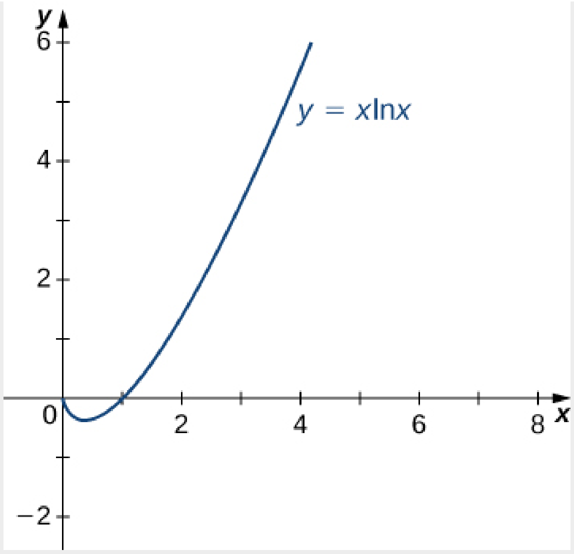
$$x \ln x = \frac{\ln x}{1/x},$$

we see that $\ln x \to -\infty$ as $x \to 0^+$ and $\frac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain
Equation:

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

We conclude that
Equation:

$$\lim_{x \to 0^+} x \ln x = 0.$$



Finding the limit at $x = 0$ of the function $f(x) = x \ln x$.

Note:
Exercise:

Problem: Evaluate $\lim\limits_{x \to 0} x \cot x$.

Solution:

1

Hint

Write $x \cot x = \frac{x \cos x}{\sin x}$

Indeterminate Form of Type $\infty - \infty$

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim\limits_{x \to \infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do

more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

Equation:

$$\lim_{x\to\infty} (f(x) - g(x)) = \lim_{x\to\infty} \left(3x^3 - 3x^2 - 5\right) = \infty.$$

On the other hand, if $n = 2$, then

Equation:

$$\lim_{x\to\infty} (f(x) - g(x)) = \lim_{x\to\infty} \left(3x^2 - 3x^2 - 5\right) = -5.$$

However, if $n = 1$, then

Equation:

$$\lim_{x\to\infty} (f(x) - g(x)) = \lim_{x\to\infty} \left(3x - 3x^2 - 5\right) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L’Hôpital’s rule.

Example:
Exercise:

Problem:
Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim\limits_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right)$.

Solution:

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

Equation:

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L’Hôpital’s rule. Taking the derivatives of the numerator and the denominator, we have

Equation:

$$\lim_{x\to 0^+}\frac{(\tan x)-x^2}{x^2\tan x}=\lim_{x\to 0^+}\frac{\left(\sec^2 x\right)-2x}{x^2\sec^2 x+2x\tan x}.$$

As $x\to 0^+$, $\left(\sec^2 x\right)-2x\to 1$ and $x^2\sec^2 x+2x\tan x\to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

Equation:

$$\lim_{x\to 0^+}\frac{\left(\sec^2 x\right)-2x}{x^2\sec^2 x+2x\tan x}=\infty.$$

Therefore,

Equation:

$$\lim_{x\to 0^+}\left(\frac{1}{x^2}-\frac{1}{\tan x}\right)=\infty.$$

Note:

Exercise:

Problem: Evaluate $\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Solution:

0

Hint

Rewrite the difference of fractions as a single fraction.

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions 0^0, ∞^0, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L’Hôpital’s rule can be used to evaluate limits involving these indeterminate forms.

Since L’Hôpital’s rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim\limits_{x\to a} f(x)^{g(x)}$ and we

arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

Equation:

$$y = f(x)^{g(x)}.$$

Then,

Equation:

$$\ln y = \ln \left(f(x)^{g(x)}\right) = g\left(x\right)\ln\left(f\left(x\right)\right).$$

Therefore,

Equation:

$$\lim_{x\to a}\left[\ln\left(y\right)\right] = \lim_{x\to a}\left[g\left(x\right)\ln\left(f\left(x\right)\right)\right].$$

Since $\lim_{x\to a} f\left(x\right) = \infty$, we know that $\lim_{x\to a}\ln\left(f\left(x\right)\right) = \infty$. Therefore, $\lim_{x\to a} g\left(x\right)\ln\left(f\left(x\right)\right)$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g\left(x\right)\ln\left(f\left(x\right)\right)$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x\to a} g\left(x\right)\ln\left(f\left(x\right)\right) = L$, where L may be ∞ or $-\infty$. Then

Equation:

$$\lim_{x\to a}\left[\ln\left(y\right)\right] = L.$$

Since the natural logarithm function is continuous, we conclude that

Equation:

$$\ln\left(\lim_{x\to a} y\right) = L,$$

which gives us

Equation:

$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^L.$$

Example:

Exercise:

Problem:

Indeterminate Form of Type ∞^0

Evaluate $\lim\limits_{x \to \infty} x^{1/x}$.

Solution:

Let $y = x^{1/x}$. Then,
Equation:

$$\ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim\limits_{x \to \infty} \frac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain
Equation:

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

Therefore, $\lim\limits_{x \to \infty} \ln y = 0$. Since the natural logarithm function is continuous, we conclude that
Equation:

$$\ln\left(\lim_{x \to \infty} y\right) = 0,$$

which leads to
Equation:

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{\ln x}{x} = e^0 = 1.$$

Hence,
Equation:

$$\lim_{x \to \infty} x^{1/x} = 1.$$

Note:
Exercise:

Problem: Evaluate $\lim_{x\to\infty} x^{1/\ln(x)}$.

Solution:

e

Hint

Let $y = x^{1/\ln(x)}$ and apply the natural logarithm to both sides of the equation.

Example:
Exercise:

Problem:
Indeterminate Form of Type 0^0

Evaluate $\lim_{x\to 0^+} x^{\sin x}$.

Solution:

Let
Equation:

$$y = x^{\sin x}.$$

Therefore,
Equation:

$$\ln y = \ln\left(x^{\sin x}\right) = \sin x \ln x.$$

We now evaluate $\lim_{x\to 0^+} \sin x \ln x$. Since $\lim_{x\to 0^+} \sin x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$, we have the indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write
Equation:

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or
Equation:

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let’s consider the first option. In this case, applying L’Hôpital’s rule, we would obtain
Equation:

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x\to 0^+} \frac{\cos x}{-1/\left(x(\ln x)^2\right)} = \lim_{x\to 0^+} \left(-x(\ln x)^2 \cos x\right).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing
Equation:

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x},$$

and applying L’Hôpital’s rule, we obtain
Equation:

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \frac{1}{\sin x}$ and $\cot x = \frac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as
Equation:

$$\lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x\to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x)\right] = \left(\lim_{x\to 0^+} \frac{\sin x}{x}\right) \cdot \left(\lim_{x\to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

We conclude that $\lim\limits_{x\to 0^+} \ln y = 0$. Therefore, $\ln\left(\lim\limits_{x\to 0^+} y\right) = 0$ and we have
Equation:

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,
Equation:

$$\lim_{x\to 0^+} x^{\sin x} = 1.$$

Note:
Exercise:

Problem: Evaluate $\lim_{x \to 0^+} x^x$.

Solution:

1

Hint

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as shown in the following table, the values of x^3 are growing much faster than the values of x^2.

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

Comparing the Growth Rates of x^2 and x^3

In fact,
Equation:

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \lim_{x \to \infty} x = \infty. \text{ or, equivalently, } \lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in the following table. In fact,
Equation:

$$\lim_{x\to\infty} \frac{x^2}{3x^2+4x+1} = \frac{1}{3}.$$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if
Equation:

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty; \text{ or, equivalently, } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that
Equation:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L’Hôpital’s rule to compare the growth rates of power, exponential, and logarithmic functions.

Example:

Exercise:

Problem:
Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate $\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)$.

a. $f(x) = x^2$ and $g(x) = e^x$
b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution:

a. Since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty}\left[\frac{x^2}{e^x}\right]$. We obtain

Equation:

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x}.$$

Since $\lim_{x\to\infty} 2x = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

Equation:

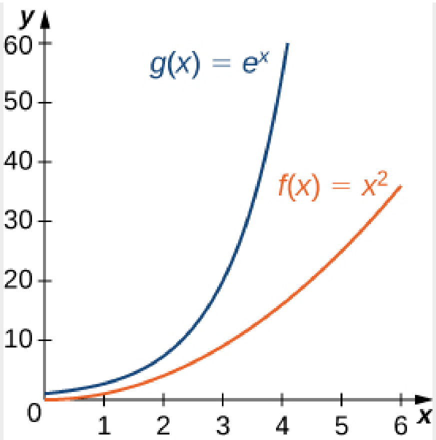
$$\lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0,$$

we conclude that

Equation:

$$\lim_{x\to\infty}\frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x\to\infty$ (See [link] and [link]).



An exponential function grows at a faster rate than a power function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

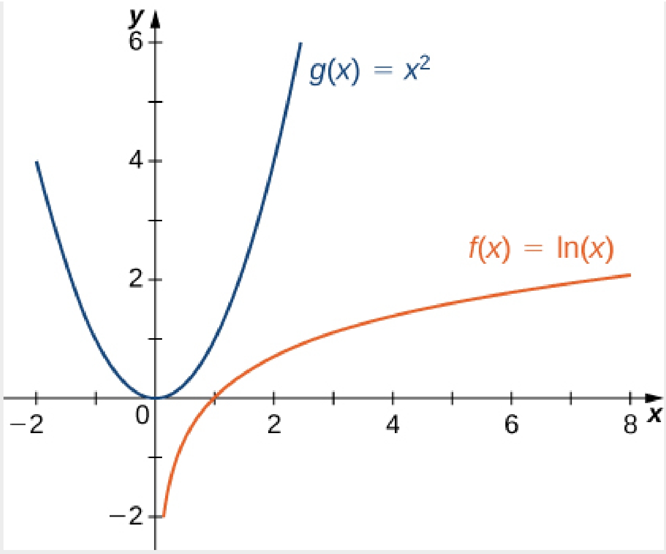
Growth rates of a power function and an exponential function.

b. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \frac{\ln x}{x^2}$. We obtain

Equation:

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see [link] and [link]).



A power function grows at a faster rate than a logarithmic function.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

Growth rates of a power function and a logarithmic function

Note:
Exercise:

Problem: Compare the growth rates of x^{100} and 2^x.

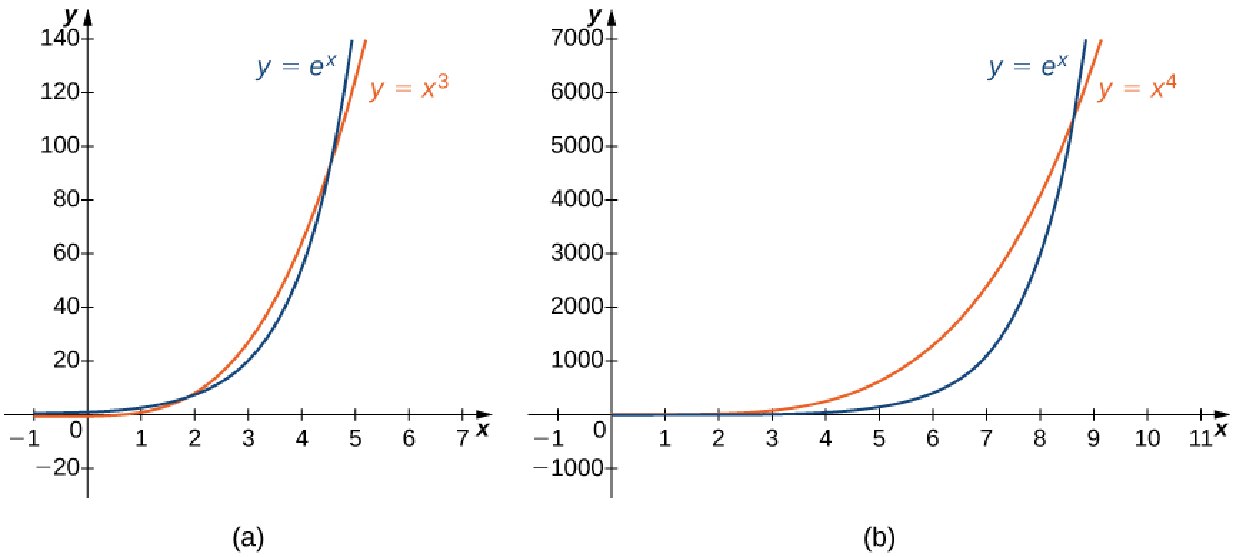
Solution:

The function 2^x grows faster than x^{100}.

Hint

Apply L'Hôpital's rule to $x^{100}/2^x$

Using the same ideas as in [link]a. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In [link] and [link], we compare e^x with x^3 and x^4 as $x \to \infty$.

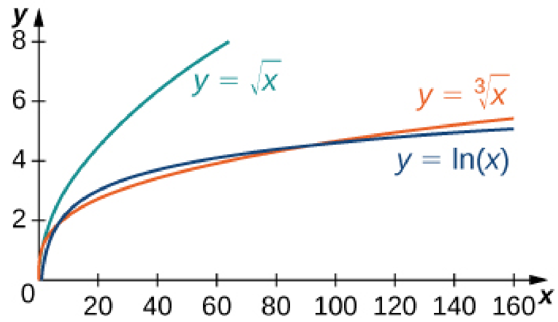


The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

An exponential function grows at a faster rate than any power function

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In [link] and [link], we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.



The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

A logarithmic function grows at a slower rate than any root function

Key Concepts

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.
- The exponential function e^x grows faster than any power function x^p, $p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function x^p, $p > 0$.

For the following exercises, evaluate the limit.

Exercise:

Problem: Evaluate the limit $\lim\limits_{x\to\infty} \frac{e^x}{x}$.

Exercise:

Problem: Evaluate the limit $\lim_{x \to \infty} \frac{e^x}{x^k}$.

Solution:

∞

Exercise:

Problem: Evaluate the limit $\lim_{x \to \infty} \frac{\ln x}{x^k}$.

Exercise:

Problem: Evaluate the limit $\lim_{x \to a} \frac{x-a}{x^2-a^2}, \quad a \neq 0.$

Solution:

$\frac{1}{2a}$

Exercise:

Problem: Evaluate the limit $\lim_{x \to a} \frac{x-a}{x^3-a^3}, \quad a \neq 0.$

Exercise:

Problem: Evaluate the limit $\lim_{x \to a} \frac{x-a}{x^n-a^n}, \quad a \neq 0.$

Solution:

$\frac{1}{na^{n-1}}$

For the following exercises, determine whether you can apply L'Hôpital's rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply L'Hôpital's rule.

Exercise:

Problem: $\lim_{x \to 0^+} x^2 \ln x$

Exercise:

Problem: $\lim_{x \to \infty} x^{1/x}$

Solution:

Cannot apply directly; use logarithms

Exercise:

Problem: $\lim_{x \to 0} x^{2/x}$

Exercise:

Problem: $\lim_{x \to 0} \frac{x^2}{1/x}$

Solution:

Cannot apply directly; rewrite as $\lim_{x \to 0} x^3$

Exercise:

Problem: $\lim_{x \to \infty} \frac{e^x}{x}$

For the following exercises, evaluate the limits with either L'Hôpital's rule or previously learned methods.

Exercise:

Problem: $\lim_{x \to 3} \frac{x^2-9}{x-3}$

Solution:

6

Exercise:

Problem: $\lim_{x \to 3} \frac{x^2-9}{x+3}$

Exercise:

Problem: $\lim_{x \to 0} \frac{(1+x)^{-2}-1}{x}$

Solution:

-2

Exercise:

Problem: $\lim_{x \to \pi/2} \frac{\cos x}{\frac{\pi}{2}-x}$

Exercise:

Problem: $\lim_{x \to \pi} \frac{x-\pi}{\sin x}$

Solution:

-1

Exercise:

Problem: $\lim_{x \to 1} \frac{x-1}{\sin x}$

Exercise:

Problem: $\lim_{x \to 0} \frac{(1+x)^n-1}{x}$

Solution:

n

Exercise:

Problem: $\lim_{x \to 0} \frac{(1+x)^n-1-nx}{x^2}$

Exercise:

Problem: $\lim_{x \to 0} \frac{\sin x-\tan x}{x^3}$

Solution:

$-\frac{1}{2}$

Exercise:

Problem: $\lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Exercise:

Problem: $\lim_{x \to 0} \frac{e^x-x-1}{x^2}$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $\lim_{x \to 0} \frac{\tan x}{\sqrt{x}}$

Exercise:

Problem: $\lim_{x \to 1} \frac{x-1}{\ln x}$

Solution:

1

Exercise:

Problem: $\lim_{x \to 0} (x+1)^{1/x}$

Exercise:

Problem: $\lim_{x \to 1} \frac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

Solution:

$\frac{1}{6}$

Exercise:

Problem: $\lim_{x \to 0^+} x^{2x}$

Exercise:

Problem: $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

Solution:

1

Exercise:

Problem: $\lim_{x \to 0} \frac{\sin x - x}{x^2}$

Exercise:

Problem: $\lim_{x \to 0^+} x \ln\left(x^4\right)$

Solution:

0

Exercise:

Problem: $\lim_{x\to\infty}(x-e^x)$

Exercise:

Problem: $\lim_{x\to\infty}x^2e^{-x}$

Solution:

0

Exercise:

Problem: $\lim_{x\to 0}\frac{3^x-2^x}{x}$

Exercise:

Problem: $\lim_{x\to 0}\frac{1+1/x}{1-1/x}$

Solution:

-1

Exercise:

Problem: $\lim_{x\to\pi/4}(1-\tan x)\cot x$

Exercise:

Problem: $\lim_{x\to\infty}xe^{1/x}$

Solution:

∞

Exercise:

Problem: $\lim_{x\to 0}x^{1/\cos x}$

Exercise:

Problem: $\lim_{x\to 0}x^{1/x}$

Solution:

1

Exercise:

Problem: $\lim\limits_{x \to 0} \left(1 - \frac{1}{x}\right)^x$

Exercise:

Problem: $\lim\limits_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$

Solution:

$\frac{1}{e}$

For the following exercises, use a calculator to graph the function and estimate the value of the limit, then use L'Hôpital's rule to find the limit directly.

Exercise:

Problem: **[T]** $\lim\limits_{x \to 0} \frac{e^x - 1}{x}$

Exercise:

Problem: **[T]** $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right)$

Solution:

0

Exercise:

Problem: **[T]** $\lim\limits_{x \to 1} \frac{x - 1}{1 - \cos(\pi x)}$

Exercise:

Problem: **[T]** $\lim\limits_{x \to 1} \frac{e^{(x-1)} - 1}{x - 1}$

Solution:

1

Exercise:

Problem: [T] $\lim\limits_{x\to 1}\frac{(x-1)^2}{\ln x}$

Exercise:

Problem: [T] $\lim\limits_{x\to \pi}\frac{1+\cos x}{\sin x}$

Solution:

0

Exercise:

Problem: [T] $\lim\limits_{x\to 0}\left(\csc x-\frac{1}{x}\right)$

Exercise:

Problem: [T] $\lim\limits_{x\to 0^+}\tan\left(x^x\right)$

Solution:

$\tan\left(1\right)$

Exercise:

Problem: [T] $\lim\limits_{x\to 0^+}\frac{\ln x}{\sin x}$

Exercise:

Problem: [T] $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{x}$

Solution:

2

Glossary

indeterminate forms
: when evaluating a limit, the forms $\frac{0}{0}$, ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is

L’Hôpital’s rule

if f and g are differentiable functions over an interval a, except possibly at a, and $\lim_{x \to a} f\left(x\right) = 0 = \lim_{x \to a} g\left(x\right)$ or $\lim_{x \to a} f\left(x\right)$ and $\lim_{x \to a} g\left(x\right)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$

Calculus of the Hyperbolic Functions

- Apply the formulas for derivatives and integrals of the hyperbolic functions.
- Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.
- Describe the common applied conditions of a catenary curve.

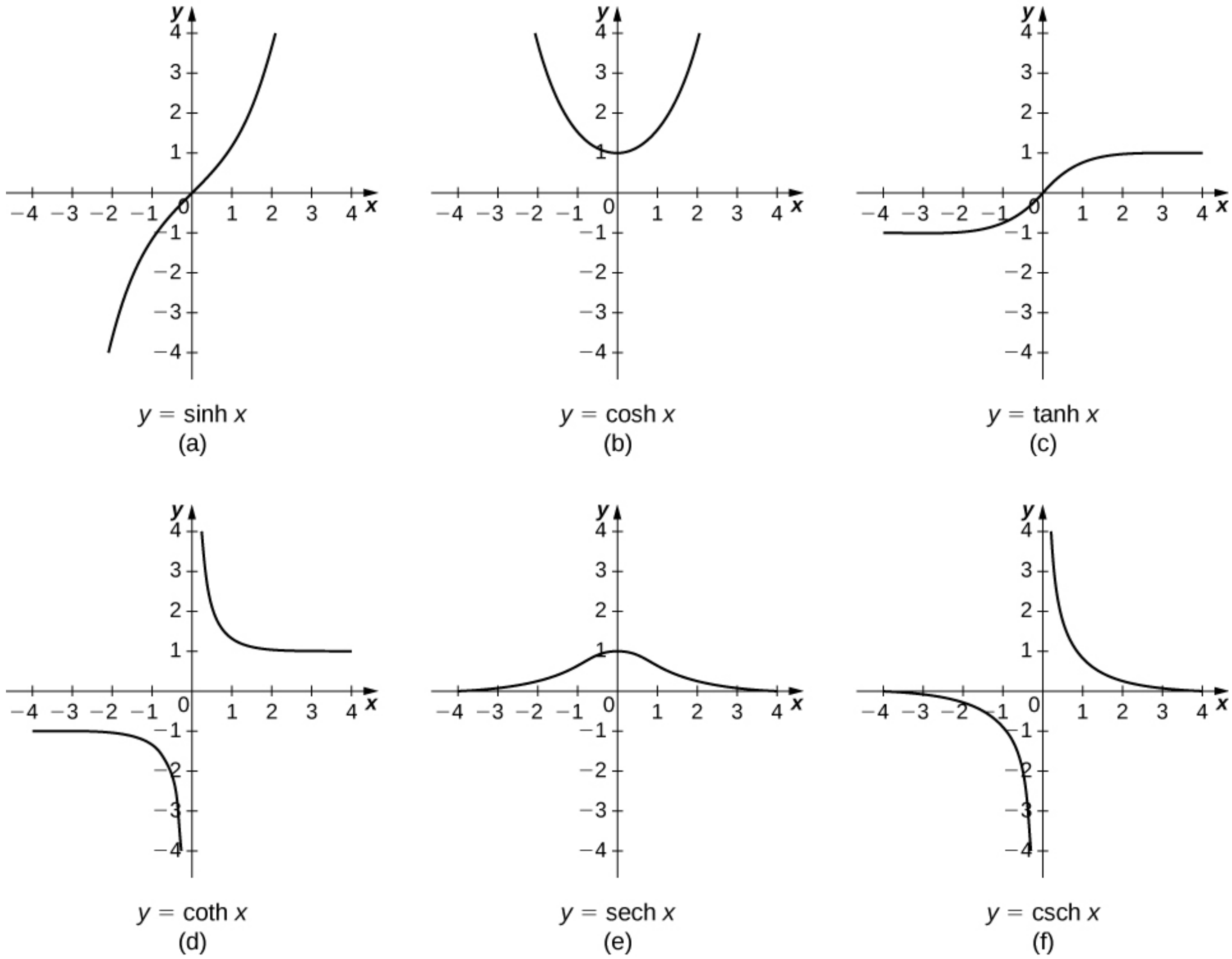
We were introduced to hyperbolic functions in Introduction to Functions and Graphs, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as
Equation:

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in the following figure.



Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

Equation:

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) \\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right] \\ &= \frac{1}{2}[e^x + e^{-x}] = \cosh x. \end{aligned}$$

Similarly, $(d/dx)\cosh x = \sinh x$. We summarize the differentiation formulas for the hyperbolic functions in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2\, x$
$\coth x$	$-\text{csch}^2\, x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{csch}\, x$	$-\text{csch}\, x \coth x$

Derivatives of the Hyperbolic Functions

Let’s take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match: $(d/dx)\sin x = \cos x$ and $(d/dx)\sinh x = \cosh x$. The derivatives of the cosine functions, however, differ in sign: $(d/dx)\cos x = -\sin x$, but $(d/dx)\cosh x = \sinh x$. As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions.

These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

Equation:

$$\begin{aligned} \int \sinh u\, du &= \cosh u + C & \int \text{csch}^2\, u\, du &= -\coth u + C \\ \int \cosh u\, du &= \sinh u + C & \int \text{sech}\, u \tanh u\, du &= -\text{sech}\, u + C \\ \int \text{sech}^2 u\, du &= \tanh u + C & \int \text{csch}\, u \coth u\, du &= -\text{csch}\, u + C \end{aligned}$$

Example:
Exercise:

Problem:
Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right)$
b. $\frac{d}{dx}(\cosh x)^2$

Solution:

Using the formulas in [link] and the chain rule, we get

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right) = \cosh\left(x^2\right) \cdot 2x$
b. $\frac{d}{dx}(\cosh x)^2 = 2\cosh x \sinh x$

Note:
Exercise:

Problem: Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\tanh\left(x^2+3x\right)\right)$
b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

Solution:

a. $\frac{d}{dx}\left(\tanh\left(x^2+3x\right)\right) = \left(\operatorname{sech}^2\left(x^2+3x\right)\right)(2x+3)$
b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right) = \frac{d}{dx}(\sinh x)^{-2} = -2(\sinh x)^{-3}\cosh x$

Hint

Use the formulas in [link] and apply the chain rule as necessary.

Example:
Exercise:

Problem:
Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\int x\cosh\left(x^2\right)dx$
b. $\int \tanh x\, dx$

Solution:

We can use u-substitution in both cases.

a. Let $u = x^2$. Then, $du = 2x\,dx$ and
Equation:

$$\int x \cosh\left(x^2\right) dx = \int \frac{1}{2} \cosh u\, du = \frac{1}{2} \sinh u + C = \frac{1}{2} \sinh\left(x^2\right) + C.$$

b. Let $u = \cosh x$. Then, $du = \sinh x\, dx$ and
Equation:

$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|\cosh x| + C.$$

Note that $\cosh x > 0$ for all x, so we can eliminate the absolute value signs and obtain
Equation:

$$\int \tanh x\, dx = \ln(\cosh x) + C.$$

Note:
Exercise:

Problem: Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x\, dx$
b. $\int \operatorname{sech}^2(3x) dx$

Solution:

a. $\int \sinh^3 x \cosh x\, dx = \frac{\sinh^4 x}{4} + C$
b. $\int \operatorname{sech}^2(3x) dx = \frac{\tanh(3x)}{3} + C$

Hint

Use the formulas above and apply u-substitution as necessary.

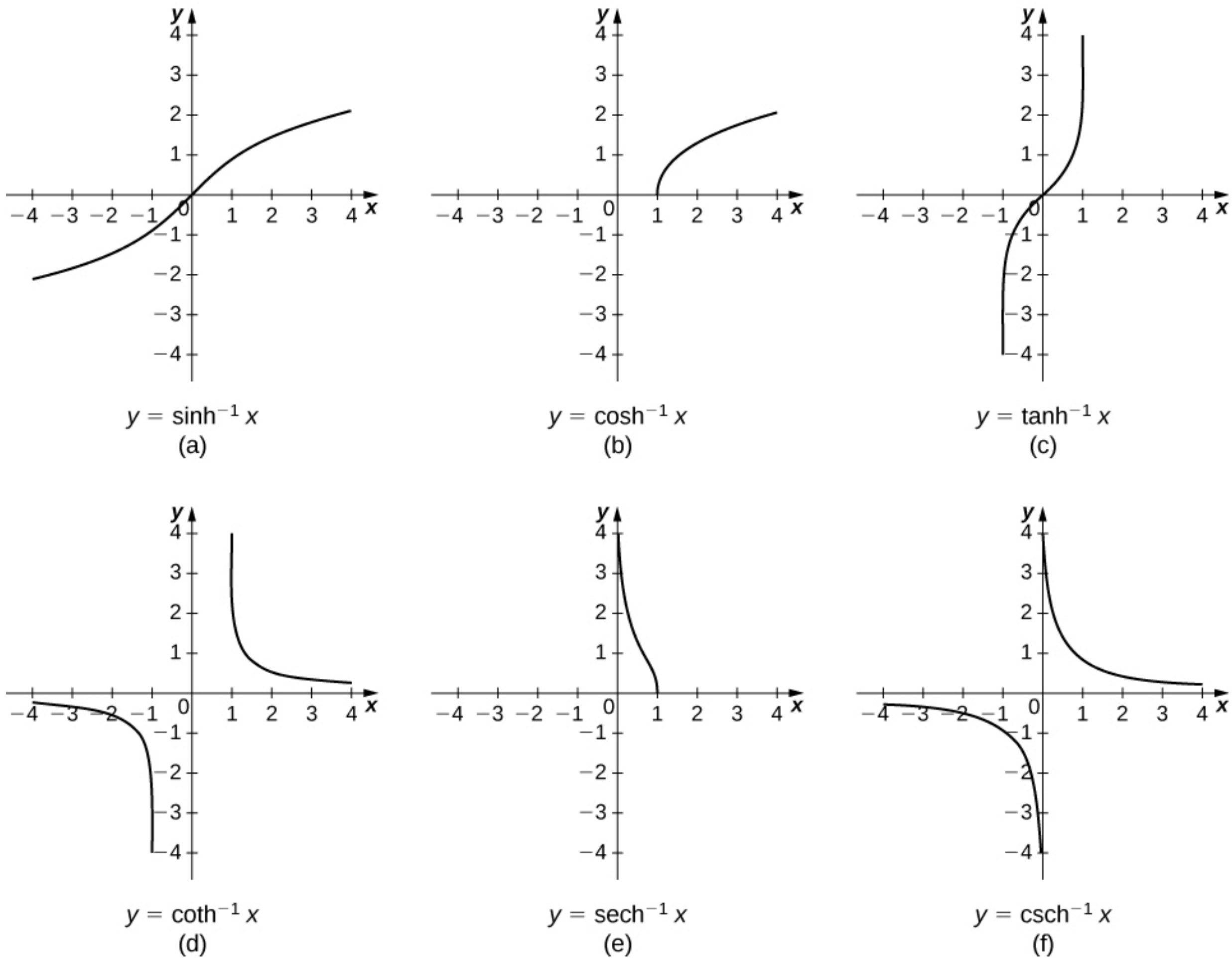
Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in the following table.

Function	**Domain**	**Range**
$\sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x$	$(1, \infty)$	$[0, \infty)$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1} x$	$(0, 1)$	$[0, \infty)$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Domains and Ranges of the Inverse Hyperbolic Functions

The graphs of the inverse hyperbolic functions are shown in the following figure.



Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

Equation:

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx}\sinh y &= \frac{d}{dx}x \\ \cosh y \frac{dy}{dx} &= 1. \end{aligned}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1+\sinh^2 y}$. Then,

Equation:

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh^{-1}x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1}x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1}x$	$\frac{1}{1-x^2}$
$\coth^{-1}x$	$\frac{1}{1-x^2}$
$\operatorname{sech}^{-1}x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{csch}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Derivatives of the Inverse Hyperbolic Functions

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/\left(1-x^2\right)$, we need to select the proper antiderivative based on the domain of the functions and the values of x. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

Equation:

$$\begin{aligned} \int \frac{1}{\sqrt{1+u^2}}du &= \sinh^{-1}u+C \\ \int \frac{1}{\sqrt{u^2-1}}du &= \cosh^{-1}u+C \\ \int \frac{1}{1-u^2}du &= \begin{cases} \tanh^{-1}u+C \text{ if } \lvert u\rvert<1 \\ \coth^{-1}u+C \text{ if } \lvert u\rvert>1 \end{cases} \end{aligned}$$

$$\begin{aligned} \int \frac{1}{u\sqrt{1-u^2}}du &= -\operatorname{sech}^{-1}\lvert u\rvert+C \\ \int \frac{1}{u\sqrt{1+u^2}}du &= -\operatorname{csch}^{-1}\lvert u\rvert+C \end{aligned}$$

Example:
Exercise:

Problem:

Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$
b. $\frac{d}{dx}\left(\tanh^{-1}x\right)^2$

Solution:

Using the formulas in [link] and the chain rule, we obtain the following results:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$
b. $\frac{d}{dx}\left(\tanh^{-1}x\right)^2 = \frac{2\left(\tanh^{-1}x\right)}{1-x^2}$

Note:
Exercise:

Problem: Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\cosh^{-1}\left(3x\right)\right)$
b. $\frac{d}{dx}\left(\coth^{-1}x\right)^3$

Solution:

a. $\frac{d}{dx}\left(\cosh^{-1}\left(3x\right)\right) = \frac{3}{\sqrt{9x^2-1}}$
b. $\frac{d}{dx}\left(\coth^{-1}x\right)^3 = \frac{3\left(\coth^{-1}x\right)^2}{1-x^2}$

Hint

Use the formulas in [link] and apply the chain rule as necessary.

Example:
Exercise:

Problem:
Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\displaystyle\int \frac{1}{\sqrt{4x^2-1}}dx$
b. $\displaystyle\int \frac{1}{2x\sqrt{1-9x^2}}dx$

Solution:

We can use u-substitution in both cases.

a. Let $u = 2x$. Then, $du = 2dx$ and we have
Equation:

$$\int \frac{1}{\sqrt{4x^2-1}}dx = \int \frac{1}{2\sqrt{u^2-1}}du = \frac{1}{2}\cosh^{-1}u + C = \frac{1}{2}\cosh^{-1}(2x) + C.$$

b. Let $u = 3x$. Then, $du = 3dx$ and we obtain
Equation:

$$\int \frac{1}{2x\sqrt{1-9x^2}}dx = \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}du = -\frac{1}{2}\operatorname{sech}^{-1}|u| + C = -\frac{1}{2}\operatorname{sech}^{-1}|3x| + C.$$

Note:
Exercise:

Problem: Evaluate the following integrals:

a. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}dx,\ \ x > 2$
b. $\displaystyle\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Solution:

a. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}dx = \cosh^{-1}\left(\frac{x}{2}\right) + C$
b. $\displaystyle\int \frac{1}{\sqrt{1-e^{2x}}}dx = -\operatorname{sech}^{-1}(e^x) + C$

Hint

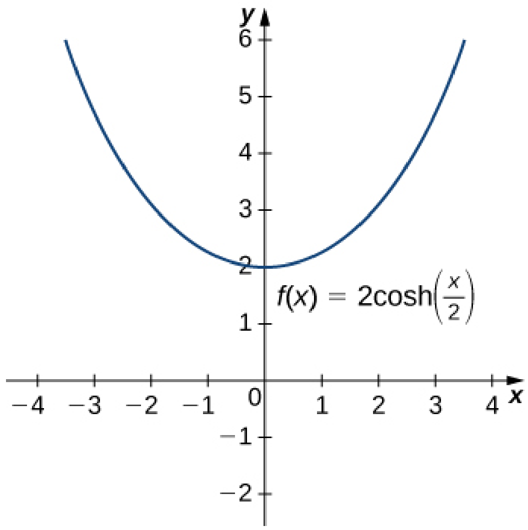
Use the formulas above and apply u-substitution as necessary.

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.

Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a\cosh\left(x/a\right)$ are catenaries. [link] shows the graph of $y = 2\cosh\left(x/2\right)$.



A hyperbolic cosine function forms the shape of a catenary.

Example:
Exercise:

Problem:
Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where x is measured in feet. Determine the length of the cable (in feet).

Solution:

Recall from Section 6.4 that the formula for arc length is
Equation:

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then
Equation:

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx.\end{aligned}$$

Now recall that $1 + \sinh^2 x = \cosh^2 x$, so we have

Equation:

$$\begin{aligned} \text{Arc Length} &= \int_{-15}^{15} \sqrt{1 + \sinh^2 \left(\frac{x}{10}\right)}\, dx \\ &= \int_{-15}^{15} \cosh\left(\tfrac{x}{10}\right) dx \\ &= 10 \sinh\left(\tfrac{x}{10}\right)\Big|_{-15}^{15} = 10\left[\sinh\left(\tfrac{3}{2}\right) - \sinh\left(-\tfrac{3}{2}\right)\right] = 20 \sinh\left(\tfrac{3}{2}\right) \\ &\approx 42.586 \text{ ft.} \end{aligned}$$

Note:

Exercise:

Problem:

Assume a hanging cable has the shape $15 \cosh(x/15)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).

Solution:

52.95 ft

Hint

Use the procedure from the previous example.

Key Concepts

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.
- The most common physical applications of hyperbolic functions are calculations involving catenaries.

Exercise:

Problem:

[T] Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

Solution:

e^x and e^{-x}

Exercise:

Problem: From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

Exercise:

Problem: Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

Solution:

Answers may vary

Exercise:

Problem: Use the quotient rule to verify that $\tanh(x)' = \text{sech}^2(x)$.

Exercise:

Problem: Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

Solution:

Answers may vary

Exercise:

Problem: Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

Exercise:

Problem:

Prove $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

Solution:

Answers may vary

Exercise:

Problem:

Take the derivative of the previous expression to find an expression for $\cosh(x+y)$.

For the following exercises, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

Exercise:

Problem: **[T]** $\cosh(3x+1)$

Solution:

$3\sinh(3x+1)$

Exercise:

Problem: **[T]** $\sinh\left(x^2\right)$

Exercise:

Problem: **[T]** $\frac{1}{\cosh(x)}$

Solution:

$-\tanh(x)\operatorname{sech}(x)$

Exercise:

Problem: **[T]** $\sinh(\ln(x))$

Exercise:

Problem: **[T]** $\cosh^2(x)+\sinh^2(x)$

Solution:

$4\cosh(x)\sinh(x)$

Exercise:

Problem: **[T]** $\cosh^2(x)-\sinh^2(x)$

Exercise:

Problem: **[T]** $\tanh\left(\sqrt{x^2+1}\right)$

Solution:

$\frac{x\operatorname{sech}^2\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$

Exercise:

Problem: **[T]** $\frac{1+\tanh(x)}{1-\tanh(x)}$

Exercise:

Problem: **[T]** $\sinh^6(x)$

Solution:

$6\sinh^5(x)\cosh(x)$

Exercise:

Problem: **[T]** $\ln(\operatorname{sech}(x)+\tanh(x))$

For the following exercises, find the antiderivatives for the given functions.

Exercise:

Problem: $\cosh(2x+1)$

Solution:

$\frac{1}{2}\sinh(2x+1)+C$

Exercise:

Problem: $\tanh(3x+2)$

Exercise:

Problem: $x\cosh\left(x^2\right)$

Solution:

$\frac{1}{2}\sinh^2\left(x^2\right)+C$

Exercise:

Problem: $3x^3\tanh\left(x^4\right)$

Exercise:

Problem: $\cosh^2(x)\sinh(x)$

Solution:

$\frac{1}{3}\cosh^3(x)+C$

Exercise:

Problem: $\tanh^2(x)\operatorname{sech}^2(x)$

Exercise:

Problem: $\frac{\sinh(x)}{1+\cosh(x)}$

Solution:

$\ln\left(1+\cosh(x)\right)+C$

Exercise:

Problem: $\coth\left(x\right)$

Exercise:

Problem: $\cosh\left(x\right)+\sinh\left(x\right)$

Solution:

$\cosh\left(x\right)+\sinh\left(x\right)+C$

Exercise:

Problem: $\left(\cosh\left(x\right)+\sinh\left(x\right)\right)^{n}$

For the following exercises, find the derivatives for the functions.

Exercise:

Problem: $\tanh^{-1}\left(4x\right)$

Solution:

$\frac{4}{1-16x^2}$

Exercise:

Problem: $\sinh^{-1}\left(x^2\right)$

Exercise:

Problem: $\sinh^{-1}\left(\cosh\left(x\right)\right)$

Solution:

$\frac{\sinh(x)}{\sqrt{\cosh^2(x)+1}}$

Exercise:

Problem: $\cosh^{-1}\left(x^3\right)$

Exercise:

Problem: $\tanh^{-1}\left(\cos\left(x\right)\right)$

Solution:

$-\csc\left(x\right)$

Exercise:

Problem: $e^{\sinh^{-1}(x)}$

Exercise:

Problem: $\ln\left(\tanh^{-1}\left(x\right)\right)$

Solution:

$-\frac{1}{(x^2-1)\tanh^{-1}(x)}$

For the following exercises, find the antiderivatives for the functions.

Exercise:

Problem: $\int \frac{dx}{4-x^2}$

Exercise:

Problem: $\int \frac{dx}{a^2-x^2}$

Solution:

$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^2+1}}$

Exercise:

Problem: $\int \frac{x\,dx}{\sqrt{x^2+1}}$

Solution:

$\sqrt{x^2+1}+C$

Exercise:

Problem: $\int -\frac{dx}{x\sqrt{1-x^2}}$

Exercise:

Problem: $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

Solution:

$\cosh^{-1}(e^x) + C$

Exercise:

Problem: $\int -\frac{2x}{x^4-1}$

For the following exercises, use the fact that a falling body with friction equal to velocity squared obeys the equation $dv/dt = g - v^2$.

Exercise:

Problem: Show that $v(t) = \sqrt{g}\tanh\left(\sqrt{g}t\right)$ satisfies this equation.

Solution:

Answers may vary

Exercise:

Problem: Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g-v^2} = dt$.

Exercise:

Problem:

[T] Estimate how far a body has fallen in 12 seconds by finding the area underneath the curve of $v(t)$.

Solution:

37.30

For the following exercises, use this scenario: A cable hanging under its own weight has a slope $S = dy/dx$ that satisfies $dS/dx = c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

Exercise:

Problem: Show that $S = \sinh(cx)$ satisfies this equation.

Exercise:

Problem: Integrate $dy/dx = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

Solution:

$y = \frac{1}{c}\cosh(cx)$

Exercise:

Problem: Sketch the cable and determine how far down it sags at $x = 0$.

For the following exercises, solve each problem.

Exercise:

Problem:

[T] A chain hangs from two posts 2 m apart to form a catenary described by the equation $y = 2\cosh\left(x/2\right) - 1$. Find the slope of the catenary at the left fence post.

Solution:

-0.521095

Exercise:

Problem:

[T] A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh\left(x/4\right) - 3$. Find the total length of the catenary (arc length).

Exercise:

Problem:

[T] A high-voltage power line is a catenary described by $y = 10\cosh\left(x/10\right)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

Solution:

10

Exercise:

Problem:

A telephone line is a catenary described by $y = a\cosh\left(x/a\right)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

Exercise:

Problem:

Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

Exercise:

Problem:

Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$.

(*Hint:* Use hyperbolic trigonometric identities.)

Exercise:

Problem:

Prove the formula for the derivative of $y = \text{sech}^{-1}(x)$ by differentiating $x = \text{sech}(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

Exercise:

Problem: Prove that $(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$.

Exercise:

Problem:

Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for y. Does your expression match the textbook?

Exercise:

Problem:

Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for y. Does your expression match the textbook?

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem:

The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

Solution:

False

Exercise:

Problem:

If the force is constant, the amount of work to move an object from $x=a$ to $x=b$ is $F(b-a)$.

Exercise:

Problem:

The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

Solution:

False

Exercise:

Problem: If the half-life of seaborgium-266 is 360 ms, then $k=(\ln(2))/360$.

For the following exercises, use the requested method to determine the volume of the solid.

Exercise:

Problem:

The volume that has a base of the ellipse $x^2/4+y^2/9=1$ and cross-sections of an equilateral triangle perpendicular to the y-axis. Use the method of slicing.

Solution:

$32\sqrt{3}$

Exercise:

Problem: $y=x^2-x$, from $x=1$ to $x=4$, rotated around the*y*-axis using the washer method

Exercise:

Problem: $x=y^2$ and $x=3y$ rotated around the *y*-axis using the washer method

---

Solution:

$\frac{162\pi}{5}$

Exercise:

Problem: $x=2y^2-y^3, x=0,$ and $y=0$ rotated around the *x*-axis using cylindrical shells

For the following exercises, find

a. the area of the region,

b. the volume of the solid when rotated around the *x*-axis, and
c. the volume of the solid when rotated around the *y*-axis. Use whichever method seems most appropriate to you.

Exercise:

Problem: $y = x^3, x = 0, y = 0, \text{ and } x = 2$

Solution:

a. 4, b. $\frac{128\pi}{7}$, c. $\frac{64\pi}{5}$

Exercise:

Problem: $y = x^2 - x$ and $x = 0$

Exercise:

Problem: **[T]** $y = \ln(x) + 2$ and $y = x$

Solution:

a. 1.949, b. 21.952, c. 17.099

Exercise:

Problem: $y = x^2$ and $y = \sqrt{x}$

Exercise:

Problem: $y = 5 + x$, $y = x^2$, $x = 0$, and $x = 1$

Solution:

a. $\frac{31}{6}$, b. $\frac{452\pi}{15}$, c. $\frac{31\pi}{6}$

Exercise:

Problem: Below $x^2 + y^2 = 1$ and above $y = 1 - x$

Exercise:

Problem: Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius 4.

Solution:

245.282

Exercise:

Problem: Find the center of mass for $\rho = \tan^2 x$ on $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Exercise:

Problem:

Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

Solution:

Mass: $\frac{1}{2}$, center of mass: $\left(\frac{18}{35}, \frac{9}{11}\right)$

For the following exercises, find the requested arc lengths.

Exercise:

Problem: The length of x for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

Exercise:

Problem: The length of y for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

Solution:

$\sqrt{17} + \frac{1}{8}\ln(33 + 8\sqrt{17})$

For the following exercises, find the surface area and volume when the given curves are revolved around the specified axis.

Exercise:

Problem:

The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the *y*-axis.

Exercise:

Problem:

The loudspeaker created by revolving $y = 1/x$ from $x = 1$ to $x = 4$ around the *x*-axis.

Solution:

Volume: $\frac{3\pi}{4}$, surface area: $\pi\left(\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(16) - \frac{\sqrt{257}}{16}\right)$

For the following exercises, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height 205 m and width 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m ^{3}.

Exercise:

Problem: Find the total force on the wall of the dam.

Exercise:

Problem:

You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and 45°F outside and the temperature of the body is 78°F. You know the cooling constant is $k = 0.00824^\circ\text{F/min}$. When did the victim die, assuming that a human's temperature is 98°F ?

Solution:

11:02 a.m.

For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

Years after 1920	**Value ($)**
1	63.90
3	100
5	110
7	160
9	381.17

Source: http://stockcharts.com/freecharts/historical/djia19201940.html

Exercise:

Problem:

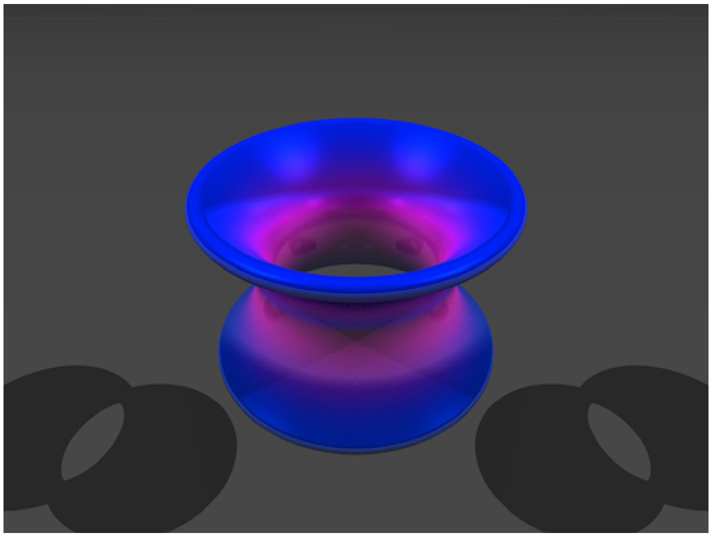
[T] The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in 2014 ?

For the following exercises, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

Exercise:

Problem:

Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-axis, as shown here.



Solution:

$\pi(1 + \sinh(1)\cosh(1))$

Exercise:

Problem:

Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x -axis.

Glossary

catenary
: a curve in the shape of the function $y = a\cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

Introduction class="introduction"

Careful planning of traffic signals can prevent or reduce the number of accidents at busy intersections. (credit: modification of work by David McKelvey, Flickr)

In a large city, accidents occurred at an average rate of one every three months at a particularly busy intersection. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the eight-month interval without an accident a result of chance? We explore this question later in this chapter and see that integration is an essential part of determining the answer (see [link]).

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

Integration by Parts

- Recognize when to use integration by parts.
- Use the integration-by-parts formula to solve integration problems.
- Use the integration-by-parts formula for definite integrals.

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x\sin(x^2)dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x\sin x\,dx$ defies us. Many students want to know whether there is a product rule for integration. There isn't, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts**.

The Integration-by-Parts Formula

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain $h'(x) = f'(x)g(x) + g'(x)f(x)$. Although at first it may seem counterproductive, let's now integrate both sides of this equation: $\int h'(x)dx = \int \big(g(x)f'(x) + f(x)g'(x)\big)dx.$

This gives us
Equation:

$$h(x) = f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx.$$

Now we solve for $\int f(x)g'(x)dx$:
Equation:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)dx$ and $dv = g'(x)dx$, we have the more compact form
Equation:

$$\int u\,dv = uv - \int v\,du.$$

Note:
Integration by Parts
Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:
Equation:

$$\int u\,dv = uv - \int v\,du.$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

Example:
Exercise:

Problem:
Using Integration by Parts

Use integration by parts with $u = x$ and $dv = \sin x\, dx$ to evaluate $\int x \sin x\, dx$.

Solution:

By choosing $u = x$, we have $du = 1dx$. Since $dv = \sin x\, dx$, we get $v = \int \sin x\, dx = -\cos x$. It is handy to keep track of these values as follows:
Equation:

$$\begin{array}{rclcrcl} u & = & x & \quad & dv & = & \sin x\, dx \\ du & = & 1dx & \quad & v & = & \int \sin x\, dx = -\cos x. \end{array}$$

Applying the integration-by-parts formula results in
Equation:

$$\begin{aligned} \int x \sin x\, dx &= (x)(-\cos x) - \int (-\cos x)(1dx) && \text{Substitute.} \\ &= -x \cos x + \int \cos x\, dx && \text{Simplify.} \\ &= -x \cos x + \sin x + C. && \text{Use } \int \cos x\, dx = \sin x + C. \end{aligned}$$

Analysis

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin x$ and $dv = x$. If we had done so, then we would have $du = \cos x$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts, we have $\int x \sin x\, dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x\, dx$. Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x\, dx = -\cos x$, we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:
Equation:

$$\begin{aligned}\int x\sin x\,dx &= (x)(-\cos x+K)-\int(-\cos x+K)(1dx)\\ &= -x\cos x+Kx+\int\cos x\,dx-\int Kdx\\ &= -x\cos x\ +Kx+\sin x-Kx+C\\ &= -x\cos x+\sin x+C.\end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x\cos x+\sin x+C$:
Equation:

$$\begin{aligned}\tfrac{d}{dx}(-x\cos x+\sin x+C) &= (-1)\cos x+(-x)(-\sin x)+\cos x\\ &= x\sin x.\end{aligned}$$

Therefore, the antiderivative checks out.

Note:
Watch this video and visit this website for examples of integration by parts.

Note:
Exercise:

Problem: Evaluate $\int xe^{2x}dx$ using the integration-by-parts formula with $u=x$ and $dv=e^{2x}dx$.

Solution:

$\int xe^{2x}dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$

Hint

Find du and v, and use the previous example as a guide.

The natural question to ask at this point is: How do we know how to choose u and dv? Sometimes it is a matter of trial and error; however, the acronym LIATE can often help to take some of the guesswork out of our choices. This acronym stands for **L**ogarithmic Functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric Functions, and **E**xponential Functions. This mnemonic serves as an aid in determining an appropriate choice for u.

The type of function in the integral that appears first in the list should be our first choice of u. For example, if an integral contains a logarithmic function and an algebraic function, we should choose u to be the logarithmic function, because L comes before A in LIATE. The integral in [link] has a trigonometric function $(\sin x)$ and an algebraic function (x). Because A comes before T in LIATE, we chose u to be the algebraic function. When we have chosen u, dv is selected to be the remaining part of the function to be integrated, together with dx.

Why does this mnemonic work? Remember that whatever we pick to be dv must be something we can integrate. Since we do not have integration formulas that allow us to integrate simple logarithmic functions and inverse trigonometric functions, it makes sense that they should not be chosen as values for dv. Consequently, they should be at the head of the list as choices for u. Thus, we put LI at the beginning of the mnemonic. (We could just as easily have started with IL, since these two types of functions won't appear together in an integration-by-parts problem.) The exponential and trigonometric functions are at the end of our list because they are fairly easy to integrate and make good choices for dv. Thus, we have TE at the end of our mnemonic. (We could just as easily have used ET at the end, since when these types of functions appear together it usually doesn't really matter which one is u and which one is dv.) Algebraic functions are generally easy both to integrate and to differentiate, and they come in the middle of the mnemonic.

Example:
Exercise:

Problem:
Using Integration by Parts

Evaluate $\int \frac{\ln x}{x^3} dx$.

Solution:

Begin by rewriting the integral:
Equation:

$$\int \frac{\ln x}{x^3} dx = \int x^{-3} \ln x \, dx.$$

Since this integral contains the algebraic function x^{-3} and the logarithmic function $\ln x$, choose $u = \ln x$, since L comes before A in LIATE. After we have chosen $u = \ln x$, we must choose $dv = x^{-3} dx$.

Next, since $u = \ln x$, we have $du = \frac{1}{x} dx$. Also, $v = \int x^{-3} dx = -\frac{1}{2} x^{-2}$. Summarizing,
Equation:

$$\begin{array}{rclcrcl} u & = & \ln x & & dv & = & x^{-3} dx \\ du & = & \frac{1}{x} dx & & v & = & \int x^{-3} dx = -\frac{1}{2} x^{-2}. \end{array}$$

Substituting into the integration-by-parts formula ([link]) gives
Equation:

$$\begin{array}{rcll} \int \frac{\ln x}{x^3} dx & = & \int x^{-3} \ln x \, dx = (\ln x)(-\frac{1}{2} x^{-2}) - \int (-\frac{1}{2} x^{-2})(\frac{1}{x} dx) & \\ & = & -\frac{1}{2} x^{-2} \ln x + \int \frac{1}{2} x^{-3} dx & \text{Simplify.} \\ & = & -\frac{1}{2} x^{-2} \ln x - \frac{1}{4} x^{-2} + C & \text{Integrate.} \\ & = & -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + C. & \text{Rewrite with positive intege} \end{array}$$

Note:
Exercise:

Problem: Evaluate $\int x \ln x \, dx$.

Solution:

$\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

Hint

Use $u = \ln x$ and $dv = x \, dx$.

In some cases, as in the next two examples, it may be necessary to apply integration by parts more than once.

Example:
Exercise:

Problem:
Applying Integration by Parts More Than Once

Evaluate $\int x^2 e^{3x} dx$.

Solution:

Using LIATE, choose $u = x^2$ and $dv = e^{3x} dx$. Thus, $du = 2x \, dx$ and $v = \int e^{3x} dx = \left(\frac{1}{3}\right) e^{3x}$.

Therefore,
Equation:

$$\begin{array}{ccccccc} u & = & x^2 & & dv & = & e^{3x} dx \\ du & = & 2x \, dx & & v & = & \int e^{3x} dx = \frac{1}{3} e^{3x}. \end{array}$$

Substituting into [link] produces
Equation:

$$\int x^2 e^{3x} dx = \frac{1}{3} x^2 e^{3x} - \int \frac{2}{3} x e^{3x} dx.$$

We still cannot integrate $\int \frac{2}{3} x e^{3x} dx$ directly, but the integral now has a lower power on x. We can evaluate this new integral by using integration by parts again. To do this, choose $u = x$ and $dv = \frac{2}{3} e^{3x} dx$. Thus, $du = dx$ and $v = \int \left(\frac{2}{3}\right) e^{3x} dx = \left(\frac{2}{9}\right) e^{3x}$. Now we have
Equation:

$$\begin{aligned} u &= x & dv &= \tfrac{2}{3}e^{3x}dx \\ du &= dx & v &= \int \frac{2}{3}e^{3x}dx = \frac{2}{9}e^{3x}. \end{aligned}$$

Substituting back into the previous equation yields
Equation:

$$\int x^2 e^{3x}dx = \frac{1}{3}x^2 e^{3x} - \left(\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x}dx\right).$$

After evaluating the last integral and simplifying, we obtain
Equation:

$$\int x^2 e^{3x}dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.$$

Example:
Exercise:

Problem:
Applying Integration by Parts When LIATE Doesn’t Quite Work

Evaluate $\int t^3 e^{t^2} dt$.

Solution:

If we use a strict interpretation of the mnemonic LIATE to make our choice of u, we end up with $u = t^3$ and $dv = e^{t^2}dt$. Unfortunately, this choice won’t work because we are unable to evaluate $\int e^{t^2}dt$. However, since we can evaluate $\int te^{t^2}dx$, we can try choosing $u = t^2$ and $dv = te^{t^2}dt$. With these choices we have
Equation:

$$\begin{aligned} u &= t^2 & dv &= te^{t^2}dt \\ du &= 2t\,dt & v &= \int te^{t^2}dt = \tfrac{1}{2}e^{t^2}. \end{aligned}$$

Thus, we obtain
Equation:

$$\begin{aligned} \int t^3 e^{t^2}dt &= \tfrac{1}{2}t^2 e^{t^2} - \int \frac{1}{2}e^{t^2}2tdt \\ &= \tfrac{1}{2}t^2 e^{t^2} - \tfrac{1}{2}e^{t^2} + C. \end{aligned}$$

Example:

Exercise:

Problem:
Applying Integration by Parts More Than Once

Evaluate $\int \sin(\ln x)dx$.

Solution:

This integral appears to have only one function—namely, $\sin(\ln x)$ —however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln x)$ and $dv = 1dx$. (The decision to use $u = \sin(\ln x)$ is easy. We can't choose $dv = \sin(\ln x)dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = (1/x)\cos(\ln x)dx$ and $v = \int 1dx = x$. After applying integration by parts to the integral and simplifying, we have

Equation:

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln x)$ and $dv = 1dx$, making $du = -(1/x)\sin(\ln x)dx$ and $v = \int 1dx = x$. Substituting, we have

Equation:

$$\int \sin(\ln x)dx = x\sin(\ln x) - \left(x\cos(\ln x) — \int -\sin(\ln x)dx\right).$$

After simplifying, we obtain

Equation:

$$\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx.$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln x)dx$. Thus, the equation becomes

Equation:

$$I = x\sin(\ln x) - x\cos(\ln x) - I.$$

First, add I to both sides of the equation to obtain

Equation:

$$2I = x\sin(\ln x) - x\cos(\ln x).$$

Next, divide by 2:

Equation:

$$I = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

Substituting $I = \int \sin(\ln x)dx$ again, we have

Equation:

$$\int \sin(\ln x)dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

From this we see that $(1/2)x\sin(\ln x) - (1/2)x\cos(\ln x)$ is an antiderivative of $\sin(\ln x)dx$. For the most general antiderivative, add $+C$:

Equation:

$$\int \sin(\ln x)dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C.$$

Analysis

If this method feels a little strange at first, we can check the answer by differentiation:

Equation:

$$\begin{aligned}&\tfrac{d}{dx}\left(\tfrac{1}{2}x\sin(\ln x) - \tfrac{1}{2}x\cos(\ln x)\right)\\ &= \tfrac{1}{2}(\sin(\ln x)) + \cos(\ln x)\cdot\tfrac{1}{x}\cdot\tfrac{1}{2}x - \left(\tfrac{1}{2}\cos(\ln x) - \sin(\ln x)\cdot\tfrac{1}{x}\cdot\tfrac{1}{2}x\right)\\ &= \sin(\ln x).\end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\int x^2 \sin x\, dx$.

Solution:

$-x^2\cos x + 2x\sin x + 2\cos x + C$

Hint

This is similar to [link].

Integration by Parts for Definite Integrals

Now that we have used integration by parts successfully to evaluate indefinite integrals, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Note:

Integration by Parts for Definite Integrals
Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then
Equation:

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du.$$

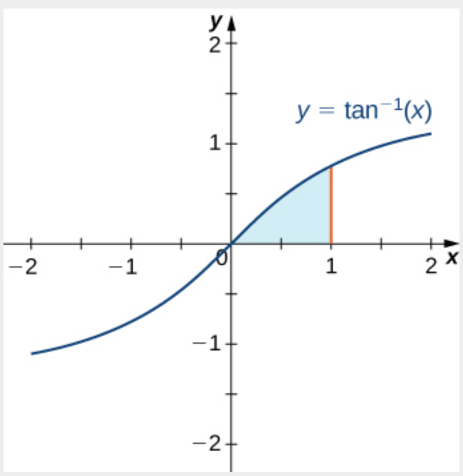
Example:
Exercise:

Problem:
Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1}x$ and below by the x-axis over the interval $[0, 1]$.

Solution:

This region is shown in [link]. To find the area, we must evaluate $\int\limits_0^1 \tan^{-1}x\,dx.$



To find the area of the shaded region, we have to use integration by parts.

For this integral, let's choose $u = \tan^{-1}x$ and $dv = dx$, thereby making $du = \frac{1}{x^2+1}dx$ and $v = x$. After applying the integration-by-parts formula ([link]) we obtain
Equation:

$$\text{Area} = x\tan^{-1}x|_0^1 - \int\limits_0^1 \frac{x}{x^2+1}dx.$$

Use u-substitution to obtain
Equation:

$$\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\ln|x^2+1|\Big|_0^1.$$

Thus,
Equation:

$$\text{Area} = x\ \tan^{-1} x\Big|_0^1 - \frac{1}{2}\ln|x^2+1|\Big|\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln\ 2.$$

At this point it might not be a bad idea to do a “reality check” on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.4388$, and from [link] we expect our area to be slightly less than 0.5, this solution appears to be reasonable.

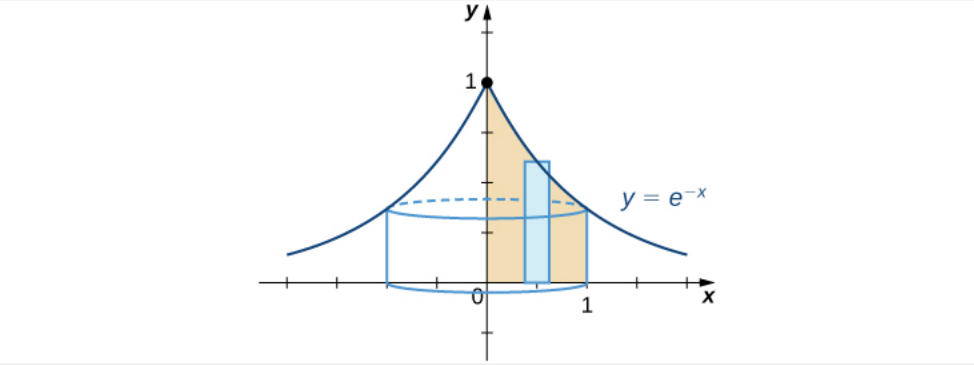
Example:
Exercise:

Problem:
Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the *x*-axis, the *y*-axis, and the line $x = 1$ about the *y*-axis.

Solution:

The best option to solving this problem is to use the shell method. Begin by sketching the region to be revolved, along with a typical rectangle (see the following graph).



We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate $2\pi \int_0^1 xe^{-x}dx$. To do this, let $u = x$ and $dv = e^{-x}$.

These choices lead to $du = dx$ and $v = \int e^{-x} = -e^{-x}$. Substituting into [link], we obtain

Equation:

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^1 xe^{-x}dx = 2\pi\left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx\right) && \text{Use integration by parts.} \\
&= -2\pi xe^{-x}\big|_0^1 - 2\pi e^{-x}\big|_0^1 && \text{Evaluate } \int_0^1 e^{-x}dx = -e^{-x}\Big|_0^1. \\
&= 2\pi - \tfrac{4\pi}{e}. && \text{Evaluate and simplify.}
\end{aligned}$$

Analysis

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius 1 and height of $1/e$ added to the volume of a cone of base radius 1 and height of $1 - \frac{1}{3}$. Consequently, the solid should have a volume a bit less than

Equation:

$$\pi(1)^2 \frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2\left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} - \frac{\pi}{3} \approx 1.8177.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

Note:
Exercise:

Problem: Evaluate $\int_0^{\pi/2} x\cos x\, dx$.

Solution:

$\frac{\pi}{2} - 1$

Hint

Use [link] with $u = x$ and $dv = \cos x\, dx$.

Key Concepts

- The integration-by-parts formula allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

Key Equations

- **Integration by parts formula**
  $\int u\, dv = uv - \int v\, du$
- **Integration by parts for definite integrals**
  $\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du$

In using the technique of integration by parts, you must carefully choose which expression is *u*. For each of the following problems, use the guidelines in this section to choose *u*. Do **not** evaluate the integrals.
Exercise:

Problem: $\int x^3 e^{2x} dx$

Solution:

$u = x^3$

Exercise:

Problem: $\int x^3 \ln(x) dx$

Exercise:

Problem: $\int y^3 \cos y dx$

Solution:

$u = y^3$

Exercise:

Problem: $\int x^2 \arctan x \, dx$

Exercise:

Problem: $\int e^{3x} \sin(2x) dx$

Solution:

$u = \sin(2x)$

Find the integral by using the simplest method. Not all problems require integration by parts.
Exercise:

Problem: $\int v \sin v dv$

Exercise:

Problem: $\int \ln x \, dx$ (*Hint:* $\int \ln x \, dx$ is equivalent to $\int 1 \cdot \ln(x) dx.$)

Solution:

$-x + x \ln x + C$

Exercise:

Problem: $\int x \cos x \, dx$

Exercise:

Problem: $\int \tan^{-1} x \, dx$

Solution:

$x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$

Exercise:

Problem: $\int x^2 e^x dx$

Exercise:

Problem: $\int x \sin(2x) dx$

Solution:

$-\frac{1}{2} x \cos(2x) + \frac{1}{4} \sin(2x) + C$

Exercise:

Problem: $\int x e^{4x} dx$

Exercise:

Problem: $\int x e^{-x} dx$

Solution:

$e^{-x} (-1 - x) + C$

Exercise:

Problem: $\int x \cos 3x \, dx$

Exercise:

Problem: $\int x^2 \cos x \, dx$

Solution:

$2x \cos x + \left(-2 + x^2\right) \sin x + C$

Exercise:

Problem: $\int x \ln x \, dx$

Exercise:

Problem: $\int \ln(2x+1) dx$

Solution:

$\frac{1}{2}(1+2x)(-1+\ln(1+2x))+C$

Exercise:

Problem: $\int x^2 e^{4x} dx$

Exercise:

Problem: $\int e^x \sin x \, dx$

Solution:

$\frac{1}{2}e^x(-\cos x+\sin x)+C$

Exercise:

Problem: $\int e^x \cos x \, dx$

Exercise:

Problem: $\int x e^{-x^2} dx$

Solution:

$-\frac{e^{-x^2}}{2}+C$

Exercise:

Problem: $\int x^2 e^{-x} dx$

Exercise:

Problem: $\int \sin(\ln(2x)) dx$

Solution:

$-\frac{1}{2}x\cos[\ln(2x)]+\frac{1}{2}x\sin[\ln(2x)]+C$

Exercise:

Problem: $\int cos(\ln x)dx$

Exercise:

Problem: $\int (\ln x)^2 dx$

Solution:

$2x - 2x\ln x + x(\ln x)^2 + C$

Exercise:

Problem: $\int \ln(x^2)dx$

Exercise:

Problem: $\int x^2 \ln x \, dx$

Solution:

$\left(-\frac{x^3}{9} + \frac{1}{3}x^3 \ln x\right) + C$

Exercise:

Problem: $\int \sin^{-1} x \, dx$

Exercise:

Problem: $\int \cos^{-1}(2x)dx$

Solution:

$-\frac{1}{2}\sqrt{1-4x^2} + x\cos^{-1}(2x) + C$

Exercise:

Problem: $\int x \arctan x \, dx$

Exercise:

Problem: $\int x^2 \sin x \, dx$

Solution:

$-\left(-2 + x^2\right)\cos x + 2x\sin x + C$

Exercise:

Problem: $\int x^3 \cos x \, dx$

Exercise:

Problem: $\int x^3 \sin x \, dx$

Solution:

$-x\left(-6+x^2\right)\cos x+3\left(-2+x^2\right)\sin x+C$

Exercise:

Problem: $\int x^3 e^x dx$

Exercise:

Problem: $\int x \sec^{-1} x \, dx$

Solution:

$\frac{1}{2}x\left(-\sqrt{1-\frac{1}{x^2}}+x\cdot\sec^{-1}x\right)+C$

Exercise:

Problem: $\int x \sec^2 x \, dx$

Exercise:

Problem: $\int x \cosh x \, dx$

Solution:

$-\cosh x+x\sinh x+C$

Compute the definite integrals. Use a graphing utility to confirm your answers.
Exercise:

Problem: $\int_{1/e}^{1} \ln x \, dx$

Exercise:

Problem: $\int_0^1 xe^{-2x} dx$ (Express the answer in exact form.)

Solution:

$\frac{1}{4} - \frac{3}{4e^2}$

Exercise:

Problem: $\int_0^1 e^{\sqrt{x}} dx \; (\text{let } u = \sqrt{x})$

Exercise:

Problem: $\int_1^e \ln(x^2) dx$

Solution:

2

Exercise:

Problem: $\int_0^{\pi} x \cos x \, dx$

Exercise:

Problem: $\int_{-\pi}^{\pi} x \sin x \, dx$ (Express the answer in exact form.)

Solution:

2π

Exercise:

Problem: $\int_0^3 \ln(x^2+1) dx$ (Express the answer in exact form.)

Exercise:

Problem: $\int_0^{\pi/2} x^2 \sin x \, dx$ (Express the answer in exact form.)

Solution:

$-2+\pi$

Exercise:

Problem: $\int_0^1 x 5^x dx$ (Express the answer using five significant digits.)

Exercise:

Problem: Evaluate $\int \cos x \ln(\sin x) dx$

Solution:

$-\sin(x) + \ln[\sin(x)]\sin x + C$

Derive the following formulas using the technique of integration by parts. Assume that *n* is a positive integer. These formulas are called *reduction formulas* because the exponent in the *x* term has been reduced by one in each case. The second integral is simpler than the original integral.

Exercise:

Problem: $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$

Exercise:

Problem: $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

Solution:

Answers vary

Exercise:

Problem: $\int x^n \sin x \, dx =$ munder

Exercise:

Problem: Integrate $\int 2x\sqrt{2x-3}dx$ using two methods:

a. Using parts, letting $dv = \sqrt{2x-3}\,dx$
b. Substitution, letting $u = 2x - 3$

Solution:

a. $\frac{2}{5}(1+x)(-3+2x)^{3/2} + C$ b. $\frac{2}{5}(1+x)(-3+2x)^{3/2} + C$

State whether you would use integration by parts to evaluate the integral. If so, identify *u* and *dv*. If not, describe the technique used to perform the integration without actually doing the problem.

Exercise:

Problem: $\int x \ln x \, dx$

Exercise:

Problem: $\int \frac{\ln^2 x}{x} dx$

Solution:

Do not use integration by parts. Choose *u* to be $\ln x$, and the integral is of the form $\int u^2 du$.

Exercise:

Problem: $\int xe^x dx$

Exercise:

Problem: $\int xe^{x^2-3} dx$

Solution:

Do not use integration by parts. Let $u = x^2 - 3$, and the integral can be put into the form $\int e^u du$.

Exercise:

Problem: $\int x^2 \sin x \, dx$

Exercise:

Problem: $\int x^2 \sin(3x^3 + 2) dx$

Solution:

Do not use integration by parts. Choose u to be $u = 3x^3 + 2$ and the integral can be put into the form $\int \sin(u) du$.

Sketch the region bounded above by the curve, the x-axis, and $x = 1$, and find the area of the region. Provide the exact form or round answers to the number of places indicated.

Exercise:

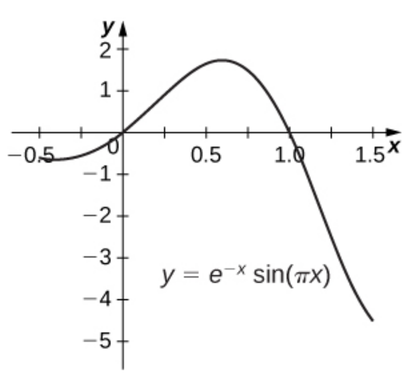
Problem: $y = 2xe^{-x}$ (Approximate answer to four decimal places.)

Exercise:

Problem: $y = e^{-x} \sin(\pi x)$ (Approximate answer to five decimal places.)

Solution:

The area under graph is 0.39535.



Find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.

Exercise:

Problem: $y = \sin x, y = 0, x = 2\pi, x = 3\pi$ about the y-axis (Express the answer in exact form.)

Exercise:

Problem: $y = e^{-x}$ $y = 0, x = -1x = 0$; about $x = 1$ (Express the answer in exact form.)

Solution:

$2\pi e$

Exercise:

Problem:

A particle moving along a straight line has a velocity of $v(t) = t^2 e^{-t}$ after *t* sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

Exercise:

Problem:

Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1$. (Round the answer to two significant digits.)

Solution:

2.05

Exercise:

Problem:

Find the area between $y = (x - 2)e^x$ and the *x*-axis from $x = 2$ to $x = 5$. (Express the answer in exact form.)

Exercise:

Problem: Find the area of the region enclosed by the curve $y = x \cos x$ and the *x*-axis for

$\frac{11\pi}{2} \leq x \leq \frac{13\pi}{2}$. (Express the answer in exact form.)

Solution:

12π

Exercise:

Problem:

Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the *x*-axis, and the vertical line $x = e^2$ about the *x*-axis. (Express the answer in exact form.)

Exercise:

Problem:

Find the volume of the solid generated by revolving the region bounded by the curve $y = 4\cos x$ and the *x*-axis, $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, about the *x*-axis. (Express the answer in exact form.)

Solution:

$8\pi^2$

Exercise:

Problem:

Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the *x*-axis, from $x = 0$ to $x = \ln(7)$, about the *y*-axis. (Express the answer in exact form.)

Glossary

integration by parts
: a technique of integration that allows the exchange of one integral for another using the formula $\int u\,dv = uv - \int v\,du$

Trigonometric Integrals

- Solve integration problems involving products and powers of $\sin x$ and $\cos x$.
- Solve integration problems involving products and powers of $\tan x$ and $\sec x$.
- Use reduction formulas to solve trigonometric integrals.

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called **trigonometric integrals**. They are an important part of the integration technique called *trigonometric substitution*, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating Products and Powers of sin*x* and cos*x*

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j x \cos x \, dx$ or $\int \cos^j x \sin x \, dx$. After rewriting these integrals, we evaluate them using u-substitution. Before describing the general process in detail, let's take a look at the following examples.

Example:
Exercise:

Problem:

Integrating $\int \cos^j x \sin x \, dx$

Evaluate $\int \cos^3 x \sin x \, dx$.

Solution:

Use u-substitution and let $u = \cos x$. In this case, $du = -\sin x \, dx$. Thus,
Equation:

$$\begin{aligned} \int \cos^3 x \sin x \, dx &= -\int u^3 \, du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C. \end{aligned}$$

Note:
Exercise:

Problem: Evaluate $\int \sin^4 x \cos x \, dx$.

Solution:

$\frac{1}{5}\sin^5 x + C$

Hint

Let $u = \sin x$.

Example:
Exercise:

Problem:

A Preliminary Example: Integrating $\int \cos^j x \sin^k x\, dx$ Where *k* is Odd

Evaluate $\int \cos^2 x \sin^3 x\, dx$.

Solution:

To convert this integral to integrals of the form $\int \cos^j x \sin x\, dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$. Thus,

Equation:

$$\begin{aligned}
\int \cos^2 x \sin^3 x\, dx &= \int \cos^2 x (1 - \cos^2 x) \sin x\, dx \quad \text{Let } u = \cos x; \text{ then } du = -\sin x\, dx. \\
&= -\int u^2 \left(1 - u^2\right) du \\
&= \int \left(u^4 - u^2\right) du \\
&= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C \\
&= \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C.
\end{aligned}$$

Note:
Exercise:

Problem: Evaluate $\int \cos^3 x \sin^2 x\, dx$.

Solution:

$\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

Hint

Write $\cos^3 x = \cos^2 x \cos x = \left(1 - \sin^2 x\right)\cos x$ and let $u = \sin x$.

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$. For integrals of this type, the identities

Equation:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1-\cos(2x)}{2}$$

and

Equation:

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1+\cos(2x)}{2}$$

are invaluable. These identities are sometimes known as *power-reducing identities* and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

Example:

Exercise:

Problem:

Integrating an Even Power of $\sin x$

Evaluate $\int \sin^2 x\, dx$.

Solution:

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

Equation:

$$\begin{aligned} \int \sin^2 x\, dx &= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right) dx \\ &= \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) + C. \end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\int \cos^2 x\, dx$.

Solution:

$\frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

Hint

Equation:

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

Note:
Problem-Solving Strategy: Integrating Products and Powers of sin *x* and cos *x*

To integrate $\int \cos^j x \sin^k x \, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x \, dx$.
2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x \, dx$. (*Note*: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Example:
Exercise:

Problem:

Integrating $\int \cos^j x \sin^k x \, dx$ where *k* is Odd

Evaluate $\int \cos^8 x \sin^5 x \, dx$.

Solution:

Since the power on $\sin x$ is odd, use strategy 1. Thus,

Equation:

$$\begin{aligned}
\int \cos^8 x \sin^5 x \, dx &= \int \cos^8 x \sin^4 x \sin x \, dx && \text{Break off } \sin x. \\
&= \int \cos^8 x (\sin^2 x)^2 \sin x \, dx && \text{Rewrite } \sin^4 x = (\sin^2 x)^2. \\
&= \int \cos^8 x (1-\cos^2 x)^2 \sin x \, dx && \text{Substitute } \sin^2 x = 1-\cos^2 x. \\
&= \int u^8 (1-u^2)^2 (-du) && \text{Let } u = \cos x \text{ and } du = -\sin x \, dx. \\
&= \int \left(-u^8 + 2u^{10} - u^{12}\right) du && \text{Expand.} \\
&= -\tfrac{1}{9}u^9 + \tfrac{2}{11}u^{11} - \tfrac{1}{13}u^{13} + C && \text{Evaluate the integral.} \\
&= -\tfrac{1}{9}\cos^9 x + \tfrac{2}{11}\cos^{11} x - \tfrac{1}{13}\cos^{13} x + C. && \text{Substitute } u = \cos x.
\end{aligned}$$

Example:
Exercise:

Problem:

Integrating $\int \cos^j x \sin^k x \, dx$ where *k* and *j* are Even

Evaluate $\int \sin^4 x \, dx$.

Solution:

Since the power on $\sin x$ is even $(k = 4)$ and the power on $\cos x$ is even $(j = 0)$, we must use strategy 3. Thus,

Equation:

$$\begin{aligned}
\int \sin^4 x \, dx &= \int \left(\sin^2 x\right)^2 dx && \text{Rewrite } \sin^4 x = \left(\sin^2 x\right)^2. \\
&= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)^2 dx && \text{Substitute } \sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x). \\
&= \int \left(\tfrac{1}{4} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{4}\cos^2(2x)\right) dx && \text{Expand } \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)^2. \\
&= \int \left(\tfrac{1}{4} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right)\right) dx.
\end{aligned}$$

Since $\cos^2(2x)$ has an even power, substitute $\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$:

Equation:

$$\begin{aligned}
&= \int \left(\tfrac{3}{8} - \tfrac{1}{2}\cos(2x) + \tfrac{1}{8}\cos(4x)\right) dx && \text{Simplify.} \\
&= \tfrac{3}{8}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x) + C && \text{Evaluate the integral.}
\end{aligned}$$

Note:
Exercise:

Problem: Evaluate $\int \cos^3 x\,dx$.

Solution:

$\sin x - \frac{1}{3}\sin^3 x + C$

Hint

Use strategy 2. Write $\cos^3 x = \cos^2 x \cos x$ and substitute $\cos^2 x = 1 - \sin^2 x$.

Note:
Exercise:

Problem: Evaluate $\int \cos^2(3x)dx$.

Solution:

$\frac{1}{2}x + \frac{1}{12}\sin(6x) + C$

Hint

Use strategy 3. Substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$. These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

Note:
Rule: Integrating Products of Sines and Cosines of Different Angles
To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions
Equation:

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

Equation:

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

Equation:

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

Example:
Exercise:

Problem:

Evaluating $\int \sin(ax)\cos(bx)dx$

Evaluate $\int \sin(5x)\cos(3x)dx$.

Solution:

Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(8x)$. Thus,

Equation:

$$\begin{aligned} \int \sin(5x)\cos(3x)dx &= \int \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(8x)dx \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{16}\sin(8x) + C. \end{aligned}$$

Note:
Exercise:

Problem: Evaluate $\int \cos(6x)\cos(5x)dx$.

Solution:

$\frac{1}{2}\sin x + \frac{1}{22}\sin(11x) + C$

Hint

Substitute $\cos(6x)\cos(5x) = \frac{1}{2}\cos x + \frac{1}{2}\cos(11x)$.

Integrating Products and Powers of tan*x* and sec*x*

Before discussing the integration of products and powers of $\tan x$ and $\sec x$, it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\int \sec^2 x\,dx = \tan x + C$
2. $\int \sec x \tan x\,dx = \sec x + C$

3. $\int \tan x \, dx = \ln|\sec x| + C$
4. $\int \sec x \, dx = \ln|\sec x + \tan x| + C.$

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x \, dx$ or $\int \sec^j x \tan x \, dx$. As we see in the following example, we can evaluate these new integrals by using *u*-substitution.

Example:
Exercise:

Problem:

Evaluating $\int \sec^j x \tan x \, dx$

Evaluate $\int \sec^5 x \tan x \, dx$.

Solution:

Start by rewriting $\sec^5 x \tan x$ as $\sec^4 x \sec x \tan x$.
Equation:

$$\begin{aligned} \int \sec^5 x \tan x \, dx &= \int \sec^4 x \sec x \tan x \, dx && \text{Let } u = \sec x;\ \text{then, } du = \sec x \tan x \, dx. \\ &= \int u^4 du && \text{Evaluate the integral.} \\ &= \tfrac{1}{5}u^5 + C && \text{Substitute } \sec x = u. \\ &= \tfrac{1}{5}\sec^5 x + C \end{aligned}$$

Note:
You can read some interesting information at this website to learn about a common integral involving the secant.

Note:
Exercise:

Problem: Evaluate $\int \tan^5 x \sec^2 x \, dx$.

Solution:

$\tfrac{1}{6}\tan^6 x + C$

Hint

Let $u = \tan x$ and $du = \sec^2 x$.

We now take a look at the various strategies for integrating products and powers of $\sec x$ and $\tan x$.

Note:

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x \, dx$

To integrate $\int \tan^k x \sec^j x \, dx$, use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$.
2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x \, dx$. (*Note*: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x (\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.
4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

Example:
Exercise:

Problem:

Integrating $\int \tan^k x \sec^j x \, dx$ when j is Even

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Solution:

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,

Equation:

$$\begin{aligned} \int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x \left(\tan^2 x + 1\right) \sec^2 x\, dx && \text{Let } u = \tan x \text{ and } du = \sec^2 x. \\ &= \int u^6 \left(u^2 + 1\right) du && \text{Expand.} \\ &= \int (u^8 + u^6) du && \text{Evaluate the integral.} \\ &= \tfrac{1}{9} u^9 + \tfrac{1}{7} u^7 + C && \text{Substitute } \tan x = u. \\ &= \tfrac{1}{9} \tan^9 x + \tfrac{1}{7} \tan^7 x + C. \end{aligned}$$

Example:
Exercise:

Problem:

Integrating $\int \tan^k x \sec^j x\, dx$ when k is Odd

Evaluate $\int \tan^5 x \sec^3 x\, dx$.

Solution:

Since the power on $\tan x$ is odd, begin by rewriting $\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$. Thus,
Equation:

$$\begin{aligned} \tan^5 x \sec^3 x &= \tan^4 x \sec^2 x \sec x \tan x. && \text{Write } \tan^4 x = \left(\tan^2 x\right)^2. \\ \int \tan^5 x \sec^3 x\, dx &= \int \left(\tan^2 x\right)^2 \sec^2 x \sec x \tan x\, dx && \text{Use } \tan^2 x = \sec^2 x - 1. \\ &= \int \left(\sec^2 x - 1\right)^2 \sec^2 x \sec x \tan x\, dx && \text{Let } u = \sec x \text{ and } du = \sec x \tan x\, dx. \\ &= \int \left(u^2 - 1\right)^2 u^2 du && \text{Expand.} \\ &= \int \left(u^6 - 2u^4 + u^2\right) du && \text{Integrate.} \\ &= \tfrac{1}{7} u^7 - \tfrac{2}{5} u^5 + \tfrac{1}{3} u^3 + C && \text{Substitute } \sec x = u. \\ &= \tfrac{1}{7} \sec^7 x - \tfrac{2}{5} \sec^5 x + \tfrac{1}{3} \sec^3 x + C. \end{aligned}$$

Example:
Exercise:

Problem:

Integrating $\int \tan^k x\, dx$ where k is Odd and $k \geq 3$

Evaluate $\int \tan^3 x\,dx$.

Solution:

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x\left(\sec^2 x - 1\right) = \tan x \sec^2 x - \tan x$. Thus,
Equation:

$$\begin{aligned}\int \tan^3 x\,dx &= \int \left(\tan x \sec^2 x - \tan x\right) dx \\ &= \int \tan x \sec^2 x\,dx - \int \tan x\,dx \\ &= \tfrac{1}{2}\tan^2 x - \ln|\sec x| + C.\end{aligned}$$

For the first integral, use the substitution $u = \tan x$. For the second integral, use the formula.

Example:
Exercise:

Problem:
Integrating $\int \sec^3 x\,dx$

Integrate $\int \sec^3 x\,dx$.

Solution:

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,
Equation:

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx && \\ &= \sec x \tan x - \int \tan^2 x \sec x\,dx && \text{Simplify.} \\ &= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\,dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\ &= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx && \text{Rewrite.} \\ &= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx. && \text{Evaluate } \int \sec x\,dx.\end{aligned}$$

We now have
Equation:

$$\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx.$$

Since the integral $\int \sec^3 x\,dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x\,dx$ by adding it to both sides. In doing so, we obtain

Equation:

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Dividing by 2, we arrive at

Equation:

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Note:

Exercise:

Problem: Evaluate $\int \tan^3 x \sec^7 x\,dx.$

Solution:

$\frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C$

Hint

Use [link] as a guide.

Reduction Formulas

Evaluating $\int \sec^n x\,dx$ for values of n where n is odd requires integration by parts. In addition, we must also know the value of $\int \sec^{n-2} x\,dx$ to evaluate $\int \sec^n x\,dx$. The evaluation of $\int \tan^n x\,dx$ also requires being able to integrate $\int \tan^{n-2} x\,dx$. To make the process easier, we can derive and apply the following **power reduction formulas**. These rules allow us to replace the integral of a power of $\sec x$ or $\tan x$ with the integral of a lower power of $\sec x$ or $\tan x$.

Note:

Rule: Reduction Formulas for $\int \sec^n x\,dx$ and $\int \tan^n x\,dx$

Equation:

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

Equation:

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of $\tan x$.

Example:
Exercise:

Problem:

Revisiting $\int \sec^3 x\,dx$

Apply a reduction formula to evaluate $\int \sec^3 x\,dx$.

Solution:

By applying the first reduction formula, we obtain
Equation:

$$\begin{aligned}\int \sec^3 x\,dx &= \tfrac{1}{2}\sec x\tan x + \tfrac{1}{2}\int \sec x\,dx \\ &= \tfrac{1}{2}\sec x\tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

Example:
Exercise:

Problem:
Using a Reduction Formula

Evaluate $\int \tan^4 x\,dx$.

Solution:

Applying the reduction formula for $\int \tan^4 x\,dx$ we have

Equation:

$$\begin{aligned} \int \tan^4 x\,dx &= \tfrac{1}{3}\tan^3 x - \int \tan^2 x\,dx \\ &= \tfrac{1}{3}\tan^3 x - \left(\tan x - \int \tan^0 x\,dx\right) && \text{Apply the reduction formula to } \int \tan^2 x\,dx. \\ &= \tfrac{1}{3}\tan^3 x - \tan x + \int 1\,dx && \text{Simplify.} \\ &= \tfrac{1}{3}\tan^3 x - \tan x + x + C. && \text{Evaluate } \int 1dx. \end{aligned}$$

Note:
Exercise:

Problem: Apply the reduction formula to $\int \sec^5 x\,dx$.

Solution:

$$\int \sec^5 x\,dx = \tfrac{1}{4}\sec^3 x\tan x - \tfrac{3}{4}\int \sec^3 x$$

Hint

Use reduction formula 1 and let $n = 5$.

Key Concepts

- Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include
  1. Applying trigonometric identities to rewrite the integral so that it may be evaluated by u-substitution
  2. Using integration by parts
  3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
  4. Applying reduction formulas

Key Equations

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions.

- **Sine Products**
  $\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$
- **Sine and Cosine Products**
  $\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$
- **Cosine Products**
  $\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$

- **Power Reduction Formula**
$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-1} x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$
- **Power Reduction Formula**
$\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx$

Fill in the blank to make a true statement.
Exercise:

Problem: $\sin^2 x +$ _______ $= 1$

Solution:

$\cos^2 x$

Exercise:

Problem: $\sec^2 x - 1 =$_______

Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.
Exercise:

Problem: $\sin^2 x =$_______

Solution:

$\frac{1-\cos(2x)}{2}$

Exercise:

Problem: $\cos^2 x =$_______

Evaluate each of the following integrals by *u*-substitution.
Exercise:

Problem: $\int \sin^3 x \cos x\, dx$

Solution:

$\frac{\sin^4 x}{4} + C$

Exercise:

Problem: $\int \sqrt{\cos x} \sin x\, dx$

Exercise:

Problem: $\int \tan^5(2x)\sec^2(2x) dx$

Solution:

$\frac{1}{12}\tan^6(2x) + C$

Exercise:

Problem: $\int \sin^7(2x)\cos(2x)dx$

Exercise:

Problem: $\int \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$

Solution:

$\sec^2\left(\frac{x}{2}\right) + C$

Exercise:

Problem: $\int \tan^2 x \sec^2 x \, dx$

Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note*: Some of the problems may be done using techniques of integration learned previously.)

Exercise:

Problem: $\int \sin^3 x \, dx$

Solution:

$-\frac{3\cos x}{4} + \frac{1}{12}\cos(3x) + C = -\cos x + \frac{\cos^3 x}{3} + C$

Exercise:

Problem: $\int \cos^3 x \, dx$

Exercise:

Problem: $\int \sin x \cos x \, dx$

Solution:

$-\frac{1}{2}\cos^2 x + C$

Exercise:

Problem: $\int \cos^5 x \, dx$

Exercise:

Problem: $\int \sin^5 x \cos^2 x \, dx$

Solution:

$-\frac{5\cos x}{64} - \frac{1}{192}\cos(3x) + \frac{3}{320}\cos(5x) - \frac{1}{448}\cos(7x) + C$

Exercise:

Problem: $\int \sin^3 x \cos^3 x \, dx$

Exercise:

Problem: $\int \sqrt{\sin x} \cos x \, dx$

Solution:

$\frac{2}{3}(\sin x)^{2/3} + C$

Exercise:

Problem: $\int \sqrt{\sin x} \cos^3 x \, dx$

Exercise:

Problem: $\int \sec x \tan x \, dx$

Solution:

$\sec x + C$

Exercise:

Problem: $\int \tan(5x) dx$

Exercise:

Problem: $\int \tan^2 x \sec x \, dx$

Solution:

$\frac{1}{2}\sec x \tan x - \frac{1}{2}\ln(\sec x + \tan x) + C$

Exercise:

Problem: $\int \tan x \sec^3 x \, dx$

Exercise:

Problem: $\int \sec^4 x \, dx$

Solution:

$\frac{2\tan x}{3} + \frac{1}{3}\sec(x)^2 \tan x = \tan x + \frac{\tan^3 x}{3} + C$

Exercise:

Problem: $\int \cot x \, dx$

Exercise:

Problem: $\int \csc x \, dx$

Solution:

$-\ln|\cot x + \csc x| + C$

Exercise:

Problem: $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$

For the following exercises, find a general formula for the integrals.

Exercise:

Problem: $\int \sin^2 ax \cos ax \, dx$

Solution:

$\frac{\sin^3(ax)}{3a} + C$

Exercise:

Problem: $\int \sin ax \cos ax \, dx.$

Use the double-angle formulas to evaluate the following integrals.

Exercise:

Problem: $\int_0^{\pi} \sin^2 x \, dx$

Solution:

$\frac{\pi}{2}$

Exercise:

Problem: $\int_0^{\pi} \sin^4 x \, dx$

Exercise:

Problem: $\int \cos^2 3x \, dx$

Solution:

$\frac{x}{2} + \frac{1}{12}\sin(6x) + C$

Exercise:

Problem: $\int \sin^2 x \cos^2 x \, dx$

Exercise:

Problem: $\int \sin^2 x \, dx + \int \cos^2 x \, dx$

Solution:

$x + C$

Exercise:

Problem: $\int \sin^2 x \cos^2(2x) dx$

For the following exercises, evaluate the definite integrals. Express answers in exact form whenever possible.

Exercise:

Problem: $\int_0^{2\pi} \cos x \sin 2x \, dx$

Solution:

0

Exercise:

Problem: $\int_0^{\pi} \sin 3x \sin 5x \, dx$

Exercise:

Problem: $\int_0^{\pi} \cos(99x)\sin(101x)dx$

Solution:

0

Exercise:

Problem: $\int_{-\pi}^{\pi} \cos^2(3x)dx$

Exercise:

Problem: $\int_0^{2\pi} \sin x \sin(2x)\sin(3x)dx$

Solution:

0

Exercise:

Problem: $\int_0^{4\pi} \cos(x/2)\sin(x/2)dx$

Exercise:

Problem: $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}dx$ (Round this answer to three decimal places.)

Solution:

Approximately 0.239

Exercise:

Problem: $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1}\,dx$

Exercise:

Problem: $\int_0^{\pi/2} \sqrt{1 - \cos(2x)}\,dx$

Solution:

$\sqrt{2}$

Exercise:

Problem:

Find the area of the region bounded by the graphs of the equations $y = \sin x, y = \sin^3 x, x = 0, \text{and } x = \frac{\pi}{2}$.

Exercise:

Problem:

Find the area of the region bounded by the graphs of the equations $y = \cos^2 x, y = \sin^2 x, x = -\frac{\pi}{4}, \text{and } x = \frac{\pi}{4}$.

Solution:

1.0

Exercise:

Problem:

A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t)\cos^2(\omega t)$. Find its position function $x = f(t)$ if $f(0) = 0$.

Exercise:

Problem: Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$.

Solution:

0

For the following exercises, solve the differential equations.

Exercise:

Problem: $\frac{dy}{dx} = \sin^2 x$. The curve passes through point $(0, 0)$.

Exercise:

Problem: $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

Solution:

$\frac{3\theta}{8} - \frac{1}{4\pi}\sin(2\pi\theta) + \frac{1}{32\pi}\sin(4\pi\theta) + C = f(x)$

Exercise:

Problem: Find the length of the curve $y = \ln(\csc x), \frac{\pi}{4} \le x \le \frac{\pi}{2}$.

Exercise:

Problem: Find the length of the curve $y = \ln(\sin x), \frac{\pi}{3} \le x \le \frac{\pi}{2}$.

Solution:

$\ln\left(\sqrt{3}\right)$

Exercise:

Problem: Find the volume generated by revolving the curve $y = \cos(3x)$ about the *x*-axis, $0 \leq x \leq \frac{\pi}{36}$.

For the following exercises, use this information: The inner product of two functions *f* and *g* over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g \rangle = \int_a^b f \cdot g dx$. Two distinct functions *f* and *g* are said to be orthogonal if $\langle f, g \rangle = 0$.

Exercise:

Problem: Show that $\{\sin(2x), \cos(3x)\}$ are orthogonal over the interval $[-\pi, \pi]$.

Solution:

$$\int_{-\pi}^{\pi} \sin(2x)\cos(3x)dx = 0$$

Exercise:

Problem: Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx$.

Exercise:

Problem: Integrate $y' = \sqrt{\tan x}\sec^4 x$.

Solution:

$$\sqrt{\tan(x)}x\left(\frac{8\tan x}{21} + \frac{2}{7}\sec x^2 \tan x\right) + C = f(x)$$

For each pair of integrals, determine which one is more difficult to evaluate. Explain your reasoning.

Exercise:

Problem: $\int \sin^{456} x \cos x \, dx$ or $\int \sin^2 x \cos^2 x \, dx$

Exercise:

Problem: $\int \tan^{350} x \sec^2 x \, dx$ or $\int \tan^{350} x \sec x \, dx$

Solution:

The second integral is more difficult because the first integral is simply a *u*-substitution type.

Glossary

power reduction formula
: a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

trigonometric integral

an integral involving powers and products of trigonometric functions

Trigonometric Substitution

- Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of **trigonometric substitution** comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

Integrals Involving $\sqrt{a^2 - x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2 - x^2}$, consider the integral $\int \sqrt{9 - x^2} dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x = 3\sin\theta$, we have $dx = 3\cos\theta d\theta$. After substituting into the integral, we have

Equation:

$$\int \sqrt{9 - x^2} dx = \int \sqrt{9 - (3\sin\theta)^2} 3\cos\theta d\theta.$$

After simplifying, we have

Equation:

$$\int \sqrt{9 - x^2} dx = \int 9\sqrt{1 - \sin^2\theta} \cos\theta d\theta.$$

Letting $1 - \sin^2\theta = \cos^2\theta$, we now have

Equation:

$$\int \sqrt{9 - x^2} dx = \int 9\sqrt{\cos^2\theta} \cos\theta d\theta.$$

Assuming that $\cos\theta \geq 0$, we have

Equation:

$$\int \sqrt{9 - x^2} dx = \int 9\cos^2\theta d\theta.$$

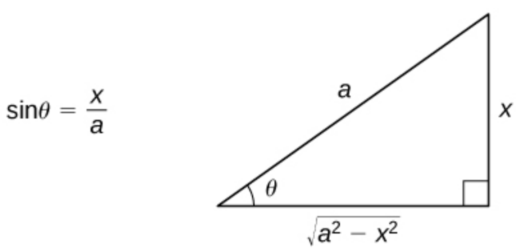
At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let’s take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2 - x^2}$, we make the substitution $x = a\sin\theta$ and $dx = a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2 - x^2}$ is $[-a, a]$. Thus, $-a \leq x \leq a$. Consequently, $-1 \leq \frac{x}{a} \leq 1$. Since the range of $\sin x$ over $[-(\pi/2), \pi/2]$ is $[-1, 1]$, there is a unique angle θ satisfying $-(\pi/2) \leq \theta \leq \pi/2$ so that $\sin\theta = x/a$, or equivalently, so that $x = a\sin\theta$. If we substitute $x = a\sin\theta$ into $\sqrt{a^2 - x^2}$, we get

Equation:

$$
\begin{aligned}
\sqrt{a^2-x^2} &= \sqrt{a^2-(a\sin\theta)^2} && \text{Let } x = a\sin\theta \text{ where } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\text{. Simplify.}\\
&= \sqrt{a^2-a^2\sin^2\theta} && \text{Factor out } a^2.\\
&= \sqrt{a^2(1-\sin^2\theta)} && \text{Substitute } 1-\sin^2 x = \cos^2 x.\\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.}\\
&= |a\cos\theta| \\
&= a\cos\theta.
\end{aligned}
$$

Since $\cos x \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $a > 0$, $|a\cos\theta| = a\cos\theta$. We can see, from this discussion, that by making the substitution $x = a\sin\theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving x. To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin\theta = \frac{x}{a}$, we can draw the reference triangle in [link] to assist in expressing the values of $\cos\theta$, $\tan\theta$, and the remaining trigonometric functions in terms of x. It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at θ for all θ satisfying $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2-x^2}$ actually appears as the length of one side of the triangle. Last, should θ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.



A reference triangle can help express the trigonometric functions evaluated at θ in terms of x.

The essential part of this discussion is summarized in the following problem-solving strategy.

Note:
Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2-x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2-x^2}}dx$, $\int \frac{x}{\sqrt{a^2-x^2}}dx$, and $\int x\sqrt{a^2-x^2}\,dx$, they can each be integrated directly either by formula or by a simple u-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta d\theta$. *Note*: This substitution yields $\sqrt{a^2-x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from [link] to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

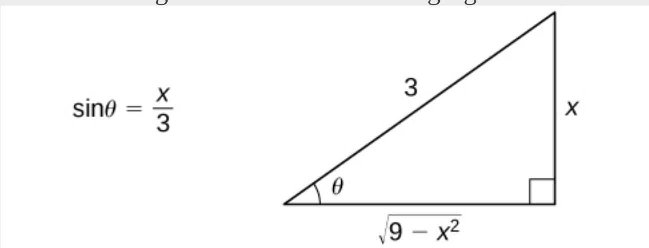
Example:
Exercise:

Problem:
Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate $\int \sqrt{9 - x^2}\, dx.$

Solution:

Begin by making the substitutions $x = 3\sin\theta$ and $dx = 3\cos\theta d\theta$. Since $\sin\theta = \frac{x}{3}$, we can construct the reference triangle shown in the following figure.



A reference triangle can be constructed for [link].

Thus,
Equation:

$$\begin{aligned}
\int \sqrt{9-x^2}\,dx &= \int \sqrt{9-(3\sin\theta)^2}\,3\cos\theta d\theta && \text{Substitute } x = 3\sin\theta \text{ and } dx = 3\cos\theta d\theta. \\
&= \int \sqrt{9(1-\sin^2\theta)}\,3\cos\theta d\theta && \text{Simplify.} \\
&= \int \sqrt{9\cos^2\theta}\,3\cos\theta d\theta && \text{Substitute } \cos^2\theta = 1-\sin^2\theta. \\
&= \int 3\,|\cos\theta|\,3\cos\theta d\theta && \text{Take the square root.} \\
&= \int 9\cos^2\theta d\theta && \text{Simplify. Since } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2},\ \cos\theta \ge 0 \text{ and } |\cos\theta| = \cos\theta. \\
&= \int 9\left(\tfrac{1}{2}+\tfrac{1}{2}\cos(2\theta)\right)d\theta && \text{Use the strategy for integrating an even pow of } \cos\theta. \\
&= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin(2\theta) + C && \text{Evaluate the integral.} \\
&= \tfrac{9}{2}\theta + \tfrac{9}{4}(2\sin\theta\cos\theta) + C && \text{Substitute } \sin(2\theta) = 2\sin\theta\cos\theta. \\
&= \tfrac{9}{2}\sin^{-1}\left(\tfrac{x}{3}\right) + \tfrac{9}{2}\cdot\tfrac{x}{3}\cdot\tfrac{\sqrt{9-x^2}}{3} + C && \text{Substitute } \sin^{-1}\left(\tfrac{x}{3}\right) = \theta \text{ and } \sin\theta = \tfrac{x}{3}. \text{ Use the reference triangle to see that } \cos\theta = \tfrac{\sqrt{9-x^2}}{3} \text{ and make this substitution.} \\
&= \tfrac{9}{2}\sin^{-1}\left(\tfrac{x}{3}\right) + \tfrac{x\sqrt{9-x^2}}{2} + C. && \text{Simplify.}
\end{aligned}$$

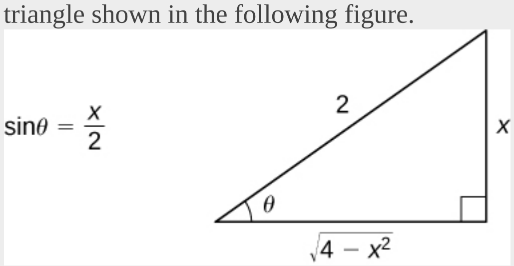
Example:
Exercise:

Problem:
Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate $\int \frac{\sqrt{4-x^2}}{x}dx$.

Solution:

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta d\theta$. Since $\sin\theta = \frac{x}{2}$, we can construct the reference triangle shown in the following figure.



A reference triangle can be constructed for [link].

Thus,
Equation:

$$\begin{aligned}
\int \frac{\sqrt{4-x^2}}{x}dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta}2\cos\theta d\theta && \text{Substitute } x = 2\sin\theta \text{ and } = 2\cos\theta d\theta. \\
&= \int \frac{2\cos^2\theta}{\sin\theta}d\theta && \text{Substitute } \cos^2\theta = 1-\sin^2\theta \text{ and simplify.} \\
&= \int \frac{2(1-\sin^2\theta)}{\sin\theta}d\theta && \text{Substitute } \sin^2\theta = 1-\cos^2\theta. \\
&= \int (2\csc\theta - 2\sin\theta)d\theta && \text{Separate the numerator, simplify, and use } \csc\theta = \tfrac{1}{\sin\theta}. \\
&= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C && \text{Evaluate the integral.} \\
&= 2\ln\left|\tfrac{2}{x} - \tfrac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C. && \text{Use the reference triangle to rewrite the expression in terms of } x \text{ and simplify.}
\end{aligned}$$

In the next example, we see that we sometimes have a choice of methods.

Example:
Exercise:

Problem:
Integrating an Expression Involving $\sqrt{a^2-x^2}$ Two Ways

Evaluate $\int x^3\sqrt{1-x^2}\,dx$ two ways: first by using the substitution $u = 1-x^2$ and then by using a trigonometric substitution.

Solution:

Method 1

Let $u = 1-x^2$ and hence $x^2 = 1-u$. Thus, $du = -2x\,dx$. In this case, the integral becomes
Equation:

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= -\tfrac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) && \text{Make the substitution.} \\
&= -\tfrac{1}{2}\int (1-u)\sqrt{u}\,du && \text{Expand the expression.} \\
&= -\tfrac{1}{2}\int \left(u^{1/2} - u^{3/2}\right)du && \text{Evaluate the integral.} \\
&= -\tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right) + C && \text{Rewrite in terms of } x. \\
&= -\tfrac{1}{3}\left(1-x^2\right)^{3/2} + \tfrac{1}{5}\left(1-x^2\right)^{5/2} + C.
\end{aligned}$$

Method 2

Let $x = \sin\theta$. In this case, $dx = \cos\theta d\theta$. Using this substitution, we have

Equation:

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3\theta\cos^2\theta d\theta \\
&= \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta d\theta && \text{Let } u=\cos\theta.\ \text{Thus, } du=-\sin\theta d\theta. \\
&= \int \left(u^4-u^2\right)du \\
&= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C && \text{Substitute } \cos\theta = u. \\
&= \tfrac{1}{5}\cos^5\theta - \tfrac{1}{3}\cos^3\theta + C && \text{Use a reference triangle to see that } \cos\theta = \sqrt{1-x^2}. \\
&= \tfrac{1}{5}\left(1-x^2\right)^{5/2} - \tfrac{1}{3}\left(1-x^2\right)^{3/2} + C.
\end{aligned}$$

Note:
Exercise:

Problem:

Rewrite the integral $\int \frac{x^3}{\sqrt{25-x^2}}dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

Solution:

$$\int 125\sin^3\theta d\theta$$

Hint

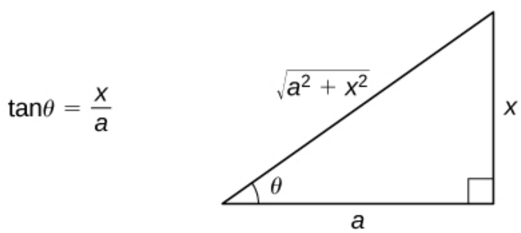
Substitute $x = 5\sin\theta$ and $dx = 5\cos\theta d\theta$.

Integrating Expressions Involving $\sqrt{a^2+x^2}$

For integrals containing $\sqrt{a^2+x^2}$, let’s first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of x, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a\tan\theta$ or $x = a\cot\theta$. Either of these substitutions would actually work, but the standard substitution is $x = a\tan\theta$ or, equivalently, $\tan\theta = x/a$. With this substitution, we make the assumption that $-(\pi/2) < \theta < \pi/2$, so that we also have $\theta = \tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Note:
Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = |a\sec\theta| = a\sec\theta$. (Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from [link] to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (*Note*: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \le 0$.)



A reference triangle can be constructed to express the trigonometric functions evaluated at θ in terms of x.

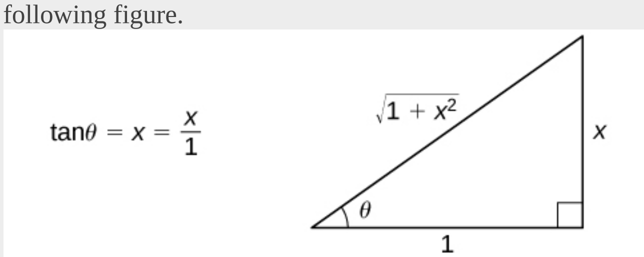
Example:
Exercise:

Problem:
Integrating an Expression Involving $\sqrt{a^2+x^2}$

Evaluate $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

Solution:

Begin with the substitution $x = \tan\theta$ and $dx = \sec^2\theta d\theta$. Since $\tan\theta = x$, draw the reference triangle in the following figure.



The reference triangle for [link].

Thus,

Equation:

$$\begin{aligned} \int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\sec^2\theta}{\sec\theta} d\theta && \text{Substitute } x = \tan\theta \text{ and } dx = \sec^2\theta d\theta. \text{ This substitution makes } \sqrt{1+x^2} = \sec\theta. \text{ Simplify.} \\ &= \int \sec\theta d\theta && \text{Evaluate the integral.} \\ &= \ln|\sec\theta + \tan\theta| + C && \text{Use the reference triangle to express the result in terms of } x. \\ &= \ln\left|\sqrt{1+x^2} + x\right| + C. \end{aligned}$$

To check the solution, differentiate:

Equation:

$$\begin{aligned} \frac{d}{dx}\left(\ln\left|\sqrt{1+x^2} + x\right|\right) &= \frac{1}{\sqrt{1+x^2}+x} \cdot \left(\frac{x}{\sqrt{1+x^2}} + 1\right) \\ &= \frac{1}{\sqrt{1+x^2}+x} \cdot \frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} \\ &= \frac{1}{\sqrt{1+x^2}}. \end{aligned}$$

Since $\sqrt{1+x^2} + x > 0$ for all values of x, we could rewrite $\ln\left|\sqrt{1+x^2} + x\right| + C = \ln\left(\sqrt{1+x^2} + x\right) + C$, if desired.

Example:

Exercise:

Problem:

Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x = \sinh\theta$ to evaluate $\int \frac{dx}{\sqrt{1+x^2}}$.

Solution:

Because $\sinh\theta$ has a range of all real numbers, and $1 + \sinh^2\theta = \cosh^2\theta$, we may also use the substitution $x = \sinh\theta$ to evaluate this integral. In this case, $dx = \cosh\theta d\theta$. Consequently,

Equation:

$$
\begin{aligned}
\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}} d\theta && \text{Substitute } x = \sinh\theta \text{ and } dx = \cosh\theta d\theta. \\
& && \text{Substitute } 1+\sinh^2\theta = \cosh^2\theta. \\
&= \int \frac{\cosh\theta}{\sqrt{\cosh^2\theta}} d\theta && \sqrt{\cosh^2\theta} = |\cosh\theta| \\
&= \int \frac{\cosh\theta}{|\cosh\theta|} d\theta && |\cosh\theta| = \cosh\theta \text{ since } \cosh\theta > 0 \text{ for all } \theta. \\
&= \int \frac{\cosh\theta}{\cosh\theta} d\theta && \text{Simplify.} \\
&= \int 1 d\theta && \text{Evaluate the integral.} \\
&= \theta + C && \text{Since } x = \sinh\theta, \text{ we know } \theta = \sinh^{-1}x. \\
&= \sinh^{-1}x + C.
\end{aligned}
$$

Analysis

This answer looks quite different from the answer obtained using the substitution $x = \tan\theta$. To see that the solutions are the same, set $y = \sinh^{-1}x$. Thus, $\sinh y = x$. From this equation we obtain:
Equation:

$$\frac{e^y - e^{-y}}{2} = x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:
Equation:

$$e^{2y} - 2xe^y - 1 = 0.$$

Use the quadratic equation to solve for e^y:
Equation:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}.$$

Simplifying, we have:
Equation:

$$e^y = x \pm \sqrt{x^2+1}.$$

Since $x - \sqrt{x^2+1} < 0$, it must be the case that $e^y = x + \sqrt{x^2+1}$. Thus,
Equation:

$$y = \ln\left(x + \sqrt{x^2+1}\right).$$

Last, we obtain
Equation:

$$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right).$$

After we make the final observation that, since $x+\sqrt{x^2+1}>0$,
Equation:

$$\ln\left(x+\sqrt{x^2+1}\right)=\ln\left|\sqrt{1+x^2}+x\right|,$$

we see that the two different methods produced equivalent solutions.

Example:
Exercise:

Problem:
Finding an Arc Length

Find the length of the curve $y=x^2$ over the interval $[0, \frac{1}{2}]$.

Solution:

Because $\frac{dy}{dx}=2x$, the arc length is given by
Equation:

$$\int_0^{1/2}\sqrt{1+(2x)^2}\,dx=\int_0^{1/2}\sqrt{1+4x^2}\,dx.$$

To evaluate this integral, use the substitution $x=\frac{1}{2}\tan\theta$ and $dx=\frac{1}{2}\sec^2\theta d\theta$. We also need to change the limits of integration. If $x=0$, then $\theta=0$ and if $x=\frac{1}{2}$, then $\theta=\frac{\pi}{4}$. Thus,
Equation:

$$\begin{aligned}
\int_0^{1/2}\sqrt{1+4x^2}\,dx &= \int_0^{\pi/4}\sqrt{1+\tan^2\theta}\frac{1}{2}\sec^2\theta d\theta && \text{After substitution, } \sqrt{1+4x^2}=\tan\theta.\text{ Substitute } 1+\tan^2\theta=\sec^2\theta \text{ and simplify.}\\
&= \tfrac{1}{2}\int_0^{\pi/4}\sec^3\theta d\theta && \text{We derived this integral in the previous section.}\\
&= \tfrac{1}{2}\left(\tfrac{1}{2}\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4} && \text{Evaluate and simplify.}\\
&= \tfrac{1}{4}(\sqrt{2}+\ln(\sqrt{2}+1)).
\end{aligned}$$

Note:
Exercise:

Problem: Rewrite $\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan\theta$.

Solution:

$$\int 32\tan^3\theta\sec^3\theta d\theta$$

Hint

Use $x = 2\tan\theta$ and $dx = 2\sec^2\theta d\theta$.

Integrating Expressions Involving $\sqrt{x^2 - a^2}$

The domain of the expression $\sqrt{x^2 - a^2}$ is $(-\infty, -a] \cup [a, +\infty)$. Thus, either $x < -a$ or $x > a$. Hence, $\frac{x}{a} \le -1$ or $\frac{x}{a} \ge 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, it makes sense to use the substitution $\sec\theta = \frac{x}{a}$ or, equivalently, $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$. The corresponding substitution for dx is $dx = a\sec\theta\tan\theta d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

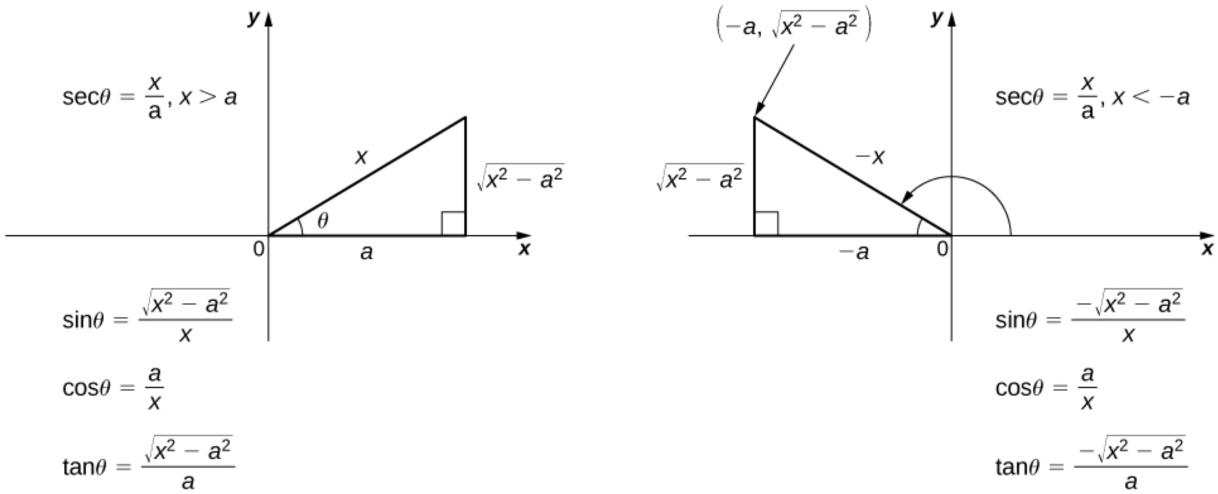
Note:

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$. This substitution yields
   Equation:

   $$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta + 1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|.$$

   For $x \ge a$, $|a\tan\theta| = a\tan\theta$ and for $x \le -a$, $|a\tan\theta| = -a\tan\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from [link] to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (*Note*: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)



Use the appropriate reference triangle to express the trigonometric functions evaluated at θ in terms of x.

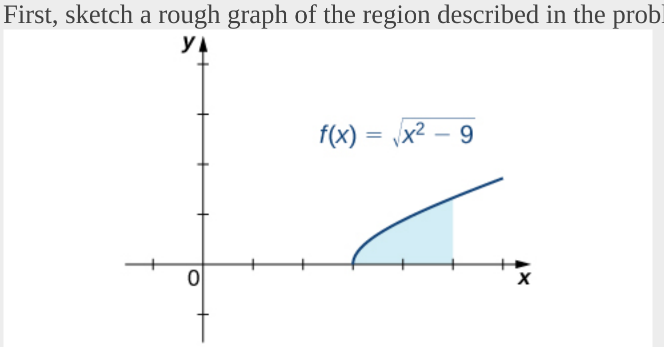
Example:
Exercise:

Problem:
Finding the Area of a Region

Find the area of the region between the graph of $f\left(x\right) = \sqrt{x^2 - 9}$ and the x-axis over the interval $\left[3, 5\right]$.

Solution:

First, sketch a rough graph of the region described in the problem, as shown in the following figure.



Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2 - 9}\, dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}\left(\frac{5}{3}\right)$. After making these substitutions and simplifying, we have

Equation:

$$
\begin{aligned}
\text{Area} &= \int_3^5 \sqrt{x^2-9}\,dx \\
&= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta d\theta && \text{Use } \tan^2\theta = 1-\sec^2\theta. \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta-1)\sec\theta d\theta && \text{Expand.} \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta-\sec\theta) d\theta && \text{Evaluate the integral.} \\
&= \left(\tfrac{9}{2}\ln|\sec\theta+\tan\theta| + \tfrac{9}{2}\sec\theta\tan\theta\right) - 9\ln|\sec\theta+\tan\theta| \Big|_0^{\sec^{-1}(5/3)} && \text{Simplify.} \\
&= \tfrac{9}{2}\sec\theta\tan\theta - \tfrac{9}{2}\ln|\sec\theta+\tan\theta| \Big|_0^{\sec^{-1}(5/3)} && \text{Evaluate. Use } \sec\left(\sec^{-1}\tfrac{5}{3}\right) \text{ and } \tan\left(\sec^{-1}\tfrac{5}{3}\right) = \tfrac{4}{3}. \\
&= \tfrac{9}{2}\cdot\tfrac{5}{3}\cdot\tfrac{4}{3} - \tfrac{9}{2}\ln\left|\tfrac{5}{3}+\tfrac{4}{3}\right| - \left(\tfrac{9}{2}\cdot 1\cdot 0 - \tfrac{9}{2}\ln|1+0|\right) \\
&= 10 - \tfrac{9}{2}\ln 3.
\end{aligned}
$$

Note:
Exercise:

Problem: Evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Assume that $x > 2$.

Solution:

$\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$

Hint

Substitute $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta d\theta$.

Key Concepts

- For integrals involving $\sqrt{a^2-x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta d\theta$.
- For integrals involving $\sqrt{a^2+x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$.
- For integrals involving $\sqrt{x^2-a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$.

Simplify the following expressions by writing each one using a single trigonometric function.
Exercise:

Problem: $4 - 4\sin^2\theta$

Exercise:

Problem: $9\sec^2\theta - 9$

Solution:

$9\tan^2\theta$

Exercise:

Problem: $a^2 + a^2\tan^2\theta$

Exercise:

Problem: $a^2 + a^2\sinh^2\theta$

Solution:

$a^2\cosh^2\theta$

Exercise:

Problem: $16\cosh^2\theta - 16$

Use the technique of completing the square to express each trinomial as the square of a binomial.

Exercise:

Problem: $4x^2 - 4x + 1$

Solution:

$4\left(x - \frac{1}{2}\right)^2$

Exercise:

Problem: $2x^2 - 8x + 3$

Exercise:

Problem: $-x^2 - 2x + 4$

Solution:

$-(x+1)^2 + 5$

Integrate using the method of trigonometric substitution. Express the final answer in terms of the variable.

Exercise:

Problem: $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}}$

Exercise:

Problem: $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$

Solution:

$\ln\left|x + \sqrt{-a^2 + x^2}\right| + C$

Exercise:

Problem: $\int \sqrt{4-x^2}\,dx$

Exercise:

Problem: $\int \frac{dx}{\sqrt{1+9x^2}}$

Solution:

$\frac{1}{3}\ln\left|\sqrt{9x^2+1}+3x\right|+C$

Exercise:

Problem: $\int \frac{x^2dx}{\sqrt{1-x^2}}$

Exercise:

Problem: $\int \frac{dx}{x^2\sqrt{1-x^2}}$

Solution:

$-\frac{\sqrt{1-x^2}}{x}+C$

Exercise:

Problem: $\int \frac{dx}{(1+x^2)^2}$

Exercise:

Problem: $\int \sqrt{x^2+9}dx$

Solution:

$9\left[\frac{x\sqrt{x^2+9}}{18}+\frac{1}{2}ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|\right]+C$

Exercise:

Problem: $\int \frac{\sqrt{x^2-25}}{x}dx$

Exercise:

Problem: $\int \frac{\theta^3d\theta}{\sqrt{9-\theta^2}}d\theta$

Solution:

$-\frac{1}{3}\sqrt{9-\theta^2}\left(18+\theta^2\right)+C$

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^6-x^2}}$

Exercise:

Problem: $\int \sqrt{x^6-x^8}dx$

Solution:

$\frac{\left(-1+x^2\right)\left(2+3x^2\right)\sqrt{x^6-x^8}}{15x^3}+C$

Exercise:

Problem: $\int \frac{dx}{\left(1+x^2\right)^{3/2}}$

Exercise:

Problem: $\int \frac{dx}{\left(x^2-9\right)^{3/2}}$

Solution:

$-\frac{x}{9\sqrt{-9+x^2}}+C$

Exercise:

Problem: $\int \frac{\sqrt{1+x^2}dx}{x}$

Exercise:

Problem: $\int \frac{x^2dx}{\sqrt{x^2-1}}$

Solution:

$\frac{1}{2}\left(\ln\left|x+\sqrt{x^2-1}\right|+x\sqrt{x^2-1}\right)+C$

Exercise:

Problem: $\int \frac{x^2dx}{x^2+4}$

Exercise:

Problem: $\int \frac{dx}{x^2\sqrt{x^2+1}}$

Solution:

$-\frac{\sqrt{1+x^2}}{x}+C$

Exercise:

Problem: $\int \frac{x^2 dx}{\sqrt{1+x^2}}$

Exercise:

Problem: $\int_{-1}^{1} \left(1-x^2\right)^{3/2} dx$

Solution:

$\frac{1}{8}\left(x\left(5-2x^2\right)\sqrt{1-x^2}+3\arcsin x\right)+C$

In the following exercises, use the substitutions $x=\sinh\theta, \cosh\theta$, or $\tanh\theta$. Express the final answers in terms of the variable *x*.

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^2-1}}$

Exercise:

Problem: $\int \frac{dx}{x\sqrt{1-x^2}}$

Solution:

$\ln x-\ln\left|1+\sqrt{1-x^2}\right|+C$

Exercise:

Problem: $\int \sqrt{x^2-1}dx$

Exercise:

Problem: $\int \frac{\sqrt{x^2-1}}{x^2}dx$

Solution:

$-\frac{\sqrt{-1+x^2}}{x}+\ln\left|x+\sqrt{-1+x^2}\right|+C$

Exercise:

Problem: $\int \frac{dx}{1-x^2}$

Exercise:

Problem: $\int \frac{\sqrt{1+x^2}}{x^2} dx$

Solution:

$-\frac{\sqrt{1+x^2}}{x} + \operatorname{arcsinh} x + C$

Use the technique of completing the square to evaluate the following integrals.

Exercise:

Problem: $\int \frac{1}{x^2 - 6x} dx$

Exercise:

Problem: $\int \frac{1}{x^2 + 2x + 1} dx$

Solution:

$-\frac{1}{1+x} + C$

Exercise:

Problem: $\int \frac{1}{\sqrt{-x^2 + 2x + 8}} dx$

Exercise:

Problem: $\int \frac{1}{\sqrt{-x^2 + 10x}} dx$

Solution:

$\frac{2\sqrt{-10+x}\sqrt{x}\ln\left|\sqrt{-10+x}+\sqrt{x}\right|}{\sqrt{(10-x)x}} + C$

Exercise:

Problem: $\int \frac{1}{\sqrt{x^2 + 4x - 12}} dx$

Exercise:

Problem: Evaluate the integral without using calculus: $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$.

Solution:

$\frac{9\pi}{2}$; area of a semicircle with radius 3

Exercise:

Problem: Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Exercise:

Problem:

Evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x = \cos\theta$ and evaluate using trigonometric substitution. Second, let $x = \sin\theta$ and use trigonometric substitution. Are the answers the same?

Solution:

$\arcsin(x) + C$ is the common answer.

Exercise:

Problem:

Evaluate the integral $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x = \sec\theta$. Next, evaluate the same integral using the substitution $x = \csc\theta$. Show that the results are equivalent.

Exercise:

Problem:

Evaluate the integral $\displaystyle\int \frac{x}{x^2+1}dx$ using the form $\displaystyle\int \frac{1}{u}du$. Next, evaluate the same integral using $x = \tan\theta$. Are the results the same?

Solution:

$\frac{1}{2}\ln\left(1+x^2\right) + C$ is the result using either method.

Exercise:

Problem:

State the method of integration you would use to evaluate the integral $\displaystyle\int x\sqrt{x^2+1}\,dx$. Why did you choose this method?

Exercise:

Problem:

State the method of integration you would use to evaluate the integral $\displaystyle\int x^2\sqrt{x^2-1}\,dx$. Why did you choose this method?

Solution:

Use trigonometric substitution. Let $x = \sec(\theta)$.

Exercise:

Problem: Evaluate $\displaystyle\int_{-1}^{1} \frac{x\,dx}{x^2+1}$

Exercise:

Problem:

Find the length of the arc of the curve over the specified interval: $y = \ln x, [1, 5]$. Round the answer to three decimal places.

Solution:

4.367

Exercise:

Problem:

Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2, y = 0, x = 0, \text{and } x = \sqrt{2}$ about the *x*-axis. (Round the answer to three decimal places).

Exercise:

Problem:

The region bounded by the graph of $f(x) = \frac{1}{1+x^2}$ and the *x*-axis between $x = 0$ and $x = 1$ is revolved about the *x*-axis. Find the volume of the solid that is generated.

Solution:

$\frac{\pi^2}{8} + \frac{\pi}{4}$

Solve the initial-value problem for *y* as a function of *x*.

Exercise:

Problem: $\left(x^2 + 36\right) \frac{dy}{dx} = 1, y(6) = 0$

Exercise:

Problem: $\left(64 - x^2\right) \frac{dy}{dx} = 1, y(0) = 3$

Solution:

$y = \frac{1}{16} \ln \left| \frac{x+8}{x-8} \right| + 3$

Exercise:

Problem: Find the area bounded by $y = \frac{2}{\sqrt{64-4x^2}}, x = 0, y = 0, \text{and } x = 2.$

Exercise:

Problem:

An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \frac{16}{\sqrt{64+x^2}}, x = 0, y = 0, x = 2$ about the *x*-axis. Find the volume of the tank (in cubic meters).

Solution:

24.6 m^3

Exercise:

Problem:

During each cycle, the velocity *v* (in feet per second) of a robotic welding device is given by $v = 2t - \frac{14}{4+t^2}$, where *t* is time in seconds. Find the expression for the displacement *s* (in feet) as a function of *t* if $s = 0$ when $t = 0$.

Exercise:

Problem: Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2$.

Solution:

$\frac{2\pi}{3}$

Glossary

trigonometric substitution
: an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral

Partial Fractions

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that

Equation:

$$\int \frac{du}{u} = \ln|u| + C \text{ and } \int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating $\int \frac{3x}{x^2 - x - 2}dx$. However, we know from material previously developed that

Equation:

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2}\right)dx = \ln|x+1| + 2\ln|x-2| + C.$$

In fact, by getting a common denominator, we see that

Equation:

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

Equation:

$$\int \frac{3x}{x^2 - x - 2}dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right)dx.$$

In this section, we examine the method of **partial fraction decomposition**, which allows us to decompose rational functions into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as: $\frac{3x}{x^2-x-2}$ as an expression such as $\frac{1}{x+1} + \frac{2}{x-2}$.

The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $\deg(P(x)) < \deg(Q(x))$. In the case when $\deg(P(x)) \geq \deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $\deg(R(x)) < \deg(Q(x))$. We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) \geq \deg(Q(x))$.

Example:

Exercise:

Problem:

Integrating $\int \frac{P(x)}{Q(x)} dx$**, where** $\deg(P(x)) \geq \deg(Q(x))$

Evaluate $\int \frac{x^2+3x+5}{x+1} dx$.

Solution:

Since $\deg\left(x^2+3x+5\right) \geq \deg(x+1)$, we perform long division to obtain

Equation:

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

Equation:

$$\begin{aligned}\int \frac{x^2+3x+5}{x+1} dx &= \int \left(x+2+\frac{3}{x+1}\right) dx \\ &= \tfrac{1}{2}x^2+2x+3\ln|x+1|+C.\end{aligned}$$

Note:

Visit this website for a review of long division of polynomials.

Note:

Exercise:

Problem: Evaluate $\int \frac{x-3}{x+2} dx$.

Solution:

$x-5\ln|x+2|+C$

Hint

Use long division to obtain $\frac{x-3}{x+2} = 1-\frac{5}{x+2}$.

To integrate $\int \frac{P(x)}{Q(x)} dx$, where $\deg(P(x)) < \deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying
Equation:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

The proof that such constants exist is beyond the scope of this course.

In this next example, we see how to use partial fractions to integrate a rational function of this type.

Example:
Exercise:

Problem:
Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}dx.$

Solution:

Since $\deg(3x+2) < \deg\left(x^3-x^2-2x\right)$, we begin by factoring the denominator of $\frac{3x+2}{x^3-x^2-2x}$. We can see that $x^3-x^2-2x = x(x-2)(x+1)$. Thus, there are constants A, B, and C satisfying
Equation:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,
Equation:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A\,(x-2)(x+1) + Bx\,(x+1) + Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining
Equation:

$$3x+2 = A\,(x-2)\,(x+1) + Bx\,(x+1) + Cx\,(x-2).$$

There are two different strategies for finding the coefficients A, B, and C. We refer to these as the *method of equating coefficients* and the *method of strategic substitution*.

Note:
Rule: Method of Equating Coefficients
Rewrite [link] in the form
Equation:

$$3x+2 = (A+B+C)x^2 + (-A+B-2C)x + (-2A).$$

Equating coefficients produces the system of equations

Equation:

$$\begin{aligned} A+B+C &= 0 \\ -A+B-2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

Equation:

$$\begin{aligned} B+C &= 1 \\ B-2C &= 2. \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces

Equation:

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B + C = 1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1$, $B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

Note:

Rule: Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A, B, and C that satisfy [link] for *all* values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A, B, and C easily. For example, if we substitute $x = 0$, the equation reduces to $2 = A\,(-2)\,(1)$. Solving for A yields $A = -1$. Next, by substituting $x = 2$, the equation reduces to $8 = B\,(2)\,(3)$, or equivalently $B = 4/3$. Last, we substitute $x = -1$ into the equation and obtain $-1 = C(-1)(-3)$. Solving, we have $C = -\frac{1}{3}$.

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of A, B, and C, we rewrite the original integral:

Equation:

$$\int \frac{3x+2}{x^3-x^2-2x}dx = \int \left(-\frac{1}{x} + \frac{4}{3}\cdot\frac{1}{(x-2)} - \frac{1}{3}\cdot\frac{1}{(x+1)}\right)dx.$$

Evaluating the integral gives us

Equation:

$$\int \frac{3x+2}{x^3-x^2-2x}dx = -\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C.$$

In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

Example:
Exercise:

Problem:
Dividing before Applying Partial Fractions

Evaluate $\int \frac{x^2+3x+1}{x^2-4}dx$.

Solution:

Since degree$(x^2+3x+1) \geq$ degree(x^2-4), we must perform long division of polynomials. This results in
Equation:

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}.$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}$. We have
Equation:

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus,
Equation:

$$3x+5 = A\,(x+2) + B\,(x-2).$$

Solving for A and B using either method, we obtain $A = 11/4$ and $B = 1/4$.

Rewriting the original integral, we have
Equation:

$$\int \frac{x^2+3x+1}{x^2-4}dx = \int \left(1 + \frac{11}{4} \cdot \frac{1}{x-2} + \frac{1}{4} \cdot \frac{1}{x+2}\right)dx.$$

Evaluating the integral produces
Equation:

$$\int \frac{x^2+3x+1}{x^2-4}dx = x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C.$$

As we see in the next example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. The trick is to convert the nonrational function to a rational function through a substitution.

Example:
Exercise:

Problem:
Applying Partial Fractions after a Substitution

Evaluate $\int \frac{\cos x}{\sin^2 x - \sin x} dx.$

Solution:

Let's begin by letting $u = \sin x$. Consequently, $du = \cos x \, dx$. After making these substitutions, we have
Equation:

$$\int \frac{\cos x}{\sin^2 x - \sin x} dx = \int \frac{du}{u^2 - u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $1/u(u-1)$ gives $\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}$.

Thus,
Equation:

$$\begin{aligned} \int \frac{\cos x}{\sin^2 x - \sin x} dx & = -\ln|u| + \ln|u-1| + C \\ & = -\ln|\sin x| + \ln|\sin x - 1| + C. \end{aligned}$$

Note:
Exercise:

Problem: Evaluate $\int \frac{x+1}{(x+3)(x-2)} dx.$

Solution:

$\frac{2}{5} \ln|x+3| + \frac{3}{5} \ln|x-2| + C$

Hint

$\frac{x+1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$

Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax+b)^n$, where n is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain
Equation:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

As we see in our next example, the basic technique used for solving for the coefficients is the same, but it requires more algebra to determine the numerators of the partial fractions.

Example:
Exercise:

Problem:
Partial Fractions with Repeated Linear Factors

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)} dx.$

Solution:

We have degree $(x-2) <$ degree $\left((2x-1)^2 (x-1)\right)$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, include $\frac{A}{2x-1} + \frac{B}{(2x-1)^2}$ in the decomposition. Thus,

Equation:

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}.$$

After getting a common denominator and equating the numerators, we have

Equation:

$$x-2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2.$$

We then use the method of equating coefficients to find the values of A, B, and C.

Equation:

$$x-2 = (2A+4C)x^2 + (-3A+B-4C)x + (A-B+C).$$

Equating coefficients yields $2A+4C=0$, $-3A+B-4C=1$, and $A-B+C=-2$. Solving this system yields $A=2$, $B=3$, and $C=-1$.

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=1/2$ into [link] easily produces the values $B=3$ and $C=-1$. At this point, it may seem that we have run out of good choices for x, however, since we already have values for B and C, we can substitute in these values and choose any value for x not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2 = A(-1)(-1) + 3(-1) + (-1)(-1)^2$ or, equivalently, $A=2$.

Now that we have the values for A, B, and C, we rewrite the original integral and evaluate it:

Equation:

$$\begin{aligned} \int \frac{x-2}{(2x-1)^2(x-1)} dx &= \int \left(\frac{2}{2x-1} + \frac{3}{(2x-1)^2} - \frac{1}{x-1} \right) dx \\ &= \ln|2x-1| - \frac{3}{2(2x-1)} - \ln|x-1| + C. \end{aligned}$$

Note:
Exercise:

Problem:

Set up the partial fraction decomposition for $\int \frac{x+2}{(x+3)^3(x-4)^2}dx$. (Do not solve for the coefficients or complete the integration.)

Solution:

$\frac{x+2}{(x+3)^3(x-4)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x-4)} + \frac{E}{(x-4)^2}$

Hint

Use the problem-solving method of [link] for guidance.

The General Method

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy.

Note:
Problem-Solving Strategy: Partial Fraction Decomposition
To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that degree $(P(x)) < \text{degree}(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that deg $(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.

   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying
   Equation:

   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

   b. If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain
   Equation:

   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

   c. For each irreducible quadratic factor ax^2+bx+c that $Q(x)$ contains, the decomposition must include
   Equation:

$$\frac{Ax+B}{ax^2+bx+c}.$$

d. For each repeated irreducible quadratic factor $\left(ax^2+bx+c\right)^n$, the decomposition must include
Equation:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}.$$

e. After the appropriate decomposition is determined, solve for the constants.
f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Simple Quadratic Factors

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic ax^2+bx+c is irreducible if $ax^2+bx+c=0$ has no real zeros—that is, if $b^2-4ac<0$.

Example:
Exercise:

Problem:
Rational Expressions with an Irreducible Quadratic Factor

Evaluate $\int \frac{2x-3}{x^3+x}dx$.

Solution:

Since $\deg(2x-3)<\deg(x^3+x)$, factor the denominator and proceed with partial fraction decomposition. Since $x^3+x=x(x^2+1)$ contains the irreducible quadratic factor x^2+1, include $\frac{Ax+B}{x^2+1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form
Equation:

$$\frac{2x-3}{x(x^2+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x}.$$

After getting a common denominator and equating the numerators, we obtain the equation
Equation:

$$2x-3=(Ax+B)x+C\left(x^2+1\right).$$

Solving for A, B, and C, we get $A=3$, $B=2$, and $C=-3$.

Thus,
Equation:

$$\frac{2x-3}{x^3+x}=\frac{3x+2}{x^2+1}-\frac{3}{x}.$$

Substituting back into the integral, we obtain

Equation:

$$\begin{aligned}\int \frac{2x-3}{x^3+x}dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x}\right)dx \\ &= 3\int \frac{x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx - 3\int \frac{1}{x}dx && \text{Split up the integral.} \\ &= \tfrac{3}{2}\ln\left|x^2+1\right| + 2\tan^{-1}x - 3\ln|x| + C. && \text{Evaluate each integral.}\end{aligned}$$

Note: We may rewrite $\ln\left|x^2+1\right| = \ln(x^2+1)$, if we wish to do so, since $x^2+1>0$.

Example:

Exercise:

Problem:

Partial Fractions with an Irreducible Quadratic Factor

Evaluate $\int \frac{dx}{x^3-8}$.

Solution:

We can start by factoring $x^3-8=(x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2-4\,(1)\,(4)=-12<0$. Using the decomposition described in the problem-solving strategy, we get

Equation:

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

Equation:

$$1 = A\left(x^2+2x+4\right) + (Bx+C)\,(x-2).$$

Applying either method, we get $A=\frac{1}{12}, B=-\frac{1}{12}$, and $C=-\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

Equation:

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{1}{x-2}dx - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx.$$

We can see that

$\int \frac{1}{x-2}dx = \ln|x-2|+C$, but $\int \frac{x+4}{x^2+2x+4}dx$ requires a bit more effort. Let's begin by completing the square on x^2+2x+4 to obtain

Equation:

$$x^2 + 2x + 4 = (x+1)^2 + 3.$$

By letting $u = x + 1$ and consequently $du = dx$, we see that

Equation:

$\int \frac{x+4}{x^2+2x+4} dx$	$= \int \frac{x+4}{(x+1)^2+3} dx$	Complete the square on the denominator.
	$= \int \frac{u+3}{u^2+3} du$	Substitute $u = x+1, x = u-1,$ and $du = dx$.
	$= \int \frac{u}{u^2+3} du + \int \frac{3}{u^2+3} du$	Split the numerator apart.
	$= \frac{1}{2} \ln \left\vert u^2+3 \right\vert + \frac{3}{\sqrt{3}} \tan^{-1} \frac{u}{\sqrt{3}} + C$	Evaluate each integral.
	$= \frac{1}{2} \ln \left\vert x^2+2x+4 \right\vert + \sqrt{3} \tan^{-1} \left(\frac{x+1}{\sqrt{3}} \right) + C.$	Rewrite in terms of x and simplify.

Substituting back into the original integral and simplifying gives

Equation:

$$\int \frac{dx}{x^3 - 8} = \frac{1}{12} \ln |x-2| - \frac{1}{24} \ln \left\vert x^2 + 2x + 4 \right\vert - \frac{\sqrt{3}}{12} \tan^{-1} \left(\frac{x+1}{\sqrt{3}} \right) + C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2 + 2x + 4 > 0$ for all x.

Example:

Exercise:

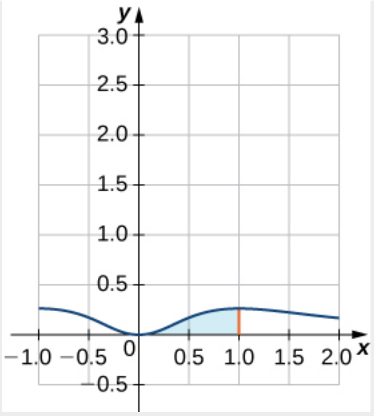
Problem:

Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x) = \frac{x^2}{(x^2+1)^2}$ and the *x*-axis over the interval $[0, 1]$ about the *y*-axis.

Solution:

Let's begin by sketching the region to be revolved (see [link]). From the sketch, we see that the shell method is a good choice for solving this problem.



We can use the shell method to find the volume of revolution obtained by revolving the region shown about the *y*-axis.

The volume is given by

Equation:

$$V = 2\pi \int_0^1 x \cdot \frac{x^2}{\left(x^2+1\right)^2} dx = 2\pi \int_0^1 \frac{x^3}{\left(x^2+1\right)^2} dx.$$

Since $\deg\left(\left(x^2+1\right)^2\right) = 4 > 3 = \deg(x^3)$, we can proceed with partial fraction decomposition. Note that $\left(x^2+1\right)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

Equation:

$$\frac{x^3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}.$$

Finding a common denominator and equating the numerators gives

Equation:

$$x^3 = (Ax+B)\left(x^2+1\right) + Cx + D.$$

Solving, we obtain $A = 1$, $B = 0$, $C = -1$, and $D = 0$. Substituting back into the integral, we have

Equation:

$$\begin{aligned} V &= 2\pi \int_0^1 \frac{x^3}{(x^2+1)^2} dx \\ &= 2\pi \int_0^1 \left(\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} \right) dx \\ &= 2\pi \left(\tfrac{1}{2} \ln(x^2+1) + \tfrac{1}{2} \cdot \tfrac{1}{x^2+1} \right) \Big|_0^1 \\ &= \pi \left(\ln 2 - \tfrac{1}{2} \right). \end{aligned}$$

Note:
Exercise:

Problem: Set up the partial fraction decomposition for $\int \frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2} dx$.

Solution:

$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$

Hint

Use the problem-solving strategy.

Key Concepts

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

Express the rational function as a sum or difference of two simpler rational expressions.
Exercise:

Problem: $\frac{1}{(x-3)(x-2)}$

Exercise:

Problem: $\frac{x^2+1}{x(x+1)(x+2)}$

Solution:

$-\frac{2}{x+1} + \frac{5}{2(x+2)} + \frac{1}{2x}$

Exercise:

Problem: $\frac{1}{x^3-x}$

Exercise:

Problem: $\frac{3x+1}{x^2}$

Solution:

$\frac{1}{x^2} + \frac{3}{x}$

Exercise:

Problem: $\frac{3x^2}{x^2+1}$ (*Hint:* Use long division first.)

Exercise:

Problem: $\frac{2x^4}{x^2-2x}$

Solution:

$2x^2 + 4x + 8 + \frac{16}{x-2}$

Exercise:

Problem: $\frac{1}{(x-1)(x^2+1)}$

Exercise:

Problem: $\frac{1}{x^2(x-1)}$

Solution:

$-\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x-1}$

Exercise:

Problem: $\frac{x}{x^2-4}$

Exercise:

Problem: $\frac{1}{x(x-1)(x-2)(x-3)}$

Solution:

$-\frac{1}{2(x-2)} + \frac{1}{2(x-1)} - \frac{1}{6x} + \frac{1}{6(x-3)}$

Exercise:

Problem: $\frac{1}{x^4-1} = \frac{1}{(x+1)(x-1)(x^2+1)}$

Exercise:

Problem: $\frac{3x^2}{x^3-1} = \frac{3x^2}{(x-1)(x^2+x+1)}$

Solution:

$\frac{1}{x-1} + \frac{2x+1}{x^2+x+1}$

Exercise:

Problem: $\frac{2x}{(x+2)^2}$

Exercise:

Problem: $\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}$

Solution:

$\frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}$

Use the method of partial fractions to evaluate each of the following integrals.

Exercise:

Problem: $\int \frac{dx}{(x-3)(x-2)}$

Exercise:

Problem: $\int \frac{3x}{x^2+2x-8} dx$

Solution:

$-\ln|2-x| + 2\ln|4+x| + C$

Exercise:

Problem: $\int \frac{dx}{x^3-x}$

Exercise:

Problem: $\int \frac{x}{x^2-4} dx$

Solution:

$\frac{1}{2}\ln\left|4-x^2\right| + C$

Exercise:

Problem: $\int \frac{dx}{x(x-1)(x-2)(x-3)}$

Exercise:

Problem: $\int \frac{2x^2+4x+22}{x^2+2x+10}dx$

Solution:

$2\left(x+\frac{1}{3}\arctan\left(\frac{1+x}{3}\right)\right)+C$

Exercise:

Problem: $\int \frac{dx}{x^2-5x+6}$

Exercise:

Problem: $\int \frac{2-x}{x^2+x}dx$

Solution:

$2\ln|x|-3\ln|1+x|+C$

Exercise:

Problem: $\int \frac{2}{x^2-x-6}dx$

Exercise:

Problem: $\int \frac{dx}{x^3-2x^2-4x+8}$

Solution:

$\frac{1}{16}\left(-\frac{4}{-2+x}-\ln|-2+x|+\ln|2+x|\right)+C$

Exercise:

Problem: $\int \frac{dx}{x^4-10x^2+9}$

Evaluate the following integrals, which have irreducible quadratic factors.

Exercise:

Problem: $\int \frac{2}{(x-4)(x^2+2x+6)}dx$

Solution:

$\frac{1}{30}\left(-2\sqrt{5}\arctan\left[\frac{1+x}{\sqrt{5}}\right]+2\ln|-4+x|-\ln\left|6+2x+x^2\right|\right)+C$

Exercise:

Problem: $\int \frac{x^2}{x^3-x^2+4x-4}dx$

Exercise:

Problem: $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}dx$

Solution:

$-\frac{3}{x}+4\ln|x+2|+x+C$

Exercise:

Problem: $\int \frac{x}{(x-1)(x^2+2x+2)^2}dx$

Use the method of partial fractions to evaluate the following integrals.

Exercise:

Problem: $\int \frac{3x+4}{(x^2+4)(3-x)}dx$

Solution:

$-\ln|3-x|+\frac{1}{2}\ln\left|x^2+4\right|+C$

Exercise:

Problem: $\int \frac{2}{(x+2)^2(2-x)}dx$

Exercise:

Problem: $\int \frac{3x+4}{x^3-2x-4}dx$ (*Hint:* Use the rational root theorem.)

Solution:

$\ln|x-2|-\frac{1}{2}\ln\left|x^2+2x+2\right|+C$

Use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

Exercise:

Problem:

$\int_0^1 \frac{e^x}{36-e^{2x}}dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

Exercise:

Problem: $\int \frac{e^x dx}{e^{2x}-e^x}dx$

Solution:

$-x + \ln|1 - e^x| + C$

Exercise:

Problem: $\int \frac{\sin x \, dx}{1 - \cos^2 x}$

Exercise:

Problem: $\int \frac{\sin x}{\cos^2 x + \cos x - 6} dx$

Solution:

$\frac{1}{5} \ln \left| \frac{\cos x + 3}{\cos x - 2} \right| + C$

Exercise:

Problem: $\int \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx$

Exercise:

Problem: $\int \frac{dt}{(e^t - e^{-t})^2}$

Solution:

$\frac{1}{2 - 2e^{2t}} + C$

Exercise:

Problem: $\int \frac{1 + e^x}{1 - e^x} dx$

Exercise:

Problem: $\int \frac{dx}{1 + \sqrt{x + 1}}$

Solution:

$2\sqrt{1 + x} - 2 \ln \left| 1 + \sqrt{1 + x} \right| + C$

Exercise:

Problem: $\int \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$

Exercise:

Problem: $\int \frac{\cos x}{\sin x (1 - \sin x)} dx$

Solution:

$\ln\left|\frac{\sin x}{1-\sin x}\right|+C$

Exercise:

Problem: $$\int \frac{e^x}{\left(e^{2x}-4\right)^2}dx$$

Exercise:

Problem: $$\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}dx$$

Solution:

$\frac{\sqrt{3}}{4}$

Exercise:

Problem: $$\int \frac{1}{2+e^{-x}}dx$$

Exercise:

Problem: $$\int \frac{1}{1+e^x}dx$$

Solution:

$x-\ln\left(1+e^x\right)+C$

Use the given substitution to convert the integral to an integral of a rational function, then evaluate.

Exercise:

Problem: $$\int \frac{1}{t-\sqrt[3]{t}}dt\, t=x^3$$

Exercise:

Problem: $$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx; x=u^6$$

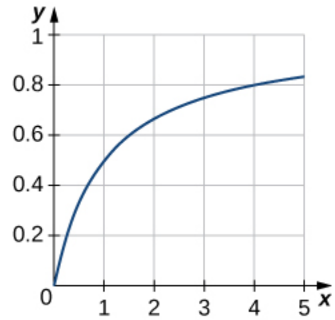
Solution:

$6x^{1/6}-3x^{1/3}+2\sqrt{x}-6\ln\left(1+x^{1/6}\right)+C$

Exercise:

Problem:

Graph the curve $y=\frac{x}{1+x}$ over the interval $[0,5]$. Then, find the area of the region bounded by the curve, the *x*-axis, and the line $x=4$.



Exercise:

Problem:

Find the volume of the solid generated when the region bounded by $y=1/\sqrt{x(3-x)}$, $y=0$, $x=1$, and $x=2$ is revolved about the *x*-axis.

Solution:

$\frac{4}{3}\pi\operatorname{arctanh}\left[\frac{1}{3}\right]=\frac{1}{3}\pi\ln 4$

Exercise:

Problem:

The velocity of a particle moving along a line is a function of time given by $v(t)=\frac{88t^2}{t^2+1}$. Find the distance that the particle has traveled after $t=5$ sec.

Solve the initial-value problem for *x* as a function of *t*.

Exercise:

Problem: $\left(t^2-7t+12\right)\frac{dx}{dt}=1,(t>4,x(5)=0)$

Solution:

$x=-\ln|t-3|+\ln|t-4|+\ln 2$

Exercise:

Problem: $(t+5)\frac{dx}{dt}=x^2+1,t>-5,x(1)=\tan 1$

Exercise:

Problem: $\left(2t^3-2t^2+t-1\right)\frac{dx}{dt}=3,x(2)=0$

Solution:

$x=\ln|t-1|-\sqrt{2}\arctan\left(\sqrt{2}t\right)-\frac{1}{2}\ln\left(t^2+\frac{1}{2}\right)+\sqrt{2}\arctan\left(2\sqrt{2}\right)+\frac{1}{2}\ln 4.5$

Exercise:

Problem: Find the *x*-coordinate of the centroid of the area bounded by

$y\left(x^2-9\right)=1,y=0,x=4,$ and $x=5$. (Round the answer to two decimal places.)

Exercise:

Problem:

Find the volume generated by revolving the area bounded by $y = \frac{1}{x^3+7x^2+6x}x = 1, x = 7,$ and $y = 0$ about the *y*-axis.

Solution:

$\frac{2}{5}\pi \ln \frac{28}{13}$

Exercise:

Problem:

Find the area bounded by $y = \frac{x-12}{x^2-8x-20}, y = 0, x = 2,$ and $x = 4$. (Round the answer to the nearest hundredth.)

Exercise:

Problem: Evaluate the integral $\int \frac{dx}{x^3+1}$.

Solution:

$\frac{\arctan\left[\frac{-1+2x}{\sqrt{3}}\right]}{\sqrt{3}} + \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln\left|1-x+x^2\right| + C$

For the following problems, use the substitutions $\tan\left(\frac{x}{2}\right) = t$, $dx = \frac{2}{1+t^2}dt$, $\sin x = \frac{2t}{1+t^2}$, and $\cos x = \frac{1-t^2}{1+t^2}$.

Exercise:

Problem: $\int \frac{dx}{3-5\sin x}$

Exercise:

Problem:

Find the area under the curve $y = \frac{1}{1+\sin x}$ between $x = 0$ and $x = \pi$. (Assume the dimensions are in inches.)

Solution:

2.0 in.^{2}

Exercise:

Problem: Given $\tan\left(\frac{x}{2}\right) = t$, derive the formulas $dx = \frac{2}{1+t^2}dt$, $\sin x = \frac{2t}{1+t^2}$, and $\cos x = \frac{1-t^2}{1+t^2}$.

Exercise:

Problem: Evaluate $\int \frac{\sqrt[3]{x-8}}{x}dx$.

Solution:

$$
\begin{aligned}
&3(-8+x)^{1/3} \\
&-2\sqrt{3}\arctan\left[\frac{-1+(-8+x)^{1/3}}{\sqrt{3}}\right] \\
&-2\ln\left[2+(-8+x)^{1/3}\right] \\
&+\ln\left[4-2(-8+x)^{1/3}+(-8+x)^{2/3}\right]+C
\end{aligned}
$$

Glossary

partial fraction decomposition
: a technique used to break down a rational function into the sum of simple rational functions

Other Strategies for Integration

- Use a table of integrals to solve integration problems.
- Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in Trigonometric Substitution, we found that, by using the substitution $x = \tan\theta$, we can arrive at

Equation:

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left(x + \sqrt{x^2+1}\right) + C.$$

However, using $x = \sinh\theta$, we obtained a different solution—namely,

Equation:

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}x + C.$$

We later showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$. In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

Example:
Exercise:

Problem:
Using a Formula from a Table to Evaluate an Integral

Use the table formula

Equation:

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

to evaluate $\int \frac{\sqrt{16-e^{2x}}}{e^x} dx$.

Solution:

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2-u^2}$. This expression is actually similar to $\sqrt{16-e^{2x}}$, where $a=4$ and $u=e^x$. Keep in mind that we must also have $du=e^x$. Multiplying the numerator and the denominator of the given integral by e^x should help to put this integral in a useful form. Thus, we now have

Equation:

$$\int \frac{\sqrt{16-e^{2x}}}{e^x} dx = \int \frac{\sqrt{16-e^{2x}}}{e^{2x}} e^x dx.$$

Substituting $u=e^x$ and $du=e^x$ produces $\int \frac{\sqrt{a^2-u^2}}{u^2} du$. From the integration table (#88 in Appendix A),

Equation:

$$\int \frac{\sqrt{a^2-u^2}}{u^2} du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C.$$

Thus,

Equation:

$$\begin{aligned}
\int \frac{\sqrt{16-e^{2x}}}{e^x} dx &= \int \frac{\sqrt{16-e^{2x}}}{e^{2x}} e^x dx && \text{Substitute } u=e^x \text{ and } du=e^x dx. \\
&= \int \frac{\sqrt{4^2-u^2}}{u^2} du && \text{Apply the formula using } a=4. \\
&= -\tfrac{\sqrt{4^2-u^2}}{u} - \sin^{-1}\tfrac{u}{4} + C && \text{Substitute } u=e^x. \\
&= -\tfrac{\sqrt{16-e^{2x}}}{u} - \sin^{-1}\left(\tfrac{e^x}{4}\right) + C.
\end{aligned}$$

Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

Example:
Exercise:

Problem:
Using a Computer Algebra System to Evaluate an Integral

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Compare this result with $\ln\left|\frac{\sqrt{x^2-4}}{2}+\frac{x}{2}\right|+C$, a result we might have obtained if we had used trigonometric substitution.

Solution:

Using Wolfram Alpha, we obtain
Equation:

$$\int \frac{dx}{\sqrt{x^2-4}} = \ln\left|\sqrt{x^2-4}+x\right|+C.$$

Notice that
Equation:

$$\ln\left|\frac{\sqrt{x^2-4}}{2}+\frac{x}{2}\right|+C = \ln\left|\frac{\sqrt{x^2-4}+x}{2}\right|+C = \ln\left|\sqrt{x^2-4}+x\right|-\ln 2+C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

Note:
You can access an [integral calculator] for more examples.

Example:
Exercise:

Problem:
Using a CAS to Evaluate an Integral

Evaluate $\int \sin^3 x\,dx$ using a CAS. Compare the result to $\frac{1}{3}\cos^3 x - \cos x + C$, the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

Solution:

Using Wolfram Alpha, we obtain
Equation:

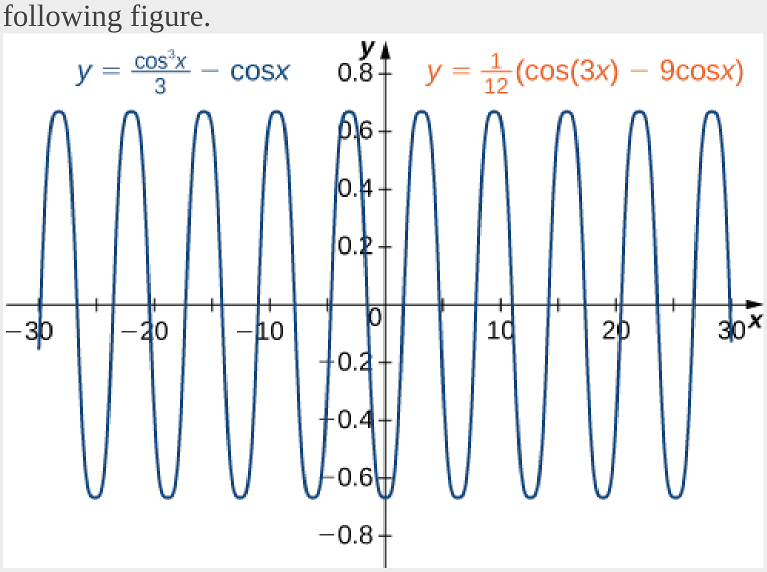
$$\int \sin^3 x \, dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C$. To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:
Equation:

$$\begin{aligned}
\tfrac{1}{12}(\cos(3x) - 9\cos x) &= \tfrac{1}{12}(\cos(x+2x) - 9\cos x) \\
&= \tfrac{1}{12}(\cos(x)\cos(2x) - \sin(x)\sin(2x) - 9\cos x) \\
&= \tfrac{1}{12}\left(\cos x\left(2\cos^2 x - 1\right) - \sin x\left(2\sin x\cos x\right) - 9\cos x\right) \\
&= \tfrac{1}{12}\left(2\cos^3 x - \cos x - 2\cos x\left(1 - \cos^2 x\right) - 9\cos x\right) \\
&= \tfrac{1}{12}(4\cos^3 x - 12\cos x) \\
&= \tfrac{1}{3}\cos^3 x - \cos x.
\end{aligned}$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.



The graphs of $y = \frac{1}{3}\cos^3 x - \cos x$ and $y = \frac{1}{12}(\cos(3x) - 9\cos x)$ are identical.

Note:
Exercise:

Problem: Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2+4}}$.

Solution:

Possible solutions include $\sinh^{-1}\left(\frac{x}{2}\right)+C$ and $\ln\left|\sqrt{x^2+4}+x\right|+C$.

Hint

Answers may vary.

Key Concepts

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

Use a table of integrals to evaluate the following integrals.
Exercise:

Problem: $\int_0^4 \frac{x}{\sqrt{1+2x}}dx$

Exercise:

Problem: $\int \frac{x+3}{x^2+2x+2}dx$

Solution:

$\frac{1}{2}\ln\left|x^2+2x+2\right|+2\arctan(x+1)+C$

Exercise:

Problem: $\int x^3\sqrt{1+2x^2}\,dx$

Exercise:

Problem: $\int \frac{1}{\sqrt{x^2+6x}}dx$

Solution:

$\cosh^{-1}\left(\frac{x+3}{3}\right)+C$

Exercise:

Problem: $\int \frac{x}{x+1}dx$

Exercise:

Problem: $\int x\cdot 2^{x^2}dx$

Solution:

$\frac{2^{x^2-1}}{\ln 2}+C$

Exercise:

Problem: $\int \frac{1}{4x^2+25}dx$

Exercise:

Problem: $\int \frac{dy}{\sqrt{4-y^2}}$

Solution:

$\arcsin\left(\frac{y}{2}\right)+C$

Exercise:

Problem: $\int \sin^3(2x)\cos(2x)dx$

Exercise:

Problem: $\int \csc(2w)\cot(2w)dw$

Solution:

$-\frac{1}{2}\csc\left(2w\right)+C$

Exercise:

Problem: $\int 2^y dy$

Exercise:

Problem: $\int_0^1 \frac{3x\,dx}{\sqrt{x^2+8}}$

Solution:

$9-6\sqrt{2}$

Exercise:

Problem: $\int_{-1/4}^{1/4} \sec^2(\pi x)\tan(\pi x)dx$

Exercise:

Problem: $\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)dx$

Solution:

$2-\frac{\pi}{2}$

Exercise:

Problem: $\int \cos^3 x\,dx$

Exercise:

Problem: $\int \tan^5(3x)dx$

Solution:

$\frac{1}{12}\tan^4(3x)-\frac{1}{6}\tan^2(3x)+\frac{1}{3}\ln|\sec(3x)|+C$

Exercise:

Problem: $\int \sin^2 y\cos^3 y dy$

Use a CAS to evaluate the following integrals. Tables can also be used to verify the answers.

Exercise:

Problem: [T] $\displaystyle\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)}$

Solution:

$2\cot\left(\frac{w}{2}\right) - 2\csc\left(\frac{w}{2}\right) + w + C$

Exercise:

Problem: [T] $\displaystyle\int \frac{dw}{1-\cos(7w)}$

Exercise:

Problem: [T] $\displaystyle\int_0^t \frac{dt}{4\cos t + 3\sin t}$

Solution:

$\frac{1}{5}\ln\left|\frac{2(5+4\sin t - 3\cos t)}{4\cos t + 3\sin t}\right|$

Exercise:

Problem: [T] $\displaystyle\int \frac{\sqrt{x^2-9}}{3x}dx$

Exercise:

Problem: [T] $\displaystyle\int \frac{dx}{x^{1/2}+x^{1/3}}$

Solution:

$6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln\left[1+x^{1/6}\right] + C$

Exercise:

Problem: [T] $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$

Exercise:

Problem: [T] $\displaystyle\int x^3 \sin x \, dx$

Solution:

$-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

Exercise:

Problem: [T] $\int x\sqrt{x^4 - 9}\,dx$

Exercise:

Problem: [T] $\int \frac{x}{1 + e^{-x^2}}dx$

Solution:

$\frac{1}{2}\left(x^2 + \ln\left|1 + e^{-x^2}\right|\right) + C$

Exercise:

Problem: [T] $\int \frac{\sqrt{3 - 5x}}{2x}dx$

Exercise:

Problem: [T] $\int \frac{dx}{x\sqrt{x - 1}}$

Solution:

$2\arctan\left(\sqrt{x - 1}\right) + C$

Exercise:

Problem: [T] $\int e^x\cos^{-1}(e^x)dx$

Use a calculator or CAS to evaluate the following integrals.

Exercise:

Problem: [T] $\int_0^{\pi/4} \cos(2x)dx$

Solution:

$0.5 = \frac{1}{2}$

Exercise:

Problem: [T] $\int_0^1 x \cdot e^{-x^2} dx$

Exercise:

Problem: [T] $\int_0^8 \frac{2x}{\sqrt{x^2+36}} dx$

Solution:

8.0

Exercise:

Problem: [T] $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2} dx$

Exercise:

Problem: [T] $\int \frac{dx}{x^2+4x+13}$

Solution:

$\frac{1}{3}\arctan\left(\frac{1}{3}(x+2)\right)+C$

Exercise:

Problem: [T] $\int \frac{dx}{1+\sin x}$

Use tables to evaluate the integrals. You may need to complete the square or change variables to put the integral into a form given in the table.

Exercise:

Problem: $\int \frac{dx}{x^2+2x+10}$

Solution:

$\frac{1}{3}\arctan\left(\frac{x+1}{3}\right)+C$

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^2 - 6x}}$

Exercise:

Problem: $\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx$

Solution:

$\ln \left(e^x + \sqrt{4 + e^{2x}} \right) + C$

Exercise:

Problem: $\int \frac{\cos x}{\sin^2 x + 2 \sin x} dx$

Exercise:

Problem: $\int \frac{\arctan \left(x^3 \right)}{x^4} dx$

Solution:

$\ln x - \frac{1}{6} \ln \left(x^6 + 1 \right) - \frac{\arctan(x^3)}{3x^3} + C$

Exercise:

Problem: $\int \frac{\ln |x| \arcsin (\ln |x|)}{x} dx$

Use tables to perform the integration.

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^2 + 16}}$

Solution:

$\ln \left| x + \sqrt{16 + x^2} \right| + C$

Exercise:

Problem: $\int \frac{3x}{2x + 7} dx$

Exercise:

Problem: $\displaystyle\int \frac{dx}{1 - \cos(4x)}$

Solution:

$-\frac{1}{4}\cot(2x) + C$

Exercise:

Problem: $\displaystyle\int \frac{dx}{\sqrt{4x + 1}}$

Exercise:

Problem:

Find the area bounded by $y\left(4 + 25x^2\right) = 5, x = 0, y = 0,$ and $x = 4$. Use a table of integrals or a CAS.

Solution:

$\frac{1}{2}\arctan 10$

Exercise:

Problem:

The region bounded between the curve $y = \frac{1}{\sqrt{1+\cos x}}, 0.3 \le x \le 1.1,$ and the *x*-axis is revolved about the *x*-axis to generate a solid. Use a table of integrals to find the volume of the solid generated. (Round the answer to two decimal places.)

Exercise:

Problem:

Use substitution and a table of integrals to find the area of the surface generated by revolving the curve $y = e^x, 0 \le x \le 3,$ about the *x*-axis. (Round the answer to two decimal places.)

Solution:

1276.14

Exercise:

Problem:

[T] Use an integral table and a calculator to find the area of the surface generated by revolving the curve $y = \frac{x^2}{2}, 0 \le x \le 1,$ about the *x*-axis. (Round the answer to two decimal places.)

Exercise:

Problem:

[T] Use a CAS or tables to find the area of the surface generated by revolving the curve $y=\cos x, 0\le x\le \frac{\pi}{2}$, about the *x*-axis. (Round the answer to two decimal places.)

Solution:

7.21

Exercise:

Problem: Find the length of the curve $y=\frac{x^2}{4}$ over $[0,8]$.

Exercise:

Problem: Find the length of the curve $y=e^x$ over $[0,\ln(2)]$.

Solution:

$\sqrt{5}-\sqrt{2}+\ln\left|\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right|$

Exercise:

Problem:

Find the area of the surface formed by revolving the graph of $y=2\sqrt{x}$ over the interval $[0,9]$ about the *x*-axis.

Exercise:

Problem: Find the average value of the function $f(x)=\frac{1}{x^2+1}$ over the interval $[-3,3]$.

Solution:

$\frac{1}{3}\arctan(3)\approx 0.416$

Exercise:

Problem:

Approximate the arc length of the curve $y=\tan(\pi x)$ over the interval $\left[0,\frac{1}{4}\right]$. (Round the answer to three decimal places.)

Glossary

computer algebra system (CAS)
: technology used to perform many mathematical tasks, including integration

integration table

a table that lists integration formulas

Numerical Integration

- Approximate the value of a definite integral by using the midpoint and trapezoidal rules.
- Determine the absolute and relative error in using a numerical integration technique.
- Estimate the absolute and relative error using an error-bound formula.
- Recognize when the midpoint and trapezoidal rules over- or underestimate the true value of an integral.
- Use Simpson's rule to approximate the value of a definite integral to a given accuracy.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in closed form (that is, in terms of known functions). Consequently, rather than evaluate definite integrals of these functions directly, we resort to various techniques of **numerical integration** to approximate their values. In this section we explore several of these techniques. In addition, we examine the process of estimating the error in using these techniques.

The Midpoint Rule

Earlier in this text we defined the definite integral of a function over an interval as the limit of Riemann sums. In general, any Riemann sum of a function $f(x)$ over an interval $[a,b]$ may be viewed as an estimate of $\int_a^b f(x)dx$. Recall that a Riemann sum of a function $f(x)$ over an interval $[a,b]$ is obtained by selecting a partition

Equation:

$$P=\{x_0,x_1,x_2,\ldots,x_n\},\ \text{where}\ a=x_0<x_1<x_2<\cdots<x_n=b$$

and a set

Equation:

$$S=\left\{x_1^*,x_2^*,\ldots,x_n^*\right\},\ \text{where}\ x_{i-1}\le x_i^*\le x_i\ \text{for all}\ i.$$

The Riemann sum corresponding to the partition P and the set S is given by $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i=x_i-x_{i-1}$, the length of the *i*th subinterval.

The **midpoint rule** for estimating a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, m_i, of each subinterval in place of x_i^*. Formally, we state a theorem regarding the convergence of the midpoint rule as follows.

Note:

The Midpoint Rule

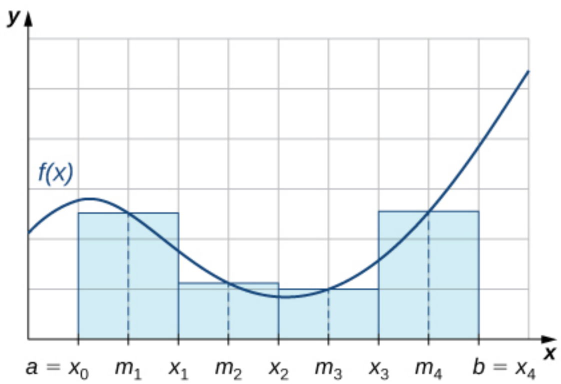
Assume that $f(x)$ is continuous on $[a,b]$. Let *n* be a positive integer and $\Delta x=\frac{b-a}{n}$. If $[a,b]$ is divided into n subintervals, each of length Δx, and m_i is the midpoint of the *i*th subinterval, set

Equation:

$$M_n=\sum_{i=1}^{n} f(m_i)\Delta x.$$

Then $\lim_{n\to\infty} M_n=\int_a^b f(x)dx$.

As we can see in [link], if $f(x) \geq 0$ over $[a, b]$, then $\sum_{i=1}^{n} f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the x-axis over $[a, b]$. The graph shows the rectangles corresponding to M_4 for a nonnegative function over a closed interval $[a, b]$.



The midpoint rule approximates the area between the graph of $f(x)$ and the x-axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

Example:
Exercise:

Problem:
Using the Midpoint Rule with M_4

Use the midpoint rule to estimate $\int_0^1 x^2 dx$ using four subintervals. Compare the result with the actual value of this integral.

Solution:

Each subinterval has length $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Therefore, the subintervals consist of
Equation:

$$\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \text{ and } \left[\frac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,
Equation:

$$M_4 = \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right) + \frac{1}{4} f\left(\frac{7}{8}\right) = \frac{1}{4} \cdot \frac{1}{64} + \frac{1}{4} \cdot \frac{9}{64} + \frac{1}{4} \cdot \frac{25}{64} + \frac{1}{4} \cdot \frac{21}{64} = \frac{21}{64}.$$

Since
Equation:

$$\int_0^1 x^2 dx = \frac{1}{3} \text{ and } \left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052,$$

we see that the midpoint rule produces an estimate that is somewhat close to the actual value of the definite integral.

Example:
Exercise:

Problem:
Using the Midpoint Rule with M_6

Use M_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ on $[1, 4]$.

Solution:

The length of $y = \frac{1}{2}x^2$ on $[1, 4]$ is
Equation:

$$\int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Since $\frac{dy}{dx} = x$, this integral becomes $\int_1^4 \sqrt{1 + x^2} dx$.

If $[1, 4]$ is divided into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$ and the midpoints of the subintervals are $\left\{\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}\right\}$. If we set $f(x) = \sqrt{1 + x^2}$,
Equation:

$$\begin{aligned} M_6 &= \tfrac{1}{2} f\left(\tfrac{5}{4}\right) + \tfrac{1}{2} f\left(\tfrac{7}{4}\right) + \tfrac{1}{2} f\left(\tfrac{9}{4}\right) + \tfrac{1}{2} f\left(\tfrac{11}{4}\right) + \tfrac{1}{2} f\left(\tfrac{13}{4}\right) + \tfrac{1}{2} f\left(\tfrac{15}{4}\right) \\ &\approx \tfrac{1}{2}(1.6008 + 2.0156 + 2.4622 + 2.9262 + 3.4004 + 3.8810) = 8.1431. \end{aligned}$$

Note:
Exercise:

Problem: Use the midpoint rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x} dx$.

Solution:

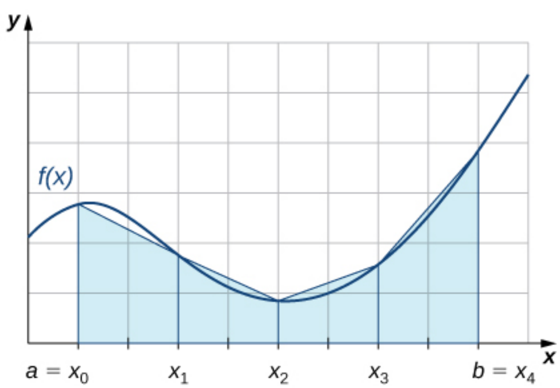
$\frac{24}{35}$

Hint

$\Delta x = \frac{1}{2}$, $m_1 = \frac{5}{4}$, and $m_2 = \frac{7}{4}$.

The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In [link], the area beneath the curve is approximated by trapezoids rather than by rectangles.



Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

The **trapezoidal rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in [link]. We assume that the length of each subinterval is given by Δx. First, recall that the area of a trapezoid with a height of h and bases of length b_1 and b_2 is given by $\text{Area} = \frac{1}{2}h(b_1 + b_2)$. We see that the first trapezoid has a height Δx and parallel bases of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in [link] is

Equation:

$$\frac{1}{2}\Delta x(f(x_0) + f(x_1)).$$

The areas of the remaining three trapezoids are

Equation:

$$\frac{1}{2}\Delta x(f(x_1) + f(x_2)), \frac{1}{2}\Delta x(f(x_2) + f(x_3)), \text{and } \frac{1}{2}\Delta x(f(x_3) + f(x_4)).$$

Consequently,

Equation:

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x(f(x_0) + f(x_1)) + \frac{1}{2}\Delta x(f(x_1) + f(x_2)) + \frac{1}{2}\Delta x(f(x_2) + f(x_3)) + \frac{1}{2}\Delta x(f(x_3) + f(x_4$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

Equation:

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x\,(f(x_0) + 2f\,(x_1) + 2f(x_2) + 2f\,(x_3) + f(x_4)).$$

Generalizing, we formally state the following rule.

Note:
The Trapezoidal Rule
Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive integer and $\Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2 \ldots, x_n\}$. Set
Equation:

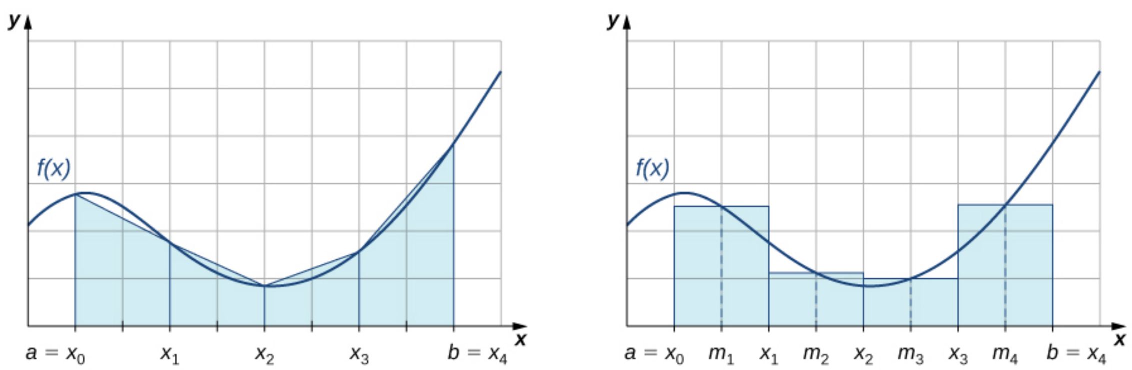
$$T_n = \frac{1}{2}\Delta x \left(f(x_0) + 2f\left(x_1\right) + 2f(x_2) + \cdots + 2f\left(x_{n-1}\right) + f(x_n)\right).$$

Then, $\lim_{n \to +\infty} T_n = \int_a^b f\left(x\right)dx.$

Before continuing, let's make a few observations about the trapezoidal rule. First of all, it is useful to note that
Equation:

$$T_n = \frac{1}{2}(L_n + R_n) \text{ where } L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \text{ and } R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

That is, L_n and R_n approximate the integral using the left-hand and right-hand endpoints of each subinterval, respectively. In addition, a careful examination of [link] leads us to make the following observations about using the trapezoidal rules and midpoint rules to estimate the definite integral of a nonnegative function. The trapezoidal rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value of a definite integral systematically over intervals where the function is concave down. On the other hand, the midpoint rule tends to average out these errors somewhat by partially overestimating and partially underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the midpoint rule tends to be more accurate than the trapezoidal rule.



The trapezoidal rule tends to be less accurate than the midpoint rule.

Example:

Exercise:

Problem:
Using the Trapezoidal Rule

Use the trapezoidal rule to estimate $\int_0^1 x^2 dx$ using four subintervals.

Solution:

The endpoints of the subintervals consist of elements of the set $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Thus,

Equation:

$$\begin{aligned} \int_0^1 x^2 dx &\approx \frac{1}{2} \cdot \frac{1}{4}\left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right) \\ &= \frac{1}{8}\left(0 + 2 \cdot \frac{1}{16} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{9}{16} + 1\right) \\ &= \frac{11}{32}. \end{aligned}$$

Note:
Exercise:

Problem: Use the trapezoidal rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x} dx$.

Solution:

$\frac{17}{24}$

Hint

Set $\Delta x = \frac{1}{2}$. The endpoints of the subintervals are the elements of the set $P = \left\{1, \frac{3}{2}, 2\right\}$.

Absolute and Relative Error

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define **absolute error** and **relative error**.

Note:
Definition
If B is our estimate of some quantity having an actual value of A, then the absolute error is given by $|A - B|$. The relative error is the error as a percentage of the absolute value and is given by $\left|\frac{A-B}{A}\right| = \left|\frac{A-B}{A}\right| \cdot 100\%$.

Example:
Exercise:

Problem:

Calculating Error in the Midpoint Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2 dx$ using the midpoint rule, found in [link].

Solution:

The calculated value is $\int_0^1 x^2 dx = \frac{1}{3}$ and our estimate from the example is $M_4 = \frac{21}{64}$. Thus, the absolute error is given by $\left|\left(\frac{1}{3}\right) - \left(\frac{21}{64}\right)\right| = \frac{1}{192} \approx 0.0052$. The relative error is

Equation:

$$\frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

Example:

Exercise:

Problem:

Calculating Error in the Trapezoidal Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2 dx$ using the trapezoidal rule, found in [link].

Solution:

The calculated value is $\int_0^1 x^2 dx = \frac{1}{3}$ and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{11}{32}\right| = \frac{1}{96} \approx 0.0104$. The relative error is given by

Equation:

$$\frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

Note:

Exercise:

Problem:

In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x} dx$ to be $\frac{24}{35}$ using T_2. The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

Solution:

0.0074, 1.1%

Hint

Use the previous examples as a guide.

In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.

Note:
Error Bounds for the Midpoint and Trapezoidal Rules
Let $f(x)$ be a continuous function over $[a, b]$, having a second derivative $f\prime\prime(x)$ over this interval. If M is the maximum value of $|f\prime\prime(x)|$ over $[a, b]$, then the upper bounds for the error in using M_n and T_n to estimate $\int_a^b f\,(x)dx$ are
Equation:

$$\text{Error in } M_n \leq \frac{M(b-a)^3}{24n^2}$$

and
Equation:

$$\text{Error in } T_n \leq \frac{M(b-a)^3}{12n^2}.$$

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

Example:
Exercise:

Problem:
Determining the Number of Intervals to Use

What value of n should be used to guarantee that an estimate of $\int_0^1 e^{x^2} dx$ is accurate to within 0.01 if we use the midpoint rule?

Solution:

We begin by determining the value of M, the maximum value of $|f\prime\prime(x)|$ over $[0, 1]$ for $f\,(x) = e^{x^2}$. Since $f'\,(x) = 2xe^{x^2}$, we have
Equation:

$$f''\,(x) = 2e^{x^2} + 4x^2e^{x^2}.$$

Thus,

Equation:

$$|f\prime\prime(x)| = 2e^{x^2}\left(1+2x^2\right) \le 2 \cdot e \cdot 3 = 6e.$$

From the error-bound [link], we have

Equation:

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \le \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2}.$$

Now we solve the following inequality for n:

Equation:

$$\frac{6e}{24n^2} \le 0.01.$$

Thus, $n \ge \sqrt{\frac{600e}{24}} \approx 8.24$. Since n must be an integer satisfying this inequality, a choice of $n = 9$ would guarantee that $\left|\int_0^1 e^{x^2}dx - M_n\right| < 0.01.$

Analysis

We might have been tempted to round 8.24 down and choose $n = 8$, but this would be incorrect because we must have an integer greater than or equal to 8.24. We need to keep in mind that the error estimates provide an upper bound only for the error. The actual estimate may, in fact, be a much better approximation than is indicated by the error bound.

Note:

Exercise:

Problem: Use [link] to find an upper bound for the error in using M_4 to estimate $\int_0^1 x^2 dx.$

Solution:

$\frac{1}{192}$

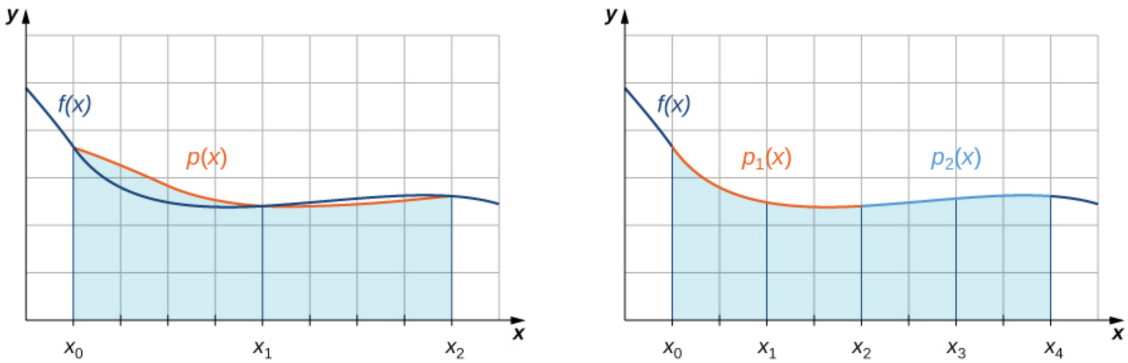
Hint

$f\prime\prime(x) = 2$, so $M = 2.$

Simpson's Rule

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With **Simpson's rule**, we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate $\int_{x_0}^{x_2} f(x)dx$ with $\int_{x_0}^{x_2} p(x)dx$, where

$p(x) = Ax^2 + Bx + C$ is the quadratic function passing through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ ([link]). Over the next pair of subintervals we approximate $\int_{x_2}^{x_4} f(x)dx$ with the integral of another quadratic function passing through $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. This process is continued with each successive pair of subintervals.



With Simpson's rule, we approximate a definite integral by integrating a piecewise quadratic function.

To understand the formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

Equation:

$$\begin{aligned} f(x_0) &= p(x_0) = A{x_0}^2 + Bx_0 + C \\ f(x_1) &= p(x_1) = A{x_1}^2 + Bx_1 + C \\ f(x_2) &= p(x_2) = A{x_2}^2 + Bx_2 + C \end{aligned}$$

$x_2 - x_0 = 2\Delta x$, where Δx is the length of a subinterval.

Equation:

$$x_2 + x_0 = 2x_1, \text{ since } x_1 = \frac{(x_2 + x_0)}{2}.$$

Thus,

Equation:

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)dx &\approx \int_{x_0}^{x_2} p(x)dx \\
&= \int_{x_0}^{x_2} (Ax^2 + Bx + C)dx \\
&= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big|_{x_0}^{x_2} && \text{Find the antiderivative.} \\
&= \frac{A}{3}\left(x_2{}^3 - x_0{}^3\right) + \frac{B}{2}\left(x_2{}^2 - x_0{}^2\right) + C(x_2 - x_0) && \text{Evaluate the antiderivati} \\
&= \frac{A}{3}(x_2 - x_0)\left(x_2{}^2 + x_2x_0 + x_0{}^2\right) \\
&+ \frac{B}{2}(x_2 - x_0)(x_2 + x_0) + C(x_2 - x_0) \\
&= \frac{x_2 - x_0}{6}\left(2A\left(x_2{}^2 + x_2x_0 + x_0{}^2\right) + 3B(x_2 + x_0) + 6C\right) && \text{Factor out } \frac{x_2 - x_0}{6}. \\
&= \frac{\Delta x}{3}\left(\left(Ax_2{}^2 + Bx_2 + C\right) + \left(Ax_0{}^2 + Bx_0 + C\right)\right. \\
&+ A\left(x_2{}^2 + 2x_2x_0 + x_0{}^2\right) + 2B(x_2 + x_0) + 4C) \\
&= \frac{\Delta x}{3}\left(f(x_2) + f(x_0) + A(x_2 + x_0)^2 + 2B(x_2 + x_0) + 4C\right) && \text{Rearrange the terms.} \\
& && \text{Factor and substitute.} \\
& && f(x_2) = Ax_0{}^2 + Bx_0 + C \\
& && f(x_0) = Ax_0{}^2 + Bx_0 + C \\
&= \frac{\Delta x}{3}\left(f(x_2) + f(x_0) + A(2x_1)^2 + 2B(2x_1) + 4C\right) && \text{Substitute } x_2 + x_0 = 2x_1. \\
&= \frac{\Delta x}{3}(f(x_2) + 4f(x_1) + f(x_0)). && \text{Expand and substitute } f(x_1) = Ax_1{}^2 + Bx_1 + .
\end{aligned}$$

If we approximate $\int_{x_2}^{x_4} f(x)dx$ using the same method, we see that we have

Equation:

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{\Delta x}{3}(f(x_4) + 4f(x_3) + f(x_2)).$$

Combining these two approximations, we get

Equation:

$$\int_{x_0}^{x_4} f(x)dx = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)).$$

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

Note:

Simpson’s Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive even integer and $\Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Set

Equation:

$$S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).$$

Then,
Equation:

$$\lim_{n \to +\infty} S_n = \int_a^b f(x)dx.$$

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson's rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that $S_{2n} = \left(\frac{2}{3}\right)M_n + \left(\frac{1}{3}\right)T_n$.

It is also possible to put a bound on the error when using Simpson's rule to approximate a definite integral. The bound in the error is given by the following rule:

Note:
Rule: Error Bound for Simpson's Rule
Let $f(x)$ be a continuous function over $[a, b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a, b]$, then the upper bound for the error in using S_n to estimate $\int_a^b f(x)dx$ is given by
Equation:

$$\text{Error in } S_n \leq \frac{M(b-a)^5}{180n^4}.$$

Example:
Exercise:

Problem:
Applying Simpson's Rule 1

Use S_2 to approximate $\int_0^1 x^3 dx$. Estimate a bound for the error in S_2.

Solution:

Since $[0, 1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The endpoints of these subintervals are $\left\{0, \frac{1}{2}, 1\right\}$. If we set $f(x) = x^3$, then

$S_4 = \frac{1}{3} \cdot \frac{1}{2}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right) = \frac{1}{6}\left(0 + 4 \cdot \frac{1}{8} + 1\right) = \frac{1}{4}$. Since $f^{(4)}(x) = 0$ and consequently $M = 0$, we see that
Equation:

$$\text{Error in } S_2 \leq \frac{0(1)^5}{180 \cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3 dx = \frac{1}{4}$.

Example:
Exercise:

Problem:
Applying Simpson's Rule 2

Use S_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ over $[1, 4]$.

Solution:

The length of $y = \frac{1}{2}x^2$ over $[1, 4]$ is $\int_1^4 \sqrt{1+x^2}\,dx$. If we divide $[1, 4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$. Setting $f(x) = \sqrt{1+x^2}$,
Equation:

$$S_6 = \frac{1}{3} \cdot \frac{1}{2}\left(f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right).$$

After substituting, we have
Equation:

$$\begin{aligned} S_6 &= \tfrac{1}{6}(1.4142 + 4 \cdot 1.80278 + 2 \cdot 2.23607 + 4 \cdot 2.69258 + 2 \cdot 3.16228 + 4 \cdot 3.64005 + 4.12311) \\ &\approx 8.14594. \end{aligned}$$

Note:
Exercise:

Problem: Use S_2 to estimate $\int_1^2 \frac{1}{x}dx$.

Solution:

$\frac{25}{36}$

Hint

$S_2 = \left(\frac{1}{3}\Delta x(f(x_0) + 4f(x_1) + f(x_2)\right)$

Key Concepts

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.

- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson's rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson's rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

Key Equations

- **Midpoint rule**
  $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$
- **Trapezoidal rule**
  $T_n = \frac{1}{2}\Delta x \left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$
- **Simpson's rule**
  $S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f($
- **Error bound for midpoint rule**
  Error in $M_n \le \frac{M(b-a)^3}{24n^2}$
- **Error bound for trapezoidal rule**
  Error in $T_n \le \frac{M(b-a)^3}{12n^2}$
- **Error bound for Simpson's rule**
  Error in $S_n \le \frac{M(b-a)^5}{180n^4}$

Approximate the following integrals using either the midpoint rule, trapezoidal rule, or Simpson's rule as indicated. (Round answers to three decimal places.)

Exercise:

Problem: $\int_1^2 \frac{dx}{x}$; trapezoidal rule; $n = 5$

Solution:

0.696

Exercise:

Problem: $\int_0^3 \sqrt{4 + x^3}\,dx$; trapezoidal rule; $n = 6$

Exercise:

Problem: $\int_0^3 \sqrt{4 + x^3}\,dx$; Simpson's rule; $n = 3$

Solution:

9.279

Exercise:

Problem: $\int_0^{12} x^2 dx$; midpoint rule; $n = 6$

Exercise:

Problem: $\int_0^1 \sin^2(\pi x)dx$; midpoint rule; $n = 3$

Solution:

0.5000

Exercise:

Problem: Use the midpoint rule with eight subdivisions to estimate $\int_2^4 x^2 dx$.

Exercise:

Problem: Use the trapezoidal rule with four subdivisions to estimate $\int_2^4 x^2 dx$.

Solution:

$T_4 = 18.75$

Exercise:

Problem:

Find the exact value of $\int_2^4 x^2 dx$. Find the error of approximation between the exact value and the value calculated using the trapezoidal rule with four subdivisions. Draw a graph to illustrate.

Approximate the integral to three decimal places using the indicated rule.

Exercise:

Problem: $\int_0^1 \sin^2(\pi x)dx$; trapezoidal rule; $n = 6$

Solution:

0.500

Exercise:

Problem: $\int_0^3 \frac{1}{1+x^3} dx$; trapezoidal rule; $n = 6$

Exercise:

Problem: $\int_0^3 \frac{1}{1+x^3} dx$; Simpson's rule; $n = 3$

Solution:

1.1614

Exercise:

Problem: $\int_0^{0.8} e^{-x^2} dx$; trapezoidal rule; $n = 4$

Exercise:

Problem: $\int_0^{0.8} e^{-x^2} dx$; Simpson's rule; $n = 4$

Solution:

0.6577

Exercise:

Problem: $\int_0^{0.4} \sin(x^2) dx$; trapezoidal rule; $n = 4$

Exercise:

Problem: $\int_0^{0.4} \sin(x^2) dx$; Simpson's rule; $n = 4$

Solution:

0.0213

Exercise:

Problem: $\int_{0.1}^{0.5} \frac{\cos x}{x} dx$; trapezoidal rule; $n = 4$

Exercise:

Problem: $\int_{0.1}^{0.5} \frac{\cos x}{x} dx$; Simpson's rule; $n = 4$

Solution:

1.5629

Exercise:

Problem:

Evaluate $\int_0^1 \frac{dx}{1 + x^2}$ exactly and show that the result is $\pi/4$. Then, find the approximate value of the integral using the trapezoidal rule with $n = 4$ subdivisions. Use the result to approximate the value of π.

Exercise:

Problem: Approximate $\int_2^4 \frac{1}{\ln x} dx$ using the midpoint rule with four subdivisions to four decimal places.

Solution:

1.9133

Exercise:

Problem: Approximate $\int_2^4 \frac{1}{\ln x} dx$ using the trapezoidal rule with eight subdivisions to four decimal places.

Exercise:

Problem: Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3 dx$ to four decimal places.

Solution:

$\text{T}(4) = 0.1088$

Exercise:

Problem:

Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3 dx$. Compare this value with the exact value and find the error estimate.

Exercise:

Problem: Using Simpson's rule with four subdivisions, find $\int_0^{\pi/2} \cos(x) dx$.

Solution:

1.0

Exercise:

Problem:

Show that the exact value of $\int_0^1 xe^{-x} dx = 1 - \frac{2}{e}$. Find the absolute error if you approximate the integral using the midpoint rule with 16 subdivisions.

Exercise:

Problem:

Given $\int_0^1 xe^{-x} dx = 1 - \frac{2}{e}$, use the trapezoidal rule with 16 subdivisions to approximate the integral and find the absolute error.

Solution:

Approximate error is 0.000325.

Exercise:

Problem:

Find an upper bound for the error in estimating $\int_0^3 (5x + 4) dx$ using the trapezoidal rule with six steps.

Exercise:

Problem:

Find an upper bound for the error in estimating $\int_4^5 \frac{1}{(x-1)^2}dx$ using the trapezoidal rule with seven subdivisions.

Solution:

$\frac{1}{7938}$

Exercise:

Problem:

Find an upper bound for the error in estimating $\int_0^3 (6x^2-1)dx$ using Simpson's rule with $n=10$ steps.

Exercise:

Problem:

Find an upper bound for the error in estimating $\int_2^5 \frac{1}{x-1}dx$ using Simpson's rule with $n=10$ steps.

Solution:

$\frac{81}{25{,}000}$

Exercise:

Problem:

Find an upper bound for the error in estimating $\int_0^\pi 2x\cos(x)dx$ using Simpson's rule with four steps.

Exercise:

Problem:

Estimate the minimum number of subintervals needed to approximate the integral $\int_1^4 \left(5x^2+8\right)dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

Solution:

475

Exercise:

Problem:

Determine a value of *n* such that the trapezoidal rule will approximate $\int_0^1 \sqrt{1+x^2}\,dx$ with an error of no more than 0.01.

Exercise:

Problem:

Estimate the minimum number of subintervals needed to approximate the integral $\int_2^3 (2x^3 + 4x)\,dx$ with an error of magnitude less than 0.0001 using the trapezoidal rule.

Solution:

174

Exercise:

Problem:

Estimate the minimum number of subintervals needed to approximate the integral $\int_3^4 \frac{1}{(x-1)^2}\,dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

Exercise:

Problem:

Use Simpson's rule with four subdivisions to approximate the area under the probability density function $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ from $x = 0$ to $x = 0.4$.

Solution:

0.1544

Exercise:

Problem:

Use Simpson's rule with $n = 14$ to approximate (to three decimal places) the area of the region bounded by the graphs of $y = 0$, $x = 0$, and $x = \pi/2$.

Exercise:

Problem:

The length of one arch of the curve $y = 3\sin(2x)$ is given by $L = \int_0^{\pi/2} \sqrt{1 + 36\cos^2(2x)}\,dx$. Estimate L using the trapezoidal rule with $n = 6$.

Solution:

6.2807

Exercise:

Problem:

The length of the ellipse $x = a\cos(t), y = b\sin(t), 0 \le t \le 2\pi$ is given by $L = 4a\int_0^{\pi/2} \sqrt{1 - e^2\cos^2(t)}\,dt$, where e is the eccentricity of the ellipse. Use Simpson's rule with $n = 6$ subdivisions to estimate the length of the ellipse when $a = 2$ and $e = 1/3$.

Exercise:

Problem:

Estimate the area of the surface generated by revolving the curve $y = \cos(2x), 0 \le x \le \frac{\pi}{4}$ about the *x*-axis. Use the trapezoidal rule with six subdivisions.

Solution:

4.606

Exercise:

Problem:

Estimate the area of the surface generated by revolving the curve $y = 2x^2$, $0 \le x \le 3$ about the *x*-axis. Use Simpson's rule with $n = 6$.

Exercise:

Problem:

The growth rate of a certain tree (in feet) is given by $y = \frac{2}{t+1} + e^{-t^2/2}$, where *t* is time in years. Estimate the growth of the tree through the end of the second year by using Simpson's rule, using two subintervals. (Round the answer to the nearest hundredth.)

Solution:

3.41 ft

Exercise:

Problem:

[T] Use a calculator to approximate $\int_0^1 \sin(\pi x)dx$ using the midpoint rule with 25 subdivisions. Compute the relative error of approximation.

Exercise:

Problem:

[T] Given $\int_1^5 (3x^2 - 2x)dx = 100$, approximate the value of this integral using the midpoint rule with 16 subdivisions and determine the absolute error.

Solution:

$T_{16} = 100.125$; absolute error = 0.125

Exercise:

Problem:

Given that we know the Fundamental Theorem of Calculus, why would we want to develop numerical methods for definite integrals?

Exercise:

Problem:

The table represents the coordinates (x, y) that give the boundary of a lot. The units of measurement are meters. Use the trapezoidal rule to estimate the number of square meters of land that is in this lot.

x	*y*	*x*	*y*
0	125	600	95
100	125	700	88
200	120	800	75
300	112	900	35
400	90	1000	0
500	90		

Solution:

about 89,250 m^2

Exercise:

Problem:

Choose the correct answer. When Simpson's rule is used to approximate the definite integral, it is necessary that the number of partitions be____

a. an even number
b. odd number
c. either an even or an odd number
d. a multiple of 4

Exercise:

Problem: The "Simpson" sum is based on the area under a ____.

Solution:

parabola

Exercise:

Problem: The error formula for Simpson's rule depends on___.

a. $f(x)$
b. $f'(x)$
c. $f^{(4)}(x)$
d. the number of steps

Glossary

absolute error
: if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A - B|$

midpoint rule

a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where m_i is the midpoint of the *i*th subinterval to approximate $\int_a^b f(x)dx$

numerical integration
the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

relative error
error as a percentage of the absolute value, given by $\left|\frac{A-B}{A}\right| = \left|\frac{A-B}{A}\right| \cdot 100\%$

Simpson's rule
a rule that approximates $\int_a^b f(x)dx$ using the integrals of a piecewise quadratic function. The approximation S_n to $\int_a^b f(x)dx$ is given by $S_n = \frac{\Delta x}{3}\begin{pmatrix} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) \\ + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \end{pmatrix}$

trapezoidal rule a rule that approximates $\int_a^b f(x)dx$ using trapezoids

Improper Integrals

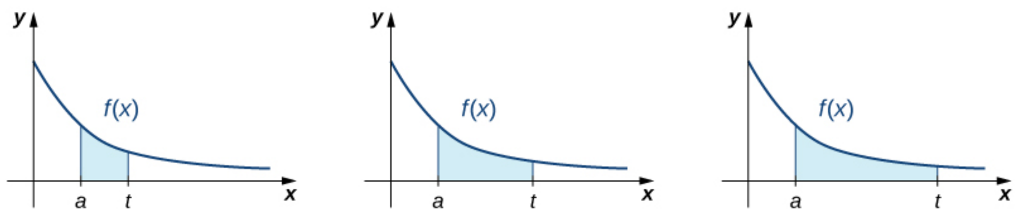
- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

Is the area between the graph of $f(x) = \frac{1}{x}$ and the *x*-axis over the interval $[1, +\infty)$ finite or infinite? If this same region is revolved about the *x*-axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the *x*-axis is finite.

In this section, we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

Integrating over an Infinite Interval

How should we go about defining an integral of the type $\int_a^{+\infty} f(x)dx$? We can integrate $\int_a^t f(x)dx$ for any value of t, so it is reasonable to look at the behavior of this integral as we substitute larger values of t. [link] shows that $\int_a^t f(x)dx$ may be interpreted as area for various values of t. In other words, we may define an improper integral as a limit, taken as one of the limits of integration increases or decreases without bound.



To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

Note:
Definition

1. Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then
   Equation:
   $$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx,$$
   provided this limit exists.
2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then
   Equation:
   $$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx,$$

provided this limit exists.
In each case, if the limit exists, then the **improper integral** is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then
Equation:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{+\infty} f(x)dx,$$

provided that $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{+\infty} f(x)dx$ both converge. If either of these two integrals diverge, then $\int_{-\infty}^{+\infty} f(x)dx$ diverges. (It can be shown that, in fact, $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx$ for any value of a.)

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite?

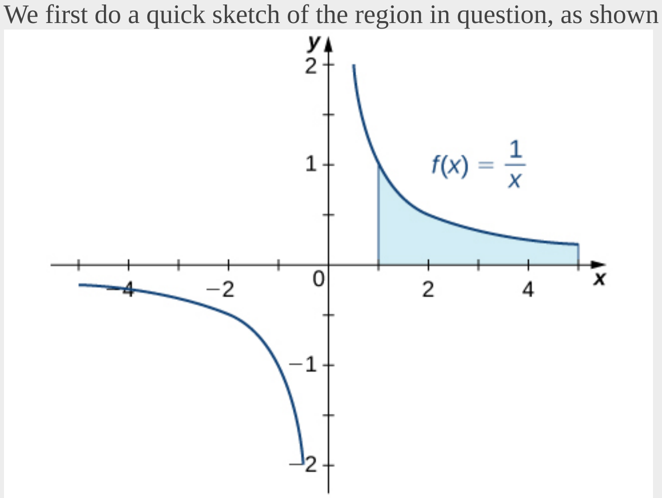
Example:
Exercise:

Problem:
Finding an Area

Determine whether the area between the graph of $f(x) = \frac{1}{x}$ and the *x*-axis over the interval $[1, +\infty)$ is finite or infinite.

Solution:

We first do a quick sketch of the region in question, as shown in the following graph.



We can find the area between the curve $f(x) = 1/x$ and the *x*-axis on an infinite interval.

We can see that the area of this region is given by $A = \int_1^\infty \frac{1}{x} dx$. Then we have

Equation:

$$\begin{aligned} A &= \int_1^\infty \frac{1}{x} dx \\ &= \lim_{t \to +\infty} \int_1^t \frac{1}{x} dx && \text{Rewrite the improper integral as a limit.} \\ &= \lim_{t \to +\infty} \ln|x| \Big|_1^t && \text{Find the antiderivative.} \\ &= \lim_{t \to +\infty} (\ln|t| - \ln 1) && \text{Evaluate the antiderivative.} \\ &= +\infty. && \text{Evaluate the limit.} \end{aligned}$$

Since the improper integral diverges to $+\infty$, the area of the region is infinite.

Example:
Exercise:

Problem:
Finding a Volume

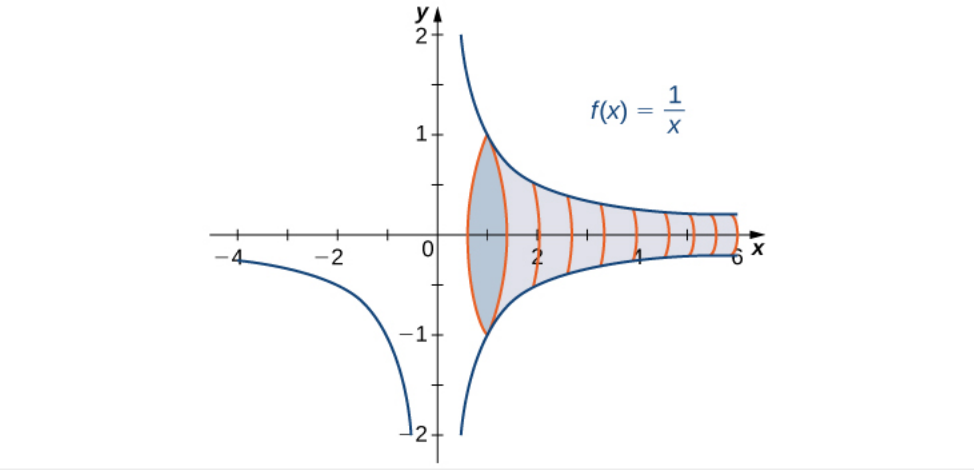
Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ about the x-axis.

Solution:

The solid is shown in [link]. Using the disk method, we see that the volume V is

Equation:

$$V = \pi \int_1^{+\infty} \frac{1}{x^2} dx.$$



The solid of revolution can be generated by rotating an infinite area about the *x*-axis.

Then we have

Equation:

$$\begin{aligned} V &= \pi \int_1^{+\infty} \frac{1}{x^2} dx \\ &= \pi \lim_{t \to +\infty} \int_1^t \frac{1}{x^2} dx && \text{Rewrite as a limit.} \\ &= \pi \lim_{t \to +\infty} -\frac{1}{x}\Big|_1^t && \text{Find the antiderivative.} \\ &= \pi \lim_{t \to +\infty} \left(-\frac{1}{t} + 1\right) && \text{Evaluate the antiderivative.} \\ &= \pi. \end{aligned}$$

The improper integral converges to π. Therefore, the volume of the solid of revolution is π.

In conclusion, although the area of the region between the *x*-axis and the graph of $f(x) = 1/x$ over the interval $[1, +\infty)$ is infinite, the volume of the solid generated by revolving this region about the *x*-axis is finite. The solid generated is known as *Gabriel's Horn*.

Note:
Visit this website to read more about Gabriel's Horn.

Example:
Exercise:

Problem:
Chapter Opener: Traffic Accidents in a City

(credit: modification of work by David McKelvey, Flickr)

In the chapter opener, we stated the following problem: Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the 8-month interval without an accident a result of chance?

Probability theory tells us that if the average time between events is k, the probability that X, the time between events, is between a and b is given by

Equation:

$$P\left(a \le x \le b\right) = \int_a^b f\left(x\right)dx \text{ where } f\left(x\right) = \begin{cases} 0 \text{ if } x < 0 \\ ke^{-kx} \text{if } x \ge 0 \end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that X, the time between accidents, is between a and b is given by

Equation:

$$P\left(a \le x \le b\right) = \int_a^b f\left(x\right)dx \text{ where } f\left(x\right) = \begin{cases} 0 \text{ if } x < 0 \\ 3e^{-3x} \text{if } x \ge 0 \end{cases}.$$

To answer the question, we must compute $P\left(X \ge 8\right) = \int_8^{+\infty} 3e^{-3x}dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

Solution:

We need to calculate the probability as an improper integral:

Equation:

$$\begin{aligned} P\left(X \ge 8\right) &= \int_8^{+\infty} 3e^{-3x}dx \\ &= \lim_{t \to +\infty} \int_8^t 3e^{-3x}dx \\ &= \lim_{t \to +\infty} -e^{-3x}\Big|_8^t \\ &= \lim_{t \to +\infty} \left(-e^{-3t} + e^{-24}\right) \\ &\approx 3.8 \times 10^{-11}. \end{aligned}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 months under the initial conditions. Since this value is very, very small, it is reasonable to conclude the changes were effective.

Example:
Exercise:

Problem:
Evaluating an Improper Integral over an Infinite Interval

Evaluate $\int_{-\infty}^0 \frac{1}{x^2+4}dx$. State whether the improper integral converges or diverges.

Solution:

Begin by rewriting $\int_{-\infty}^{0} \frac{1}{x^2+4} dx$ as a limit using [link] from the definition. Thus,

Equation:

$$\begin{aligned} \int_{-\infty}^{0} \frac{1}{x^2+4} dx &= \lim_{x \to -\infty} \int_{t}^{0} \frac{1}{x^2+4} dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to -\infty} \tfrac{1}{2} \tan^{-1} \tfrac{x}{2} \Big|_{t}^{0} && \text{Find the antiderivative.} \\ &= \tfrac{1}{2} \lim_{t \to -\infty} \left(\tan^{-1} 0 - \tan^{-1} \tfrac{t}{2}\right) && \text{Evaluate the antiderivative.} \\ &= \tfrac{\pi}{4}. && \text{Evaluate the limit and simplify.} \end{aligned}$$

The improper integral converges to $\frac{\pi}{4}$.

Example:
Exercise:

Problem:
Evaluating an Improper Integral on $(-\infty, +\infty)$

Evaluate $\int_{-\infty}^{+\infty} xe^x dx$. State whether the improper integral converges or diverges.

Solution:

Start by splitting up the integral:

Equation:

$$\int_{-\infty}^{+\infty} xe^x dx = \int_{-\infty}^{0} xe^x dx + \int_{0}^{+\infty} xe^x dx.$$

If either $\int_{-\infty}^{0} xe^x dx$ or $\int_{0}^{+\infty} xe^x dx$ diverges, then $\int_{-\infty}^{+\infty} xe^x dx$ diverges. Compute each integral separately. For the first integral,

Equation:

$$\begin{aligned} \int_{-\infty}^{0} xe^x dx &= \lim_{t\to-\infty} \int_t^0 xe^x dx && \text{Rewrite as a limit.} \\ &= \lim_{t\to-\infty} (xe^x - e^x)\Big|_t^0 && \text{Use integration by parts to find the antiderivative. (Here } u = x \text{ and } dv = e^x.\text{)} \\ &= \lim_{t\to-\infty} \left(-1 - te^t + e^t\right) && \text{Evaluate the antiderivative.} \\ &= -1. && \text{Evaluate the limit. } \textit{Note: } \lim_{t\to-\infty} te^t \text{ is indeterminate of the form } 0\cdot\infty. \text{ Thus, } \\ & && \lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \tfrac{t}{e^{-t}} = \lim_{t\to-\infty} \tfrac{-1}{e^{-t}} = \lim_{t\to-\infty} -e^t = 0 \text{ by L'Hôpital's Rule.} \end{aligned}$$

The first improper integral converges. For the second integral,

Equation:

$$\begin{aligned} \int_0^{+\infty} xe^x dx &= \lim_{t\to+\infty} \int_0^t xe^x dx && \text{Rewrite as a limit.} \\ &= \lim_{t\to+\infty} (xe^x - e^x)\Big|_0^t && \text{Find the antiderivative.} \\ &= \lim_{t\to+\infty} \left(te^t - e^t + 1\right) && \text{Evaluate the antiderivative.} \\ &= \lim_{t\to+\infty} \left((t-1)e^t + 1\right) && \text{Rewrite. (} te^t - e^t \text{ is indeterminate.)} \\ &= +\infty. && \text{Evaluate the limit.} \end{aligned}$$

Thus, $\int_0^{+\infty} xe^x dx$ diverges. Since this integral diverges, $\int_{-\infty}^{+\infty} xe^x dx$ diverges as well.

Note:
Exercise:

Problem: Evaluate $\int_{-3}^{+\infty} e^{-x} dx$. State whether the improper integral converges or diverges.

Solution:

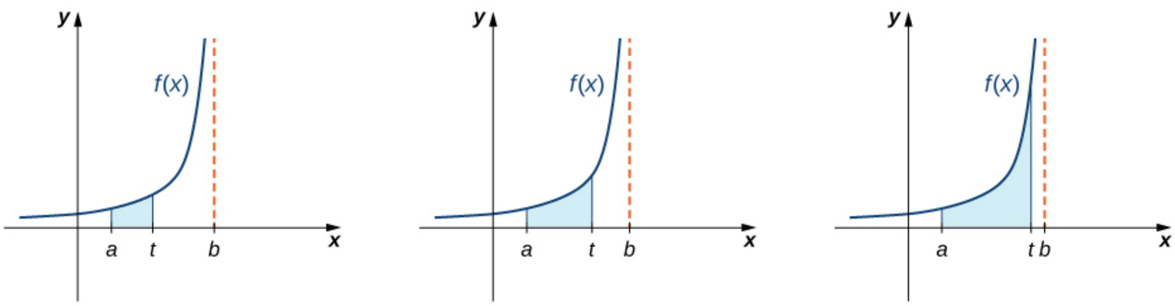
e^3, converges

Hint

$\int_{-3}^{+\infty} e^{-x} dx = \lim_{t\to+\infty} \int_{-3}^{t} e^{-x} dx$

Integrating a Discontinuous Integrand

Now let’s examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)dx$, where $f(x)$ is continuous over $[a, b)$ and discontinuous at b. Since the function $f(x)$ is continuous over $[a, t]$ for all values of t satisfying $a < t < b$, the integral $\int_a^t f(x)dx$ is defined for all such values of t. Thus, it makes sense to consider the values of $\int_a^t f(x)dx$ as t approaches b for $a < t < b$. That is, we define $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$, provided this limit exists. [link] illustrates $\int_a^t f(x)dx$ as areas of regions for values of t approaching b.



As t approaches b from the left, the value of the area from a to t approaches the area from a to b.

We use a similar approach to define $\int_a^b f(x)dx$, where $f(x)$ is continuous over $(a, b]$ and discontinuous at a. We now proceed with a formal definition.

Note:
Definition

1. Let $f(x)$ be continuous over $[a, b)$. Then,
   Equation:
   $$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx.$$
2. Let $f(x)$ be continuous over $(a, b]$. Then,
   Equation:
   $$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx.$$

   In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.
3. If $f(x)$ is continuous over $[a, b]$ except at a point c in (a, b), then
   Equation:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

provided both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)dx$ diverges.

The following examples demonstrate the application of this definition.

Example:
Exercise:

Problem:
Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}dx$, if possible. State whether the integral converges or diverges.

Solution:

The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using [link] from the definition, rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}dx$ as a limit:

Equation:

$$\begin{aligned}
\int_0^4 \frac{1}{\sqrt{4-x}}dx &= \lim_{t\to 4^-}\int_0^t \frac{1}{\sqrt{4-x}}dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to 4^-}\left(-2\sqrt{4-x}\right)\Big|_0^t && \text{Find the antiderivative.}\\
&= \lim_{t\to 4^-}\left(-2\sqrt{4-t}+4\right) && \text{Evaluate the antiderivative.}\\
&= 4. && \text{Evaluate the limit.}
\end{aligned}$$

The improper integral converges.

Example:
Exercise:

Problem:
Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x\ln x\,dx$. State whether the integral converges or diverges.

Solution:

Since $f(x) = x \ln x$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite the integral in limit form using [link]:
Equation:

$$\begin{aligned} \int_0^2 x \ln x\, dx &= \lim_{t\to 0^+} \int_t^2 x \ln x\, dx && \text{Rewrite as a limit.} \\ &= \lim_{t\to 0^+} \left(\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2\right)\Big|_t^2 && \text{Evaluate } \int x \ln x\, dx \text{ using integration by parts with } u = \ln x \text{ and } dv = x. \\ &= \lim_{t\to 0^+} \left(2\ln 2 - 1 - \tfrac{1}{2}t^2 \ln t + \tfrac{1}{4}t^2\right). && \text{Evaluate the antiderivative.} \\ &= 2\ln 2 - 1. && \text{Evaluate the limit. } \lim_{t\to 0^+} t^2 \ln t \text{ is indeterminate. To evaluate it, rewrite as a quotient and apply L'Hôpital's rule.} \end{aligned}$$

The improper integral converges.

Example:
Exercise:

Problem:
Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^{1} \frac{1}{x^3} dx$. State whether the improper integral converges or diverges.

Solution:

Since $f(x) = 1/x^3$ is discontinuous at zero, using [link], we can write
Equation:

$$\int_{-1}^{1} \frac{1}{x^3} dx = \int_{-1}^{0} \frac{1}{x^3} dx + \int_{0}^{1} \frac{1}{x^3} dx.$$

If either of the two integrals diverges, then the original integral diverges. Begin with $\int_{-1}^{0} \frac{1}{x^3} dx$:

Equation:

$$\begin{aligned} \int_{-1}^{0} \frac{1}{x^3} dx &= \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^3} dx && \text{Rewrite as a limit.} \\ &= \lim_{t\to 0^-} \left(-\tfrac{1}{2x^2}\right)\Big|_{-1}^{t} && \text{Find the antiderivative.} \\ &= \lim_{t\to 0^-} \left(-\tfrac{1}{2t^2} + \tfrac{1}{2}\right) && \text{Evaluate the antiderivative.} \\ &= +\infty. && \text{Evaluate the limit.} \end{aligned}$$

Therefore, $\int_{-1}^{0} \frac{1}{x^3} dx$ diverges. Since $\int_{-1}^{0} \frac{1}{x^3} dx$ diverges, $\int_{-1}^{1} \frac{1}{x^3} dx$ diverges.

Note:
Exercise:

Problem: Evaluate $\int_{0}^{2} \frac{1}{x} dx$. State whether the integral converges or diverges.

Solution:

$+\infty$, diverges

Hint

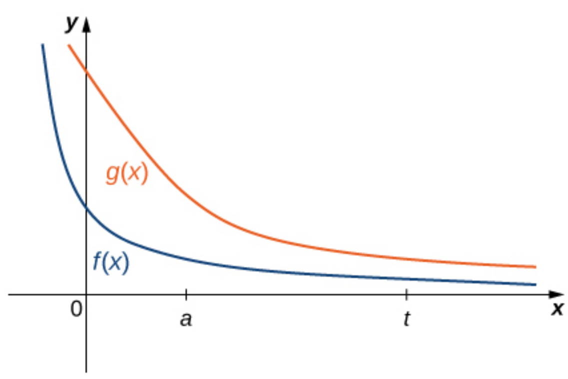
Write $\int_{0}^{2} \frac{1}{x} dx$ in limit form using [link].

A Comparison Theorem

It is not always easy or even possible to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge a$ ([link]). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

Equation:

$$0 \le \int_{a}^{t} f(x) dx \le \int_{a}^{t} g(x) dx \text{ for } t \ge a.$$



If $0 \le f(x) \le g(x)$ for $x \ge a$, then for $t \ge a$, $\int_{a}^{t} f(x) dx \le \int_{a}^{t} g(x) dx$.

Thus, if
Equation:

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx = +\infty,$$

then

$\int_a^{+\infty} g(x)dx = \lim_{t\to+\infty} \int_a^t g(x)dx = +\infty$ as well. That is, if the area of the region between the graph of $f(x)$ and the *x*-axis over $[a, +\infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the *x*-axis over $[a, +\infty)$ is infinite too.

On the other hand, if

$\int_a^{+\infty} g(x)dx = \lim_{t\to+\infty} \int_a^t g(x)dx = L$ for some real number L, then

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx$ must converge to some value less than or equal to L, since $\int_a^t f(x)dx$ increases as t increases and $\int_a^t f(x)dx \le L$ for all $t \ge a$.

If the area of the region between the graph of $g(x)$ and the *x*-axis over $[a, +\infty)$ is finite, then the area of the region between the graph of $f(x)$ and the *x*-axis over $[a, +\infty)$ is also finite.

These conclusions are summarized in the following theorem.

Note:
A Comparison Theorem
Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If $\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx = +\infty$, then $\int_a^{+\infty} g(x)dx = \lim_{t\to+\infty} \int_a^t g(x)dx = +\infty$.
ii. If $\int_a^{+\infty} g(x)dx = \lim_{t\to+\infty} \int_a^t g(x)dx = L$, where L is a real number, then $\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx = M$ for some real number $M \le L$.

Example:
Exercise:

Problem:
Applying the Comparison Theorem

Use a comparison to show that $\int_1^{+\infty} \frac{1}{xe^x}dx$ converges.

Solution:

We can see that
Equation:

$$0 \leq \frac{1}{xe^x} \leq \frac{1}{e^x} = e^{-x},$$

so if $\int_1^{+\infty} e^{-x} dx$ converges, then so does $\int_1^{+\infty} \frac{1}{xe^x} dx$. To evaluate $\int_1^{+\infty} e^{-x} dx$, first rewrite it as a limit:
Equation:

$$\begin{aligned}
\int_1^{+\infty} e^{-x} dx &= \lim_{t \to +\infty} \int_1^t e^{-x} dx \\
&= \lim_{t \to +\infty} (-e^{-x})\Big|_1^t \\
&= \lim_{t \to +\infty} \left(-e^{-t} + e^1\right) \\
&= e^1.
\end{aligned}$$

Since $\int_1^{+\infty} e^{-x} dx$ converges, so does $\int_1^{+\infty} \frac{1}{xe^x} dx$.

Example:
Exercise:

Problem:
Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^{+\infty} \frac{1}{x^p} dx$ diverges for all $p < 1$.

Solution:

For $p < 1$, $1/x \leq 1/(x^p)$ over $[1, +\infty)$. In [link], we showed that $\int_1^{+\infty} \frac{1}{x} dx = +\infty$. Therefore, $\int_1^{+\infty} \frac{1}{x^p} dx$ diverges for all $p < 1$.

Note:
Exercise:

Problem: Use a comparison to show that $\int_e^{+\infty} \frac{\ln x}{x} dx$ diverges.

Solution:

Since $\int_e^{+\infty} \frac{1}{x} dx = +\infty$, $\int_e^{+\infty} \frac{\ln x}{x} dx$ diverges.

Hint

$\frac{1}{x} \leq \frac{\ln x}{x}$ on $[e, +\infty)$

Note:
Laplace Transforms
In the last few chapters, we have looked at several ways to use integration for solving real-world problems. For this next project, we are going to explore a more advanced application of integration: integral transforms. Specifically, we describe the Laplace transform and some of its properties. The Laplace transform is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.
The Laplace transform is defined in terms of an integral as
Equation:

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project we limit ourselves to functions of real numbers.
Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have
Equation:

$$L\{t\} = \int_0^\infty te^{-st}dt.$$

This is an improper integral, so we express it in terms of a limit, which gives
Equation:

$$L\{t\} = \int_0^\infty te^{-st}dt = \lim_{z\to\infty} \int_0^z te^{-st}dt.$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to *t*, so we treat the variable *s* as a constant. We have
Equation:

$$\begin{array}{rclcrcl} u & = & t & \quad & dv & = & e^{-st}dt \\ du & = & dt & \quad & v & = & -\frac{1}{s}e^{-st}. \end{array}$$

Then we obtain
Equation:

$$
\begin{aligned}
\lim_{z\to\infty}\int_0^z te^{-st}dt &= \lim_{z\to\infty}\left[\left[-\tfrac{t}{s}e^{-st}\right]\Big|_0^z+\tfrac{1}{s}\int_0^z e^{-st}dt\right] \\
&= \lim_{z\to\infty}\left[\left[-\tfrac{z}{s}e^{-sz}+\tfrac{0}{s}e^{-0s}\right]+\tfrac{1}{s}\int_0^z e^{-st}dt\right] \\
&= \lim_{z\to\infty}\left[\left[-\tfrac{z}{s}e^{-sz}+0\right]-\tfrac{1}{s}\left[\tfrac{e^{-st}}{s}\right]\Big|_0^z\right] \\
&= \lim_{z\to\infty}\left[\left[-\tfrac{z}{s}e^{-sz}\right]-\tfrac{1}{s^2}\left[e^{-sz}-1\right]\right] \\
&= \lim_{z\to\infty}\left[-\tfrac{z}{se^{sz}}\right]-\lim_{z\to\infty}\left[\tfrac{1}{s^2e^{sz}}\right]+\lim_{z\to\infty}\tfrac{1}{s^2} \\
&= 0-0+\tfrac{1}{s^2} \\
&= \tfrac{1}{s^2}.
\end{aligned}
$$

1. Calculate the Laplace transform of $f(t)=1$.
2. Calculate the Laplace transform of $f(t)=e^{-3t}$.
3. Calculate the Laplace transform of $f(t)=t^2$. (Note, you will have to integrate by parts twice.)

   Laplace transforms are often used to solve differential equations. Differential equations are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

   Let's start with the definition of the Laplace transform. We have

   Equation:

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\lim_{z\to\infty}\int_0^z e^{-st}f(t)dt.$$

4. Use integration by parts to evaluate $\lim_{z\to\infty}\int_0^z e^{-st}f(t)dt$. (Let $u=f(t)$ and $dv=e^{-st}dt.$)

   After integrating by parts and evaluating the limit, you should see that

   Equation:

$$L\{f(t)\}=\frac{f(0)}{s}+\frac{1}{s}\left[L\left\{f'(t)\right\}\right].$$

   Then,

   Equation:

$$L\left\{f'(t)\right\}=sL\{f(t)\}-f(0).$$

   Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

   The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t)=\int_0^t f(u)du$. Then,

   Equation:

$$L\{g(t)\}=\int_0^\infty e^{-st}g(t)dt=\lim_{z\to\infty}\int_0^z e^{-st}g(t)dt.$$

5. Use integration by parts to evaluate $\lim_{z\to\infty}\int_0^z e^{-st}g(t)dt$. (Let $u=g(t)$ and $dv=e^{-st}dt$. Note, by the way, that we have defined $g(t)$, $du=f(t)dt.$)

   As you might expect, you should see that

   Equation:

$$L\{g(t)\} = \frac{1}{s} \cdot L\{f(t)\}.$$

Integration in the time domain simplifies to division by s in the frequency domain.

Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

Key Equations

- **Improper integrals**
  $$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$$
  $$\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$
  $$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^{+\infty} f(x)dx$$

Evaluate the following integrals. If the integral is not convergent, answer “divergent.”

Exercise:

Problem: $\int_2^4 \frac{dx}{(x-3)^2}$

Solution:

divergent

Exercise:

Problem: $\int_0^{\infty} \frac{1}{4+x^2} dx$

Exercise:

Problem: $\int_0^2 \frac{1}{\sqrt{4-x^2}} dx$

Solution:

$\frac{\pi}{2}$

Exercise:

Problem: $\int_1^{\infty} \frac{1}{x \ln x} dx$

Exercise:

Problem: $\int_1^{\infty} xe^{-x} dx$

Solution:

$\frac{2}{e}$

Exercise:

Problem: $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx$

Exercise:

Problem:

Without integrating, determine whether the integral $\int_1^{\infty} \frac{1}{\sqrt{x^3+1}} dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3+1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

Solution:

Converges

Exercise:

Problem: Without integrating, determine whether the integral $\int_1^{\infty} \frac{1}{\sqrt{x+1}} dx$ converges or diverges.

Determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

Exercise:

Problem: $\int_0^{\infty} e^{-x} \cos x \, dx$

Solution:

Converges to 1/2.

Exercise:

Problem: $\int_1^{\infty} \frac{\ln x}{x} dx$

Exercise:

Problem: $\int_0^{1} \frac{\ln x}{\sqrt{x}} dx$

Solution:

-4

Exercise:

Problem: $\int_0^1 \ln x \, dx$

Exercise:

Problem: $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$

Solution:

π

Exercise:

Problem: $\int_1^5 \frac{dx}{\sqrt{x-1}}$

Exercise:

Problem: $\int_{-2}^{2} \frac{dx}{(1+x)^2}$

Solution:

diverges

Exercise:

Problem: $\int_0^{\infty} e^{-x} dx$

Exercise:

Problem: $\int_0^{\infty} \sin x \, dx$

Solution:

diverges

Exercise:

Problem: $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

Exercise:

Problem: $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Solution:

1.5

Exercise:

Problem: $\int_0^2 \frac{dx}{x^3}$

Exercise:

Problem: $\int_{-1}^2 \frac{dx}{x^3}$

Solution:

diverges

Exercise:

Problem: $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Exercise:

Problem: $\int_0^3 \frac{1}{x-1} dx$

Solution:

diverges

Exercise:

Problem: $\int_1^\infty \frac{5}{x^3} dx$

Exercise:

Problem: $\int_3^5 \frac{5}{(x-4)^2} dx$

Solution:

diverges

Determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

Exercise:

Problem: $\int_1^\infty \frac{dx}{x^2+4x}$; compare with $\int_1^\infty \frac{dx}{x^2}$.

Exercise:

Problem: $\int_1^{\infty} \frac{dx}{\sqrt{x}+1}$; compare with $\int_1^{\infty} \frac{dx}{2\sqrt{x}}$.

Solution:

Both integrals diverge.

Evaluate the integrals. If the integral diverges, answer "diverges."

Exercise:

Problem: $\int_1^{\infty} \frac{dx}{x^e}$

Exercise:

Problem: $\int_0^{1} \frac{dx}{x^{\pi}}$

Solution:

diverges

Exercise:

Problem: $\int_0^{1} \frac{dx}{\sqrt{1-x}}$

Exercise:

Problem: $\int_0^{1} \frac{dx}{1-x}$

Solution:

diverges

Exercise:

Problem: $\int_{-\infty}^{0} \frac{dx}{x^2+1}$

Exercise:

Problem: $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$

Solution:

π

Exercise:

Problem: $\int_0^1 \frac{\ln x}{x} dx$

Exercise:

Problem: $\int_0^e \ln(x) dx$

Solution:

0.0

Exercise:

Problem: $\int_0^\infty xe^{-x} dx$

Exercise:

Problem: $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2} dx$

Solution:

0.0

Exercise:

Problem: $\int_0^\infty e^{-x} dx$

Evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.

Exercise:

Problem: $\int_0^9 \frac{dx}{\sqrt{9-x}}$

Solution:

6.0

Exercise:

Problem: $\int_{-27}^{1} \frac{dx}{x^{2/3}}$

Exercise:

Problem: $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Solution:

$\frac{\pi}{2}$

Exercise:

Problem: $\int_{6}^{24} \frac{dt}{t\sqrt{t^2-36}}$

Exercise:

Problem: $\int_{0}^{4} x\ln(4x)dx$

Solution:

$8\ln(16)-4$

Exercise:

Problem: $\int_{0}^{3} \frac{x}{\sqrt{9-x^2}}dx$

Exercise:

Problem: Evaluate $\int_{.5}^{t} \frac{dx}{\sqrt{1-x^2}}$. (Be careful!) (Express your answer using three decimal places.)

Solution:

1.047

Exercise:

Problem: Evaluate $\int_{1}^{4} \frac{dx}{\sqrt{x^2-1}}$. (Express the answer in exact form.)

Exercise:

Problem: Evaluate $\int_{2}^{\infty} \frac{dx}{(x^2-1)^{3/2}}$.

Solution:

$-1+\frac{2}{\sqrt{3}}$

Exercise:

Problem: Find the area of the region in the first quadrant between the curve $y=e^{-6x}$ and the *x*-axis.

Exercise:

Problem: Find the area of the region bounded by the curve $y=\frac{7}{x^2}$, the *x*-axis, and on the left by $x=1$.

Solution:

7.0

Exercise:

Problem: Find the area under the curve $y=\frac{1}{(x+1)^{3/2}}$, bounded on the left by $x=3$.

Exercise:

Problem: Find the area under $y=\frac{5}{1+x^2}$ in the first quadrant.

Solution:

$\frac{5\pi}{2}$

Exercise:

Problem:

Find the volume of the solid generated by revolving about the *x*-axis the region under the curve $y=\frac{3}{x}$ from $x=1$ to $x=\infty$.

Exercise:

Problem:

Find the volume of the solid generated by revolving about the *y*-axis the region under the curve $y=6e^{-2x}$ in the first quadrant.

Solution:

3π

Exercise:

Problem:

Find the volume of the solid generated by revolving about the *x*-axis the area under the curve $y=3e^{-x}$ in the first quadrant.

The Laplace transform of a continuous function over the interval $[0,\infty)$ is defined by $F(s)=\int_0^{\infty} e^{-sx}f(x)dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The domain of *F* is the set of all real numbers *s* such that the improper integral converges. Find the Laplace transform *F* of each of the following functions and give the domain of *F*.

Exercise:

Problem: $f(x)=1$

Solution:

$\frac{1}{s}, s>0$

Exercise:

Problem: $f(x)=x$

Exercise:

Problem: $f(x)=\cos(2x)$

Solution:

$\frac{s}{s^2+4}$, $s > 0$

Exercise:

Problem: $f(x) = e^{ax}$

Exercise:

Problem: Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π.

Solution:

Answers will vary.

A function is a probability density function if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)dt = 1$. The probability that a random variable *x* lies between *a* and *b* is given by $P(a \leq x \leq b) = \int_a^b f(t)dt$.

Exercise:

Problem: Show that $f(x) = \begin{cases} 0 \, \text{if} \, x < 0 \\ 7e^{-7x} \text{if} \, x \geq 0 \end{cases}$ is a probability density function.

Exercise:

Problem:

Find the probability that *x* is between 0 and 0.3. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

Solution:

0.8775

Chapter Review Exercises

For the following exercises, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

Exercise:

Problem: $\int e^x \sin(x) dx$ cannot be integrated by parts.

Exercise:

Problem: $\int \frac{1}{x^4+1} dx$ cannot be integrated using partial fractions.

Solution:

False

Exercise:

Problem: In numerical integration, increasing the number of points decreases the error.

Exercise:

Problem: Integration by parts can always yield the integral.

Solution:

False

For the following exercises, evaluate the integral using the specified method.

Exercise:

Problem: $\int x^2 \sin(4x)dx$ using integration by parts

Exercise:

Problem: $\int \dfrac{1}{x^2\sqrt{x^2+16}}dx$ using trigonometric substitution

---

Solution:

$-\frac{\sqrt{x^2+16}}{16x} + C$

Exercise:

Problem: $\int \sqrt{x}\ln(x)dx$ using integration by parts

Exercise:

Problem: $\int \dfrac{3x}{x^3+2x^2-5x-6}dx$ using partial fractions

---

Solution:

$\frac{1}{10}(4\ln(2-x)+5\ln(x+1)-9\ln(x+3))+C$

Exercise:

Problem: $\int \dfrac{x^5}{(4x^2+4)^{5/2}}dx$ using trigonometric substitution

Exercise:

Problem: $\int \dfrac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)dx$ using a table of integrals or a CAS

Solution:

$-\frac{\sqrt{4-\sin^2(x)}}{\sin(x)} - \frac{x}{2} + C$

For the following exercises, integrate using whatever method you choose.
Exercise:

Problem: $\int \sin^2(x)\cos^2(x)dx$

Exercise:

Problem: $\int x^3\sqrt{x^2+2}\,dx$

Solution:

$\frac{1}{15}\left(x^2+2\right)^{3/2}\left(3x^2-4\right)+C$

Exercise:

Problem: $\int \frac{3x^2+1}{x^4-2x^3-x^2+2x}dx$

Exercise:

Problem: $\int \frac{1}{x^4+4}dx$

Solution:

$\frac{1}{16}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right)-\frac{1}{8}\tan^{-1}(1-x)+\frac{1}{8}\tan^{-1}(x+1)+C$

Exercise:

Problem: $\int \frac{\sqrt{3+16x^4}}{x^4}dx$

For the following exercises, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson’s rule using four subintervals, rounding to three decimals.
Exercise:

Problem: [T] $\int_1^2 \sqrt{x^5+2}dx$

Solution:

$\mathrm{M}_4=3.312, \mathrm{T}_4=3.354, S_4=3.326$

Exercise:

Problem: [T] $\int_0^{\sqrt{\pi}} e^{-\sin(x^2)}dx$

Exercise:

Problem: [T] $\int_1^4 \frac{\ln(1/x)}{x}dx$

Solution:

$\mathrm{M}_4 = -0.982, \mathrm{T}_4 = -0.917, S_4 = -0.952$

For the following exercises, evaluate the integrals, if possible.
Exercise:

Problem: $\int_1^{\infty} \frac{1}{x^n} dx$, for what values of n does this integral converge or diverge?

Exercise:

Problem: $\int_1^{\infty} \frac{e^{-x}}{x} dx$

Solution:

approximately 0.2194

For the following exercises, consider the gamma function given by $\Gamma(a) = \int_0^{\infty} e^{-y} y^{a-1} dy$.

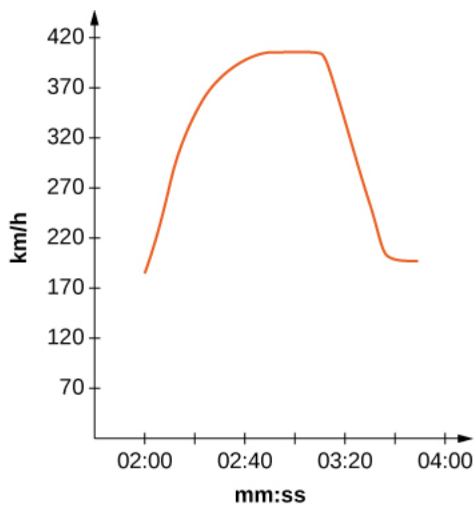
Exercise:

Problem: Show that $\Gamma(a) = (a-1)\Gamma(a-1)$.

Exercise:

Problem: Extend to show that $\Gamma(a) = (a-1)!$, assuming a is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.



Exercise:

Problem:

[T] Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = a\exp^{bx}\sin(cx) + d$. (*Hint:* Consider the time units.)

Exercise:

Problem:

[T] Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.

Solution:

Answers may vary. Ex: 9.405 km

Glossary

improper integral
: an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

Introduction class="introduction"

The white-tailed deer (*Odocoileus virginianus*) of the eastern United States. Differential equations can be used to study animal populations. (credit: modification of work by Rachel Kramer, Flickr)

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications.

Suppose we wish to study a population of deer over time and determine the total number of animals in a given area. We can first observe the population over a period of time, estimate the total number of deer, and then use various assumptions to derive a mathematical model for different scenarios. Some factors that are often considered are environmental impact, threshold population values, and predators. In this chapter we see how differential equations can be used to predict populations over time (see [link]).

Another goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.

Basics of Differential Equations

- Identify the order of a differential equation.
- Explain what is meant by a solution to a differential equation.
- Distinguish between the general solution and a particular solution of a differential equation.
- Identify an initial-value problem.
- Identify whether a given function is a solution to a differential equation or an initial-value problem.

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a *differential equation*. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

Techniques for solving differential equations can take many different forms, including direct solution, use of graphs, or computer calculations. We introduce the main ideas in this chapter and describe them in a little more detail later in the course. In this section we study what differential equations are, how to verify their solutions, some methods that are used for solving them, and some examples of common and useful equations.

General Differential Equations

Consider the equation $y' = 3x^2$, which is an example of a differential equation because it includes a derivative. There is a relationship between the variables x and y: y is an unknown function of x. Furthermore, the left-hand side of the equation is the derivative of y. Therefore we can interpret this equation as follows: Start with some function $y = f(x)$ and take its derivative. The answer must be equal to $3x^2$. What function has a derivative that is equal to $3x^2$? One such function is $y = x^3$, so this function is considered a **solution to a differential equation**.

Note:
Definition
A **differential equation** is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Note:
Go to this website to explore more on this topic.

Some examples of differential equations and their solutions appear in [link].

Equation	**Solution**
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Examples of Differential Equations and Their Solutions

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero. For example, $y = x^2 + 4$ is also a solution to the first differential equation in [link]. We will return to this idea a little bit later in this section. For now, let's focus on what it means for a function to be a solution to a differential equation.

Example:
Exercise:

Problem:
Verifying Solutions of Differential Equations

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

Solution:

To verify the solution, we first calculate y' using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$. Next we substitute y and y' into the left-hand side of the differential equation:
Equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

The resulting expression can be simplified by first distributing to eliminate the parentheses, giving
Equation:

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11$, which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

Note:
Exercise:

Problem:

Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$.

Hint

First calculate y' then substitute both y' and y into the left-hand side.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its order.

Note:
Definition
The **order of a differential equation** is the highest order of any derivative of the unknown function that appears in the equation.

Example:
Exercise:

Problem:
Identifying the Order of a Differential Equation

What is the order of each of the following differential equations?

a. $y' - 4y = x^2 - 3x + 4$
b. $x^2 y''' - 3xy'' + xy' - 3y = \sin x$
c. $\frac{4}{x} y^{(4)} - \frac{6}{x^2} y'' + \frac{12}{x^4} y = x^3 - 3x^2 + 4x - 12$

Solution:

a. The highest derivative in the equation is y', so the order is 1.
b. The highest derivative in the equation is y''', so the order is 3.
c. The highest derivative in the equation is $y^{(4)}$, so the order is 4.

Note:
Exercise:

Problem: What is the order of the following differential equation?
Equation:

$$\left(x^4-3x\right)y^{(5)}-\left(3x^2+1\right)y'+3y=\sin x\cos x$$

Solution:

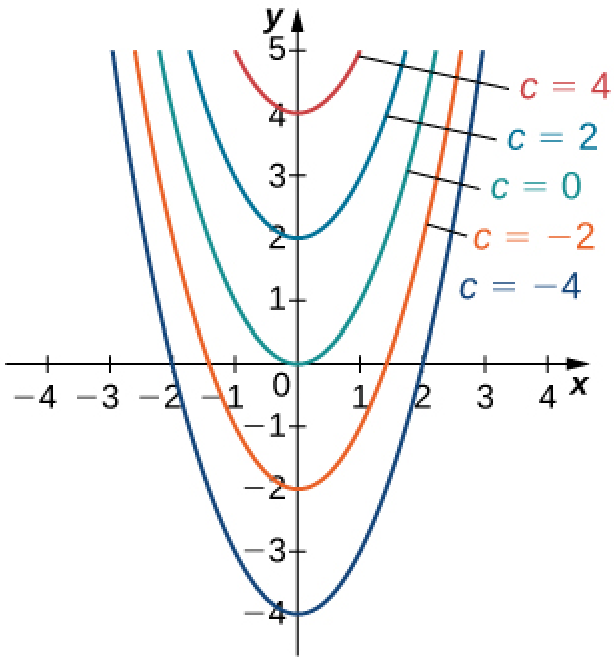
5

Hint

What is the highest derivative in the equation?

General and Particular Solutions

We already noted that the differential equation $y'=2x$ has at least two solutions: $y=x^2$ and $y=x^2+4$. The only difference between these two solutions is the last term, which is a constant. What if the last term is a different constant? Will this expression still be a solution to the differential equation? In fact, any function of the form $y=x^2+C$, where C represents any constant, is a solution as well. The reason is that the derivative of x^2+C is $2x$, regardless of the value of C. It can be shown that any solution of this differential equation must be of the form $y=x^2+C$. This is an example of a **general solution** to a differential equation. A graph of some of these solutions is given in [link]. (*Note*: in this graph we used even integer values for C ranging between -4 and 4. In fact, there is no restriction on the value of C; it can be an integer or not.)



Family of solutions to the differential equation $y' = 2x$.

In this example, we are free to choose any solution we wish; for example, $y = x^2 - 3$ is a member of the family of solutions to this differential equation. This is called a **particular solution** to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

Example:
Exercise:

Problem:
Finding a Particular Solution

Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$.

Solution:

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of C, we substitute the values $x = 2$ and $y = 7$ into this equation and solve for C:

Equation:

$$
\begin{aligned}
&y = x^2 + C \\
&7 = 2^2 + C = 4 + C \\
&C = 3.
\end{aligned}
$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

Note:

Exercise:

Problem: Find the particular solution to the differential equation

Equation:

$$y' = 4x + 3$$

passing through the point $(1, 7)$, given that $y = 2x^2 + 3x + C$ is a general solution to the differential equation.

Solution:

$y = 2x^2 + 3x + 2$

Hint

First substitute $x = 1$ and $y = 7$ into the equation, then solve for C.

Initial-Value Problems

Usually a given differential equation has an infinite number of solutions, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is an **initial value**, which is an ordered pair that is used to find a particular solution.

A differential equation together with one or more initial values is called an **initial-value problem**. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation. For example, if we have the differential equation $y' = 2x$, then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem. The differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem. These problems are so named because often the independent variable in the unknown function is t, which represents time. Thus, a value of $t = 0$ represents the beginning of the problem.

Example:
Exercise:

Problem:
Verifying a Solution to an Initial-Value Problem

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem
Equation:

$$y' + 2y = 3e^t, \quad y(0) = 3.$$

Solution:

For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that y satisfies the differential equation, we start by calculating y'. This gives $y' = -4e^{-2t} + e^t$. Next we substitute both y and y' into the left-hand side of the differential equation and simplify:

Equation:

$$\begin{aligned} y' + 2y &= \left(-4e^{-2t} + e^t\right) + 2\left(2e^{-2t} + e^t\right) \\ &= -4e^{-2t} + e^t + 4e^{-2t} + 2e^t \\ &= 3e^t. \end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation. Next we calculate $y(0)$:

Equation:

$$\begin{aligned} y(0) &= 2e^{-2(0)} + e^0 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

Note:
Exercise:

Problem:

Verify that $y = 3e^{2t} + 4\sin t$ is a solution to the initial-value problem

Equation:

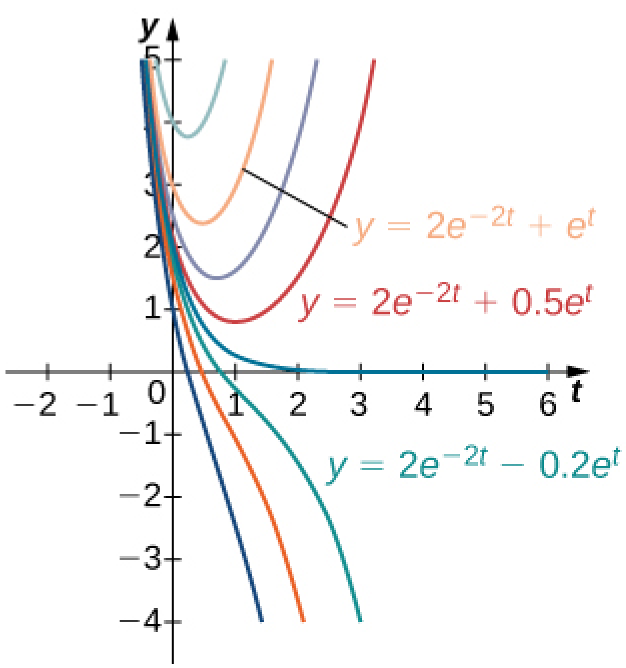
$$y' - 2y = 4\cos t - 8\sin t, \quad y(0) = 3.$$

Hint

First verify that y solves the differential equation. Then check the initial value.

In [link], the initial-value problem consisted of two parts. The first part was the differential equation $y' + 2y = 3e^x$, and the second part was the initial value $y(0) = 3$. These two equations together formed the initial-value problem.

The same is true in general. An initial-value problem will consists of two parts: the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of C. The family of solutions to the differential equation in [link] is given by $y = 2e^{-2t} + Ce^t$. This family of solutions is shown in [link], with the particular solution $y = 2e^{-2t} + e^t$ labeled.



A family of solutions to the differential equation $y' + 2y = 3e^t$. The particular solution $y = 2e^{-2t} + e^t$ is labeled.

Example:
Exercise:

Problem:
Solving an Initial-value Problem

Solve the following initial-value problem:
Equation:

$$y' = 3e^x + x^2 - 4, \quad y(0) = 5.$$

Solution:

The first step in solving this initial-value problem is to find a general family of solutions. To do this, we find an antiderivative of both sides of the differential equation
Equation:

$$\int y' dx = \int \left(3e^x + x^2 - 4\right) dx,$$

namely,
Equation:

$$y + C_1 = 3e^x + \frac{1}{3}x^3 - 4x + C_2.$$

We are able to integrate both sides because the y term appears by itself. Notice that there are two integration constants: C_1 and C_2. Solving [link] for y gives

Equation:

$$y = 3e^x + \frac{1}{3}x^3 - 4x + C_2 - C_1.$$

Because C_1 and C_2 are both constants, $C_2 - C_1$ is also a constant. We can therefore define $C = C_2 - C_1$, which leads to the equation

Equation:

$$y = 3e^x + \frac{1}{3}x^3 - 4x + C.$$

Next we determine the value of C. To do this, we substitute $x = 0$ and $y = 5$ into [link] and solve for C:

Equation:

$$\begin{array}{rcl} 5 & = & 3e^0 + \frac{1}{3}0^3 - 4\left(0\right) + C \\ 5 & = & 3 + C \\ C & = & 2. \end{array}$$

Now we substitute the value $C = 2$ into [link]. The solution to the initial-value problem is $y = 3e^x + \frac{1}{3}x^3 - 4x + 2$.

Analysis

The difference between a general solution and a particular solution is that a general solution involves a family of functions, either explicitly or implicitly defined, of the independent variable. The initial value or values determine which particular solution in the family of solutions satisfies the desired conditions.

Note:
Exercise:

Problem: Solve the initial-value problem
Equation:

$$y' = x^2 - 4x + 3 - 6e^x, \quad y(0) = 8.$$

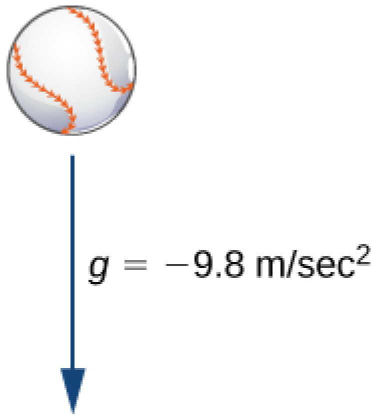
Solution:

$y = \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + 14$

Hint

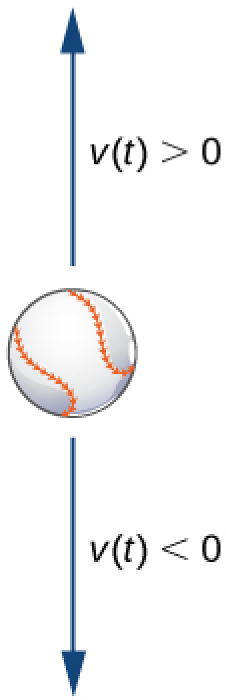
First take the antiderivative of both sides of the differential equation. Then substitute $x = 0$ and $y = 8$ into the resulting equation and solve for C.

In physics and engineering applications, we often consider the forces acting upon an object, and use this information to understand the resulting motion that may occur. For example, if we start with an object at Earth's surface, the primary force acting upon that object is gravity. Physicists and engineers can use this information, along with Newton's second law of motion (in equation form $F = ma$, where F represents force, m represents mass, and a represents acceleration), to derive an equation that can be solved.



For a baseball falling in air, the only force acting on it is gravity (neglecting air resistance).

In [link] we assume that the only force acting on a baseball is the force of gravity. This assumption ignores air resistance. (The force due to air resistance is considered in a later discussion.) The acceleration due to gravity at Earth's surface, g, is approximately $9.8\,\text{m/s}^2$. We introduce a frame of reference, where Earth's surface is at a height of 0 meters. Let $v(t)$ represent the velocity of the object in meters per second. If $v(t) > 0$, the ball is rising, and if $v(t) < 0$, the ball is falling ([link]).



Possible velocities for the rising/falling baseball.

Our goal is to solve for the velocity $v(t)$ at any time t. To do this, we set up an initial-value problem. Suppose the mass of the ball is m, where m is measured in kilograms. We use Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration $(F = ma)$. Acceleration is the derivative of velocity, so $a(t) = v'(t)$. Therefore the force acting on the baseball is given by $F = m\,v'(t)$. However, this force must be equal to the force of gravity acting on the object, which (again using Newton's second law) is given by $F_g = -mg$, since this force acts in a downward direction. Therefore we obtain the equation $F = F_g$, which becomes $m\,v'(t) = -mg$. Dividing both sides of the equation by m gives the equation

Equation:

$$v'(t) = -g.$$

Notice that this differential equation remains the same regardless of the mass of the object.

We now need an initial value. Because we are solving for velocity, it makes sense in the context of the problem to assume that we know the **initial velocity**, or the velocity at time $t = 0$. This is denoted by $v(0) = v_0$.

Example:
Exercise:

Problem:
Velocity of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of $10\,\mathrm{m/s}$, and the only force acting on it is gravity. The ball has a mass of $0.15\,\mathrm{kg}$ at Earth's surface.

a. Find the velocity $v\,(t)$ of the baseball at time t.
b. What is its velocity after 2 seconds?

Solution:

a. From the preceding discussion, the differential equation that applies in this situation is
Equation:

$$v'(t) = -g,$$

where $g = 9.8\,\mathrm{m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10\,\mathrm{m/s}$. Therefore the initial-value problem is $v'(t) = -9.8\,\mathrm{m/s}^2, v(0) = 10\,\mathrm{m/s}$.
The first step in solving this initial-value problem is to take the antiderivative of both sides of the differential equation. This gives

Equation:

$$\begin{aligned}\int v'(t)\,dt &= \int -9.8dt \\ v(t) &= -9.8t + C.\end{aligned}$$

The next step is to solve for C. To do this, substitute $t = 0$ and $v(0) = 10$:

Equation:

$$\begin{aligned}v(t) &= -9.8t + C \\ v(0) &= -9.8\,(0) + C \\ 10 &= C.\end{aligned}$$

Therefore $C = 10$ and the velocity function is given by $v(t) = -9.8t + 10.$

b. To find the velocity after 2 seconds, substitute $t = 2$ into $v\,(t)$.

Equation:

$$\begin{aligned}v(t) &= -9.8t + 10 \\ v(2) &= -9.8\,(2) + 10 \\ v\,(2) &= -9.6.\end{aligned}$$

The units of velocity are meters per second. Since the answer is negative, the object is falling at a speed of $9.6\ \mathrm{m/s}$.

Note:

Exercise:

Problem:

Suppose a rock falls from rest from a height of 100 meters and the only force acting on it is gravity. Find an equation for the velocity $v(t)$ as a function of time, measured in meters per second.

Solution:

$v(t) = -9.8t$

Hint

What is the initial velocity of the rock? Use this with the differential equation in [link] to form an initial-value problem, then solve for $v(t)$.

A natural question to ask after solving this type of problem is how high the object will be above Earth's surface at a given point in time. Let $s(t)$ denote the height above Earth's surface of the object, measured in meters. Because velocity is the derivative of position (in this case height), this assumption gives the equation $s'(t) = v(t)$. An initial value is necessary; in this case the initial height of the object works well. Let the initial height be given by the equation $s(0) = s_0$. Together these assumptions give the initial-value problem

Equation:

$$s'(t) = v(t), \quad s(0) = s_0.$$

If the velocity function is known, then it is possible to solve for the position function as well.

Example:

Exercise:

Problem:

Height of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth’s surface with an initial velocity of $10\,\text{m}/\text{s}$, and the only force acting on it is gravity. The ball has a mass of 0.15 kilogram at Earth’s surface.

a. Find the position $s(t)$ of the baseball at time t.
b. What is its height after 2 seconds?

Solution:

a. We already know the velocity function for this problem is $v(t) = -9.8t + 10$. The initial height of the baseball is 3 meters, so $s_0 = 3$. Therefore the initial-value problem for this example is
To solve the initial-value problem, we first find the antiderivatives:
Equation:

$$\begin{aligned}\int s'(t)\,dt &= \int -9.8t + 10dt \\ s(t) &= -4.9t^2 + 10t + C.\end{aligned}$$

Next we substitute $t = 0$ and solve for C:
Equation:

$$\begin{aligned}s(t) &= -4.9t^2 + 10t + C \\ s(0) &= -4.9(0)^2 + 10(0) + C \\ 3 &= C.\end{aligned}$$

Therefore the position function is $s(t) = -4.9t^2 + 10t + 3$.
b. The height of the baseball after 2 s is given by $s(2)$:
Equation:

$$\begin{aligned} s(2) &= -4.9(2)^2 + 10(2) + 3 \\ &= -4.9(4) + 23 \\ &= 3.4. \end{aligned}$$

Therefore the baseball is 3.4 meters above Earth's surface after 2 seconds. It is worth noting that the mass of the ball cancelled out completely in the process of solving the problem.

Key Concepts

- A differential equation is an equation involving a function $y = f(x)$ and one or more of its derivatives. A solution is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.
- The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.
- A differential equation coupled with an initial value is called an initial-value problem. To solve an initial-value problem, first find the general solution to the differential equation, then determine the value of the constant. Initial-value problems have many applications in science and engineering.

Determine the order of the following differential equations.

Exercise:

Problem: $y' + y = 3y^2$

Solution:

1

Exercise:

Problem: $(y')^2 = y' + 2y$

Exercise:

Problem: $y''' + y''y' = 3x^2$

Solution:

3

Exercise:

Problem: $y' = y'' + 3t^2$

Exercise:

Problem: $\frac{dy}{dt} = t$

Solution:

1

Exercise:

Problem: $\frac{dy}{dx} + \frac{d^2y}{dx^2} = 3x^4$

Exercise:

Problem: $\left(\frac{dy}{dt}\right)^2 + 8\frac{dy}{dt} + 3y = 4t$

Solution:

1

Verify that the following functions are solutions to the given differential equation.

Exercise:

Problem: $y = \frac{x^3}{3}$ solves $y' = x^2$

Exercise:

Problem: $y = 2e^{-x} + x - 1$ solves $y' = x - y$

Exercise:

Problem: $y = e^{3x} - \frac{e^x}{2}$ solves $y' = 3y + e^x$

Exercise:

Problem: $y = \frac{1}{1-x}$ solves $y' = y^2$

Exercise:

Problem: $y = e^{x^2/2}$ solves $y' = xy$

Exercise:

Problem: $y = 4 + \ln x$ solves $xy' = 1$

Exercise:

Problem: $y = 3 - x + x \ln x$ solves $y' = \ln x$

Exercise:

Problem: $y = 2e^x - x - 1$ solves $y' = y + x$

Exercise:

Problem: $y = e^x + \frac{\sin x}{2} - \frac{\cos x}{2}$ solves $y' = \cos x + y$

Exercise:

Problem: $y = \pi e^{-\cos x}$ solves $y' = y \sin x$

Verify the following general solutions and find the particular solution.

Exercise:

Problem:

Find the particular solution to the differential equation $y' = 4x^2$ that passes through $(-3, -30)$, given that $y = C + \frac{4x^3}{3}$ is a general solution.

Exercise:

Problem:

Find the particular solution to the differential equation $y' = 3x^3$ that passes through $(1, 4.75)$, given that $y = C + \frac{3x^4}{4}$ is a general solution.

Solution:

$y = 4 + \frac{3x^4}{4}$

Exercise:

Problem:

Find the particular solution to the differential equation $y' = 3x^2 y$ that passes through $(0, 12)$, given that $y = Ce^{x^3}$ is a general solution.

Exercise:

Problem:

Find the particular solution to the differential equation $y' = 2xy$ that passes through $\left(0, \frac{1}{2}\right)$, given that $y = Ce^{x^2}$ is a general solution.

Solution:

$y=\frac{1}{2}e^{x^2}$

Exercise:

Problem:

Find the particular solution to the differential equation $y'=(2xy)^2$ that passes through $\left(1,-\frac{1}{2}\right)$, given that $y=-\frac{3}{C+4x^3}$ is a general solution.

Exercise:

Problem:

Find the particular solution to the differential equation $y'x^2=y$ that passes through $\left(1,\frac{2}{e}\right)$, given that $y=Ce^{-1/x}$ is a general solution.

Solution:

$y=2e^{-1/x}$

Exercise:

Problem:

Find the particular solution to the differential equation $8\frac{dx}{dt}=-2\cos(2t)-\cos(4t)$ that passes through (π,π), given that $x=C-\frac{1}{8}\sin(2t)-\frac{1}{32}\sin(4t)$ is a general solution.

Exercise:

Problem:

Find the particular solution to the differential equation $\frac{du}{dt}=\tan u$ that passes through $\left(1,\frac{\pi}{2}\right)$, given that $u=\sin^{-1}\left(e^{C+t}\right)$ is a general solution.

Solution:

$u=\sin^{-1}\left(e^{-1+t}\right)$

Exercise:

Problem:

Find the particular solution to the differential equation $\frac{dy}{dt} = e^{(t+y)}$ that passes through $(1, 0)$, given that $y = -\ln(C - e^t)$ is a general solution.

Exercise:

Problem:

Find the particular solution to the differential equation $y'(1 - x^2) = 1 + y$ that passes through $(0, -2)$, given that $y = C\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$ is a general solution.

Solution:

$$y = -\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$$

For the following problems, find the general solution to the differential equation.

Exercise:

Problem: $y' = 3x + e^x$

Exercise:

Problem: $y' = \ln x + \tan x$

Solution:

$$y = C - x + x\ln x - \ln(\cos x)$$

Exercise:

Problem: $y' = \sin x e^{\cos x}$

Exercise:

Problem: $y' = 4^x$

Solution:

$y = C + \frac{4^x}{\ln(4)}$

Exercise:

Problem: $y' = \sin^{-1}(2x)$

Exercise:

Problem: $y' = 2t\sqrt{t^2 + 16}$

Solution:

$y = \frac{2}{3}\sqrt{t^2 + 16}\left(t^2 + 16\right) + C$

Exercise:

Problem: $x' = \coth t + \ln t + 3t^2$

Exercise:

Problem: $x' = t\sqrt{4 + t}$

Solution:

$x = \frac{2}{15}\sqrt{4 + t}\left(3t^2 + 4t - 32\right) + C$

Exercise:

Problem: $y' = y$

Exercise:

Problem: $y' = \frac{y}{x}$

Solution:

$y = Cx$

Solve the following initial-value problems starting from $y(t=0) = 1$ and $y(t=0) = -1$. Draw both solutions on the same graph.

Exercise:

Problem: $\frac{dy}{dt} = 2t$

Exercise:

Problem: $\frac{dy}{dt} = -t$

Solution:

$y = 1 - \frac{t^2}{2}, y = -\frac{t^2}{2} - 1$

Exercise:

Problem: $\frac{dy}{dt} = 2y$

Exercise:

Problem: $\frac{dy}{dt} = -y$

Solution:

$y = e^{-t}, y = -e^{-t}$

Exercise:

Problem: $\frac{dy}{dt} = 2$

Solve the following initial-value problems starting from $y_0 = 10$. At what time does y increase to 100 or drop to 1?

Exercise:

Problem: $\frac{dy}{dt} = 4t$

Solution:

$y = 2\left(t^2 + 5\right), t = 3\sqrt{5}$

Exercise:

Problem: $\frac{dy}{dt} = 4y$

Exercise:

Problem: $\frac{dy}{dt} = -2y$

Solution:

$y = 10e^{-2t}, t = -\frac{1}{2}\ln\left(\frac{1}{10}\right)$

Exercise:

Problem: $\frac{dy}{dt} = e^{4t}$

Exercise:

Problem: $\frac{dy}{dt} = e^{-4t}$

Solution:

$y=\frac{1}{4}\left(41-e^{-4t}\right)$, never

Recall that a family of solutions includes solutions to a differential equation that differ by a constant. For the following problems, use your calculator to graph a family of solutions to the given differential equation. Use initial conditions from $y(t=0)=-10$ to $y(t=0)=10$ increasing by 2. Is there some critical point where the behavior of the solution begins to change?

Exercise:

Problem: **[T]** $y'=y(x)$

Exercise:

Problem: **[T]** $xy'=y$

Solution:

Solution changes from increasing to decreasing at $y(0)=0$

Exercise:

Problem: **[T]** $y'=t^3$

Exercise:

Problem:

[T] $y'=x+y$ (*Hint:* $y=Ce^x-x-1$ is the general solution)

Solution:

Solution changes from increasing to decreasing at $y(0)=0$

Exercise:

Problem: **[T]** $y'=x\ln x+\sin x$

Exercise:

Problem:

Find the general solution to describe the velocity of a ball of mass 1 lb that is thrown upward at a rate a ft/sec.

Solution:

$v(t) = -32t + a$

Exercise:

Problem:

In the preceding problem, if the initial velocity of the ball thrown into the air is $a = 25$ ft/s, write the particular solution to the velocity of the ball. Solve to find the time when the ball hits the ground.

Exercise:

Problem:

You throw two objects with differing masses m_1 and m_2 upward into the air with the same initial velocity a ft/s. What is the difference in their velocity after 1 second?

Solution:

0 ft/s

Exercise:

Problem:

[T] You throw a ball of mass 1 kilogram upward with a velocity of $a = 25$ m/s on Mars, where the force of gravity is $g = -3.711$ m/s^2. Use your calculator to approximate how much longer the ball is in the air on Mars.

Exercise:

Problem:

[T] For the previous problem, use your calculator to approximate how much higher the ball went on Mars.

Solution:

4.86 meters

Exercise:

Problem:

[T] A car on the freeway accelerates according to $a = 15\cos(\pi t)$, where t is measured in hours. Set up and solve the differential equation to determine the velocity of the car if it has an initial speed of 51 mph. After 40 minutes of driving, what is the driver's velocity?

Exercise:

Problem:

[T] For the car in the preceding problem, find the expression for the distance the car has traveled in time t, assuming an initial distance of 0. How long does it take the car to travel 100 miles? Round your answer to hours and minutes.

Solution:

$x = 50t - \frac{15}{\pi^2}\cos(\pi t) + \frac{3}{\pi^2}$, 2 hours 1 minute

Exercise:

Problem:

[T] For the previous problem, find the total distance traveled in the first hour.

Exercise:

Problem:

Substitute $y = Be^{3t}$ into $y' - y = 8e^{3t}$ to find a particular solution.

Solution:

$y = 4e^{3t}$

Exercise:

Problem:

Substitute $y = a\cos(2t) + b\sin(2t)$ into $y' + y = 4\sin(2t)$ to find a particular solution.

Exercise:

Problem:

Substitute $y = a + bt + ct^2$ into $y' + y = 1 + t^2$ to find a particular solution.

Solution:

$y = 1 - 2t + t^2$

Exercise:

Problem:

Substitute $y = ae^t\cos t + be^t\sin t$ into $y' = 2e^t\cos t$ to find a particular solution.

Exercise:

Problem:

Solve $y' = e^{kt}$ with the initial condition $y(0) = 0$ and solve $y' = 1$ with the same initial condition. As k approaches 0, what do you notice?

Solution:

$y = \frac{1}{k}\left(e^{kt} - 1\right)$ and $y = x$

Glossary

differential equation
: an equation involving a function $y = y(x)$ and one or more of its derivatives

general solution (or family of solutions)
: the entire set of solutions to a given differential equation

initial value(s)
: a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

initial velocity
: the velocity at time $t = 0$

initial-value problem
: a differential equation together with an initial value or values

order of a differential equation
: the highest order of any derivative of the unknown function that appears in the equation

particular solution
: member of a family of solutions to a differential equation that satisfies a particular initial condition

solution to a differential equation
: a function $y = f(x)$ that satisfies a given differential equation

Direction Fields and Numerical Methods

- Draw the direction field for a given first-order differential equation.
- Use a direction field to draw a solution curve of a first-order differential equation.
- Use Euler's Method to approximate the solution to a first-order differential equation.

For the rest of this chapter we will focus on various methods for solving differential equations and analyzing the behavior of the solutions. In some cases it is possible to predict properties of a solution to a differential equation without knowing the actual solution. We will also study numerical methods for solving differential equations, which can be programmed by using various computer languages or even by using a spreadsheet program, such as Microsoft Excel.

Creating Direction Fields

Direction fields (also called slope fields) are useful for investigating first-order differential equations. In particular, we consider a first-order differential equation of the form

Equation:

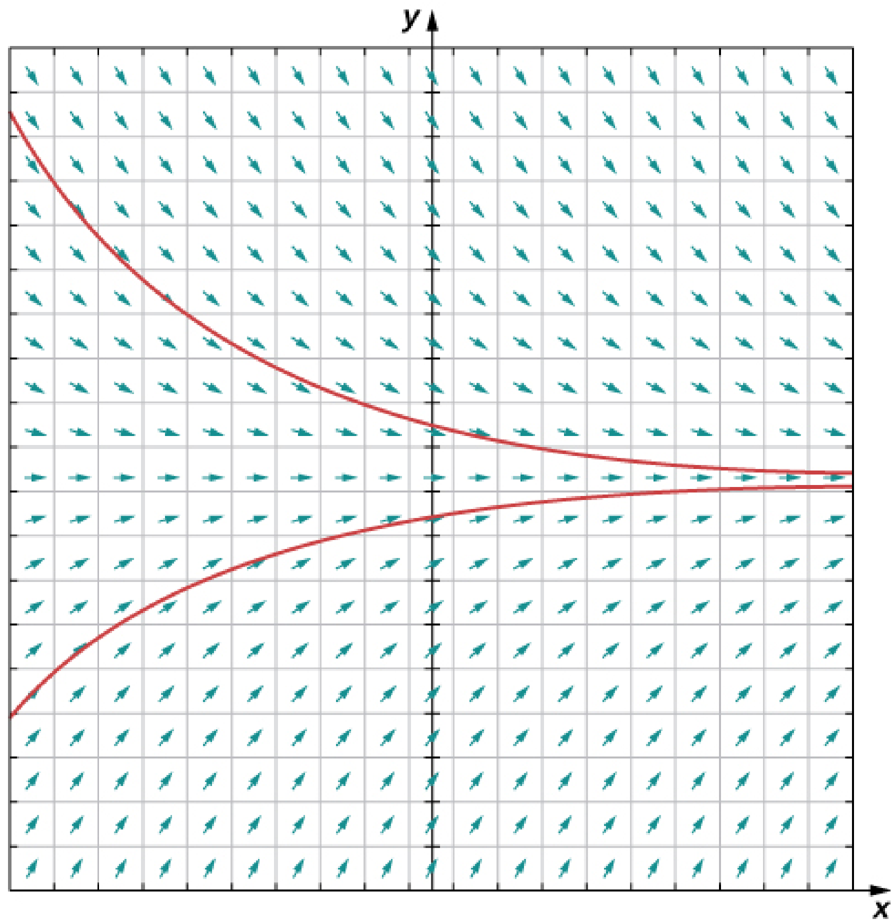
$$y\prime = f\left(x, y\right).$$

An applied example of this type of differential equation appears in Newton's law of cooling, which we will solve explicitly later in this chapter. First, though, let us create a direction field for the differential equation

Equation:

$$T'\left(t\right) = -0.4(T - 72).$$

Here $T\left(t\right)$ represents the temperature (in degrees Fahrenheit) of an object at time t, and the ambient temperature is 72°F. [link] shows the direction field for this equation.



Direction field for the differential equation $T'(t) = -0.4(T - 72)$. Two solutions are plotted: one with initial temperature less than $72^{\circ}\mathrm{F}$ and the other with initial temperature greater than $72^{\circ}\mathrm{F}$.

The idea behind a direction field is the fact that the derivative of a function evaluated at a given point is the slope of the tangent line to the graph of that function at the same point. Other examples of differential equations for which we can create a direction field include

Equation:

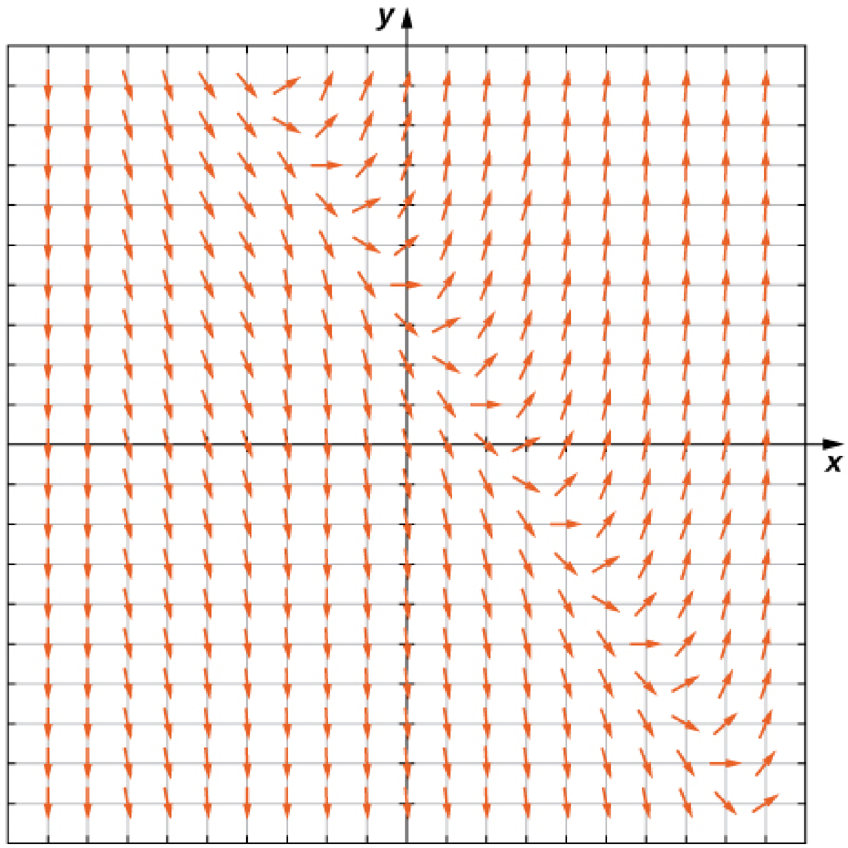
$$y' = 3x + 2y - 4$$
$$y' = x^2 - y^2$$
$$y' = \frac{2x+4}{y-2}.$$

To create a direction field, we start with the first equation: $y\prime = 3x + 2y - 4$. We let (x_0, y_0) be any ordered pair, and we substitute these numbers into the right-hand side of the differential equation. For example, if we choose $x = 1$ and $y = 2$, substituting into the right-hand side of the differential equation yields

Equation:

$$\begin{aligned} y' &= 3x + 2y - 4 \\ &= 3(1) + 2(2) - 4 = 3. \end{aligned}$$

This tells us that if a solution to the differential equation $y\prime = 3x + 2y - 4$ passes through the point $(1, 2)$, then the slope of the solution at that point must equal 3. To start creating the direction field, we put a short line segment at the point $(1, 2)$ having slope 3. We can do this for any point in the domain of the function $f(x, y) = 3x + 2y - 4$, which consists of all ordered pairs (x, y) in $\mathbb{R}^2$. Therefore any point in the Cartesian plane has a slope associated with it, assuming that a solution to the differential equation passes through that point. The direction field for the differential equation $y' = 3x + 2y - 4$ is shown in [link].



Direction field for the differential equation
$y\prime = 3x + 2y - 4.$

We can generate a direction field of this type for any differential equation of the form $y\prime = f(x, y)$.

Note:
Definition
A **direction field (slope field)** is a mathematical object used to graphically represent solutions to a first-order differential equation. At each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point.

Using Direction Fields

We can use a direction field to predict the behavior of solutions to a differential equation without knowing the actual solution. For example, the direction field in [link] serves as a guide to the behavior of solutions to the differential equation $y\prime = 3x + 2y - 4.$

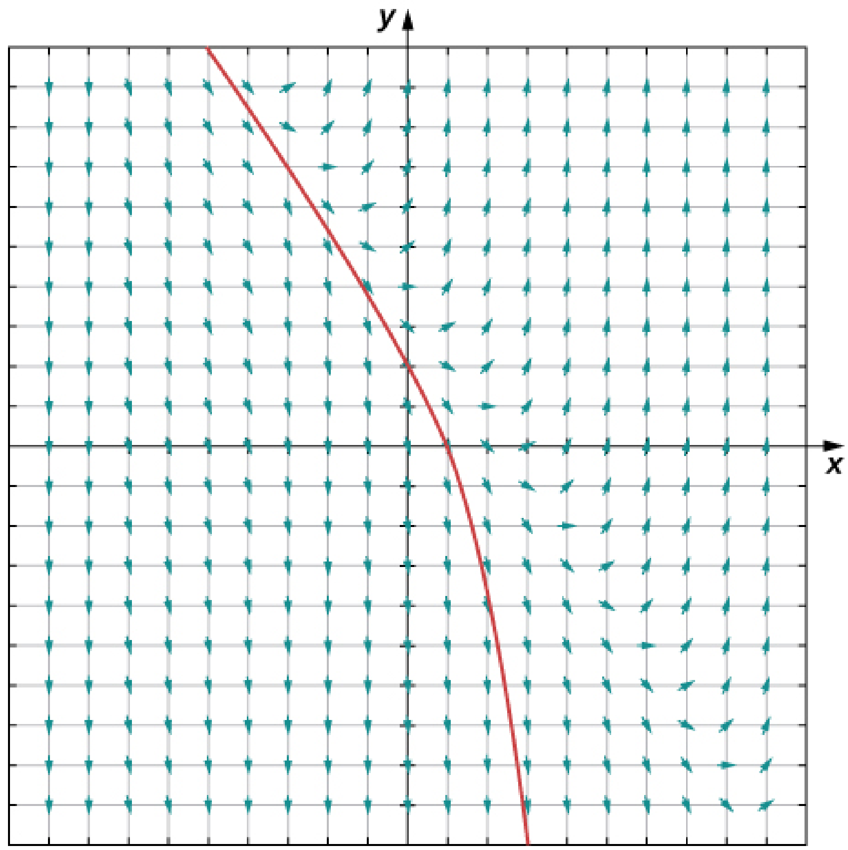
To use a direction field, we start by choosing any point in the field. The line segment at that point serves as a signpost telling us what direction to go from there. For example, if a solution to the differential equation passes through the point $(0, 1)$, then the slope of the solution passing through that point is given by $y\prime = 3\left(0\right) + 2\left(1\right) - 4 = -2.$ Now let x increase slightly, say to $x = 0.1.$ Using the method of linear approximations gives a formula for the approximate value of y for $x = 0.1.$ In particular,

Equation:

$$\begin{aligned} L\left(x\right) &= y_0 + f'\left(x_0\right)\left(x - x_0\right) \\ &= 1 - 2(x_0 - 0) \\ &= 1 - 2x_0. \end{aligned}$$

Substituting $x_0 = 0.1$ into $L(x)$ gives an approximate y value of $0.8.$

At this point the slope of the solution changes (again according to the differential equation). We can keep progressing, recalculating the slope of the solution as we take small steps to the right, and watching the behavior of the solution. [link] shows a graph of the solution passing through the point $(0, 1).$



Direction field for the differential equation $y\prime = 3x + 2y - 4$ with the solution passing through the point $(0, 1)$.

The curve is the graph of the solution to the initial-value problem
Equation:

$$y\prime = 3x + 2y - 4, \quad y(0) = 1.$$

This curve is called a **solution curve** passing through the point $(0, 1)$. The exact solution to this initial-value problem is
Equation:

$$y = -\frac{3}{2}x + \frac{5}{4} - \frac{1}{4}e^{2x},$$

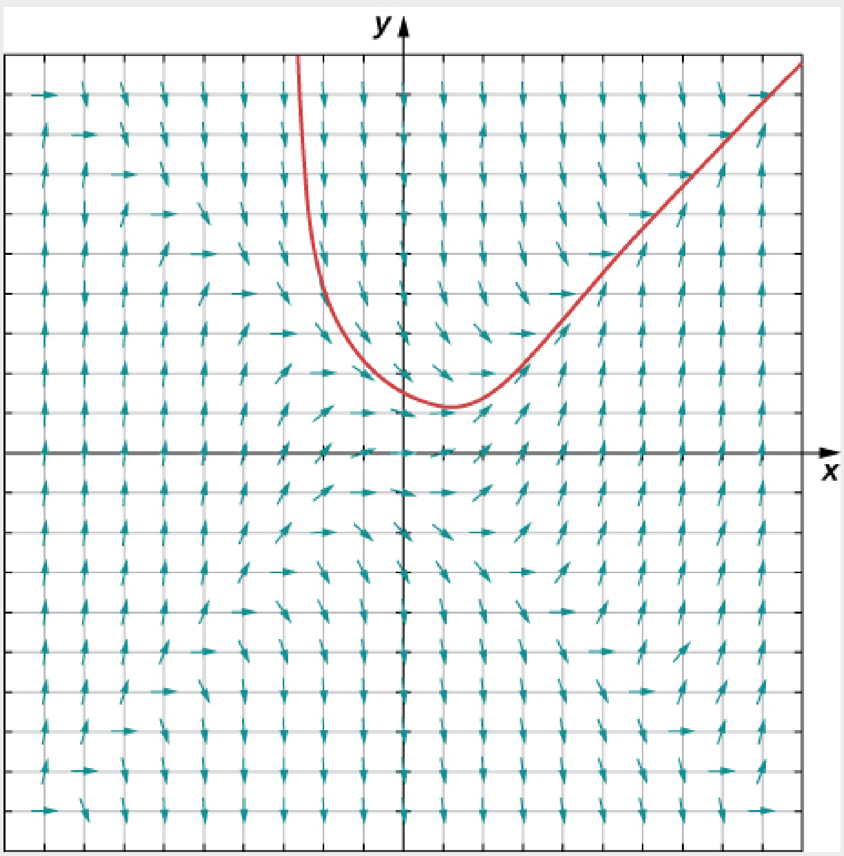
and the graph of this solution is identical to the curve in [link].

Note:
Exercise:

Problem:

Create a direction field for the differential equation $y\prime = x^2 - y^2$ and sketch a solution curve passing through the point $(-1, 2)$.

Solution:



Hint

Use x and y values ranging from -5 to 5. For each coordinate pair, calculate $y\prime$ using the right-hand side of the differential equation.

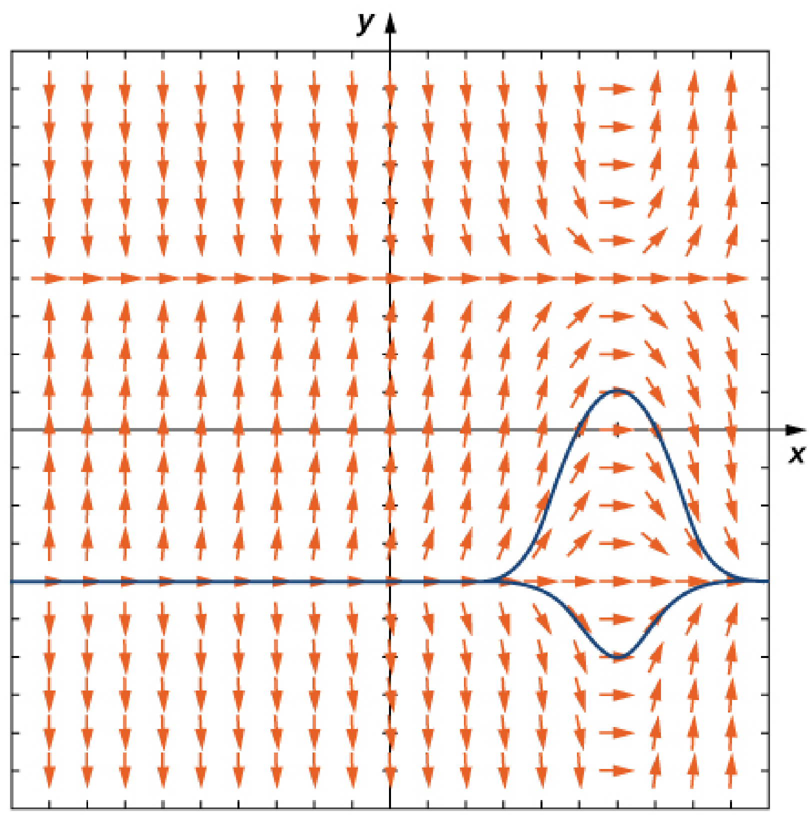
Note:
Go to this Java applet and this website to see more about slope fields.

Now consider the direction field for the differential equation $y\prime = (x-3)(y^2-4)$, shown in [link]. This direction field has several interesting properties. First of all, at $y=-2$ and $y=2$, horizontal dashes appear all the way across the graph. This means that if $y=-2$, then $y\prime = 0$. Substituting this expression into the right-hand side of the differential equation gives

Equation:

$$\begin{aligned}(x-3)(y^2-4) &= (x-3)(\text{msup}-4)\\ &= (x-3)(0)\\ &= 0\\ &= y\prime.\end{aligned}$$

Therefore $y=-2$ is a solution to the differential equation. Similarly, $y=2$ is a solution to the differential equation. These are the only constant-valued solutions to the differential equation, as we can see from the following argument. Suppose $y=k$ is a constant solution to the differential equation. Then $y'=0$. Substituting this expression into the differential equation yields $0=(x-3)\left(k^2-4\right)$. This equation must be true for all values of x, so the second factor must equal zero. This result yields the equation $k^2-4=0$. The solutions to this equation are $k=-2$ and $k=2$, which are the constant solutions already mentioned. These are called the equilibrium solutions to the differential equation.



Direction field for the differential equation $y\prime = (x-3)(y^2-4)$ showing two solutions. These solutions are very close together, but one is barely above the equilibrium solution $x = -2$ and the other is barely below the same equilibrium solution.

Note:
Definition
Consider the differential equation $y\prime = f(x,y)$. An **equilibrium solution** is any solution to the differential equation of the form $y = c$, where c is a constant.

To determine the equilibrium solutions to the differential equation $y\prime = f(x,y)$, set the right-hand side equal to zero. An equilibrium solution of the differential

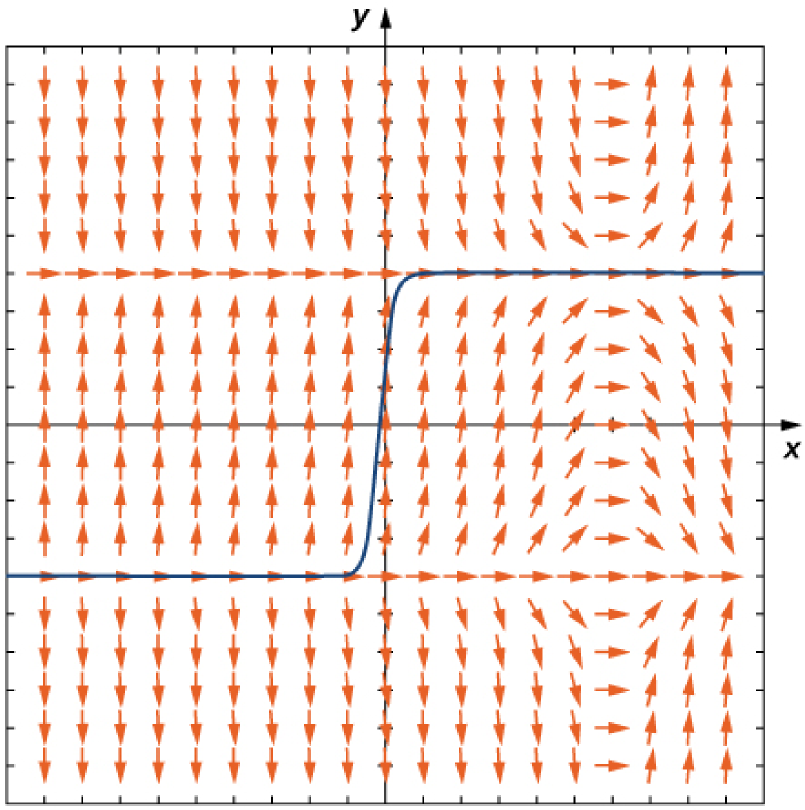
equation is any function of the form $y = k$ such that $f(x, k) = 0$ for all values of x in the domain of f.

An important characteristic of equilibrium solutions concerns whether or not they approach the line $y = k$ as an asymptote for large values of x.

Note:
Definition
Consider the differential equation $y' = f(x, y)$, and assume that all solutions to this differential equation are defined for $x \geq x_0$. Let $y = k$ be an equilibrium solution to the differential equation.

1. $y = k$ is an **asymptotically stable solution** to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem
   Equation:
   $$y' = f(x, y), \quad y(x_0) = c$$
   approaches k as x approaches infinity.
2. $y = k$ is an **asymptotically unstable solution** to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem
   Equation:
   $$y' = f(x, y), \quad y(x_0) = c$$
   never approaches k as x approaches infinity.
3. $y = k$ is an **asymptotically semi-stable solution** to the differential equation if it is neither asymptotically stable nor asymptotically unstable.

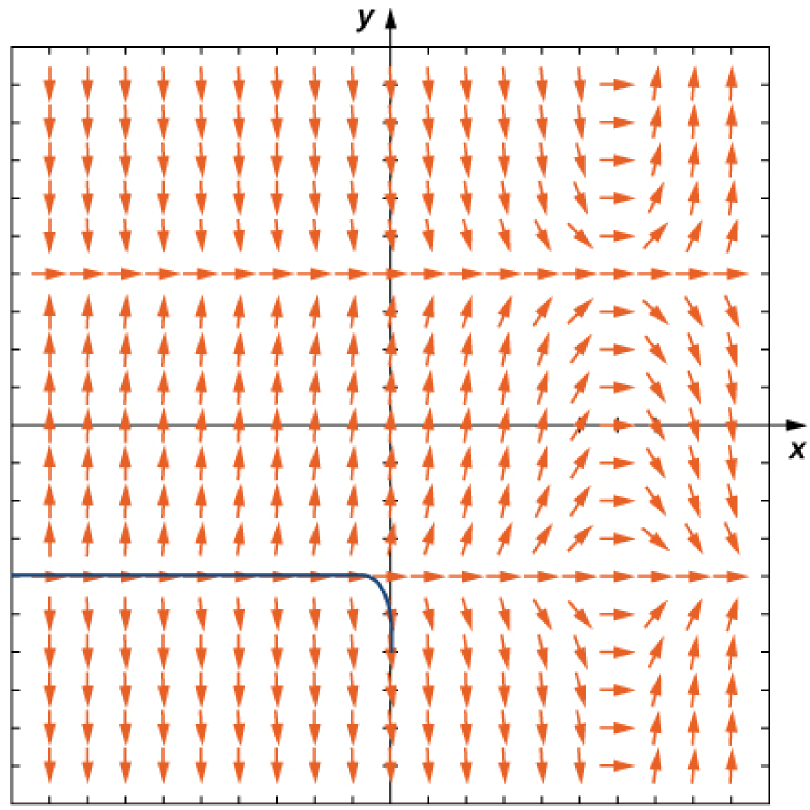
Now we return to the differential equation $y' = (x - 3)(y^2 - 4)$, with the initial condition $y(0) = 0.5$. The direction field for this initial-value problem, along with the corresponding solution, is shown in [link].



Direction field for the initial-value problem
$y\prime = (x-3)(y^2-4), y(0) = 0.5.$

The values of the solution to this initial-value problem stay between $y = -2$ and $y = 2$, which are the equilibrium solutions to the differential equation. Furthermore, as x approaches infinity, y approaches 2. The behavior of solutions is similar if the initial value is higher than 2, for example, $y\,(0) = 2.3$. In this case, the solutions decrease and approach $y = 2$ as x approaches infinity. Therefore $y = 2$ is an asymptotically stable solution to the differential equation.

What happens when the initial value is below $y = -2$? This scenario is illustrated in [link], with the initial value $y(0) = -3$.



Direction field for the initial-value problem
$y\prime = (x - 3)(y^2 - 4), \quad y(0) = -3.$

The solution decreases rapidly toward negative infinity as x approaches infinity. Furthermore, if the initial value is slightly higher than -2, then the solution approaches 2, which is the other equilibrium solution. Therefore in neither case does the solution approach $y = -2$, so $y = -2$ is called an asymptotically unstable, or unstable, equilibrium solution.

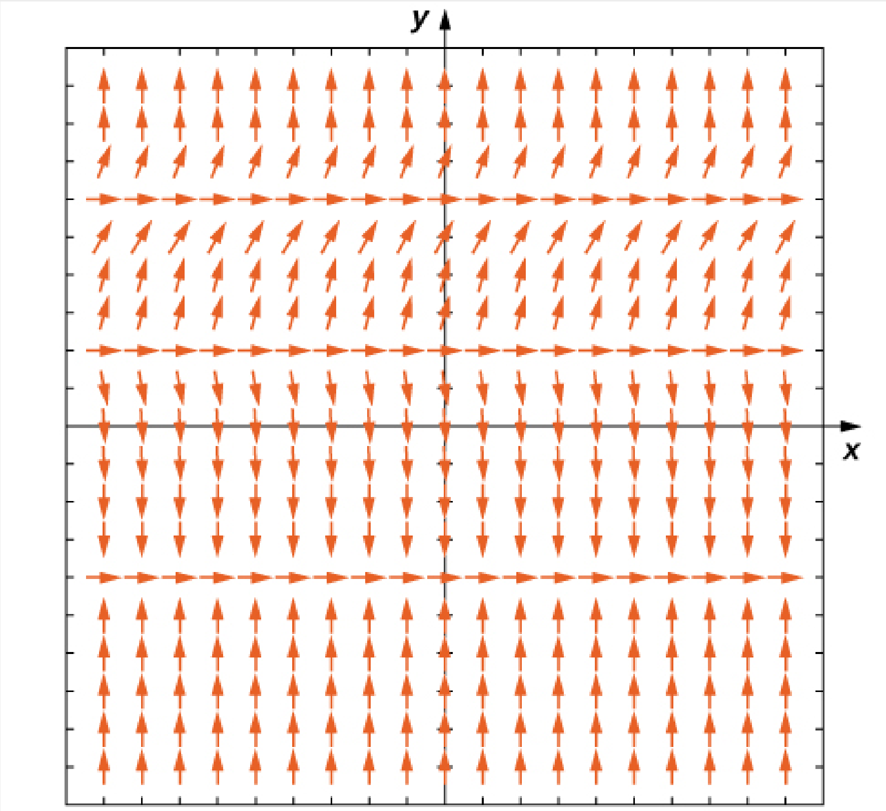
Example:
Exercise:

Problem:
Stability of an Equilibrium Solution

Create a direction field for the differential equation $y\prime = (y-3)^2(y^2+y-2)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Solution:

The direction field is shown in [link].



Direction field for the differential equation $y\prime = (y-3)^2(y^2+y-2)$.

The equilibrium solutions are $y=-2, y=1$, and $y=3$. To classify each of the solutions, look at an arrow directly above or below each of these values. For example, at $y=-2$ the arrows directly below this solution point up, and the arrows directly above the solution point down. Therefore all initial conditions close to $y=-2$ approach $y=-2$, and the solution is stable. For the solution $y=1$, all initial conditions above and below $y=1$ are repelled (pushed away) from $y=1$, so this solution is unstable. The solution $y=3$ is

semi-stable, because for initial conditions slightly greater than 3, the solution approaches infinity, and for initial conditions slightly less than 3, the solution approaches $y = 3$.

Analysis

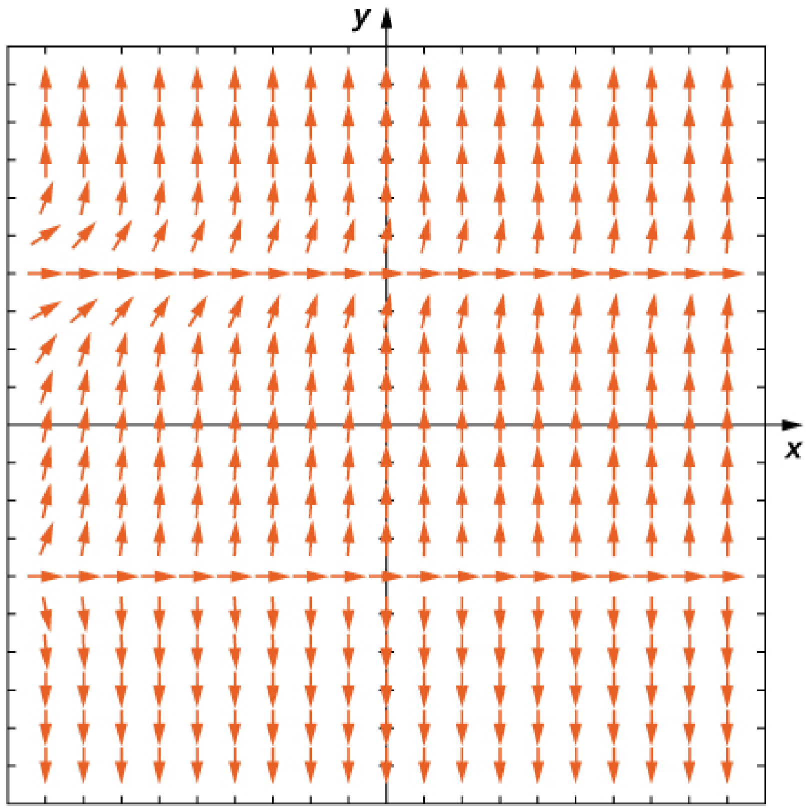
It is possible to find the equilibrium solutions to the differential equation by setting the right-hand side equal to zero and solving for y. This approach gives the same equilibrium solutions as those we saw in the direction field.

Note:
Exercise:

Problem:

Create a direction field for the differential equation $y\prime = (x + 5)(y + 2)(y^2 - 4y + 4)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Solution:



The equilibrium solutions are $y=-2$ and $y=2$. For this equation, $y=-2$ is an unstable equilibrium solution, and $y=2$ is a semi-stable equilibrium solution.

Hint

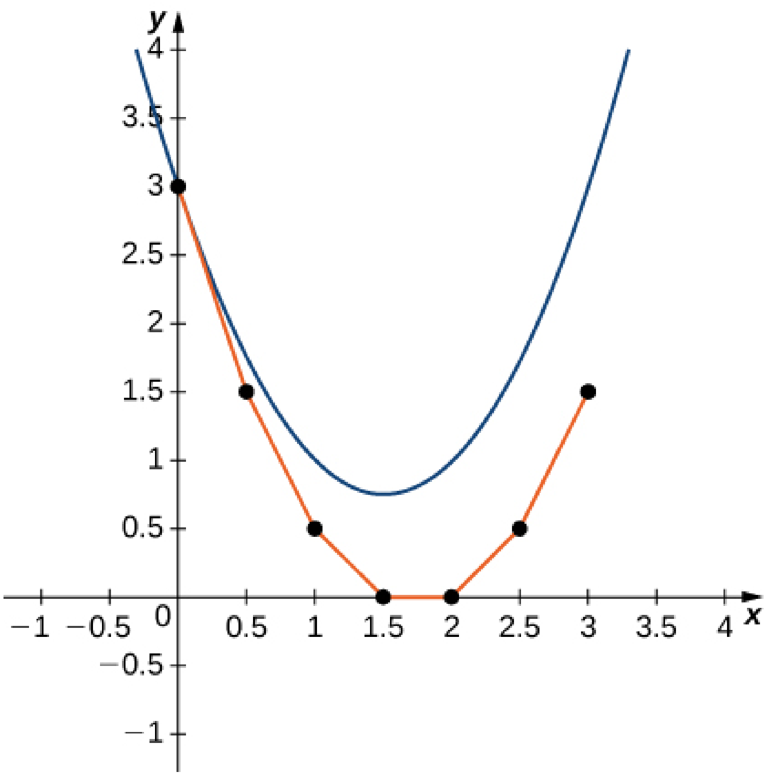
First create the direction field and look for horizontal dashes that go all the way across. Then examine the slope lines directly above and below the equilibrium solutions.

Euler’s Method

Consider the initial-value problem
Equation:

$$y' = 2x - 3, \quad y\left(0\right) = 3.$$

Integrating both sides of the differential equation gives $y = x^2 - 3x + C$, and solving for C yields the particular solution $y = x^2 - 3x + 3$. The solution for this initial-value problem appears as the parabola in [link].



Euler’s Method for the initial-value problem $y' = 2x - 3, \quad y(0) = 3.$

The red graph consists of line segments that approximate the solution to the initial-value problem. The graph starts at the same initial value of $(0, 3)$. Then the slope of the solution at any point is determined by the right-hand side of the differential equation, and the length of the line segment is determined by increasing the x value by 0.5 each time (the *step size*). This approach is the basis of Euler’s Method.

Before we state Euler’s Method as a theorem, let’s consider another initial-value problem:
Equation:

$$y' = x^2 - y^2, \quad y(-1) = 2.$$

The idea behind direction fields can also be applied to this problem to study the behavior of its solution. For example, at the point $(-1, 2)$, the slope of the solution is given by $y\prime = (-1)^2 - 2^2 = -3$, so the slope of the tangent line to the solution at that point is also equal to -3. Now we define $x_0 = -1$ and $y_0 = 2$. Since the slope of the solution at this point is equal to -3, we can use the method of linear approximation to approximate y near $(-1, 2)$.

Equation:

$$L(x) = y_0 + f'(x_0)(x - x_0).$$

Here $x_0 = -1, y_0 = 2$, and $f'(x_0) = -3$, so the linear approximation becomes

Equation:

$$\begin{aligned} L(x) &= 2 - 3(x - (-1)) \\ &= 2 - 3x - 3 \\ &= -3x - 1. \end{aligned}$$

Now we choose a **step size**. The step size is a small value, typically 0.1 or less, that serves as an increment for x; it is represented by the variable h. In our example, let $h = 0.1$. Incrementing x_0 by h gives our next x value:

Equation:

$$x_1 = x_0 + h = -1 + 0.1 = -0.9.$$

We can substitute $x_1 = -0.9$ into the linear approximation to calculate y_1.

Equation:

$$\begin{aligned} y_1 &= L(x_1) \\ &= -3(-0.9) - 1 \\ &= 1.7. \end{aligned}$$

Therefore the approximate y value for the solution when $x = -0.9$ is $y = 1.7$. We can then repeat the process, using $x_1 = -0.9$ and $y_1 = 1.7$ to calculate x_2 and y_2. The new slope is given by $y\prime = (-0.9)^2 - (1.7)^2 = -2.08$. First, $x_2 = x_1 + h = -0.9 + 0.1 = -0.8$. Using linear approximation gives

Equation:

$$\begin{aligned} L(x) &= y_1 + f'(x_1)(x - x_1) \\ &= 1.7 - 2.08(x - (-0.9)) \\ &= 1.7 - 2.08x - 1.872 \\ &= -2.08x - 0.172. \end{aligned}$$

Finally, we substitute $x_2 = -0.8$ into the linear approximation to calculate y_2.
Equation:

$$\begin{aligned} y_2 &= L(x_2) \\ &= -2.08x_2 - 0.172 \\ &= -2.08(-0.8) - 0.172 \\ &= 1.492. \end{aligned}$$

Therefore the approximate value of the solution to the differential equation is $y = 1.492$ when $x = -0.8$.

What we have just shown is the idea behind **Euler's Method**. Repeating these steps gives a list of values for the solution. These values are shown in [link], rounded off to four decimal places.

n	0	1	2	3	4	5
x_n	-1	-0.9	-0.8	-0.7	-0.6	-0.5
y_n	2	1.7	1.492	1.3334	1.2046	1.0955
n	6	7	8	9	10	
x_n	-0.4	-0.3	-0.2	-0.1	0	
y_n	1.0004	1.9164	1.8414	1.7746	1.7156	

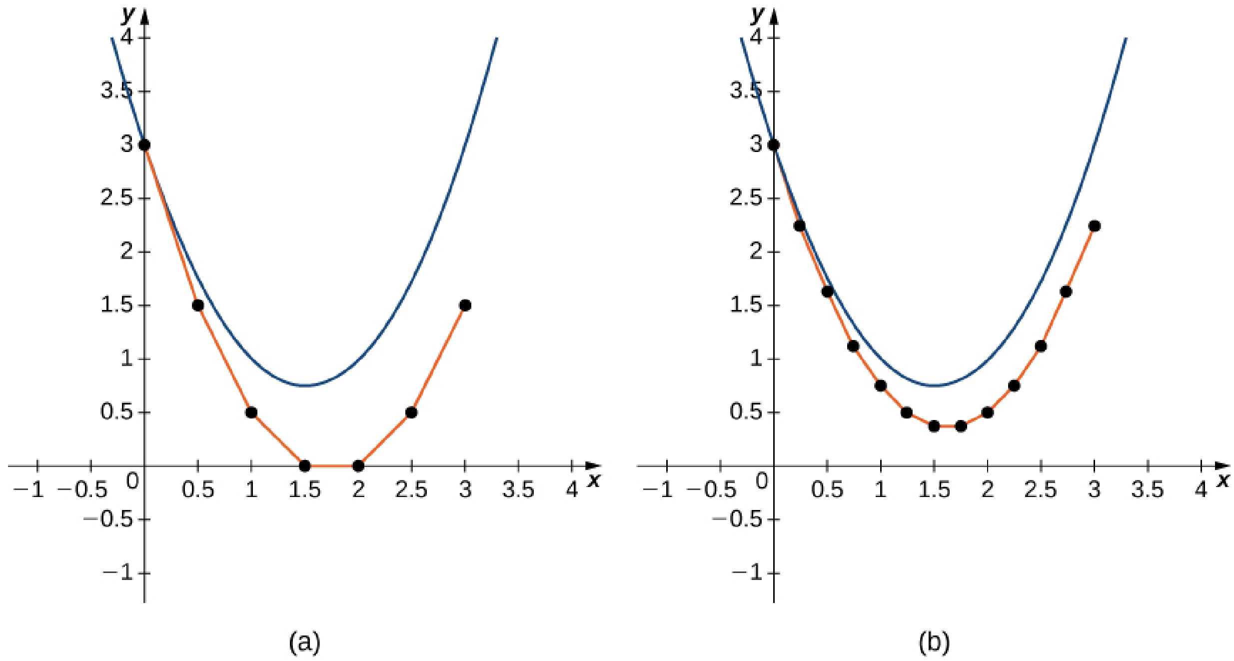
Using Euler's Method to Approximate Solutions to a Differential Equation

Note:
Euler's Method
Consider the initial-value problem
$y\prime = f(x, y), \quad y(x_0) = y_0.$
To approximate a solution to this problem using Euler's method, define
Equation:

$$x_n = x_0 + nh$$
$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}).$$

Here $h > 0$ represents the step size and n is an integer, starting with 1. The number of steps taken is counted by the variable n.

Typically h is a small value, say 0.1 or 0.05. The smaller the value of h, the more calculations are needed. The higher the value of h, the fewer calculations are needed. However, the tradeoff results in a lower degree of accuracy for larger step size, as illustrated in [link].



Euler’s method for the initial-value problem $y' = 2x - 3, y(0) = 3$ with (a) a step size of $h = 0.5$; and (b) a step size of $h = 0.25$.

Example:
Exercise:

Problem:
Using Euler’s Method

Consider the initial-value problem
Equation:

$$y' = 3x^2 - y^2 + 1, \quad y(0) = 2.$$

Use Euler’s method with a step size of 0.1 to generate a table of values for the solution for values of x between 0 and 1.

Solution:

We are given $h = 0.1$ and $f(x, y) = 3x^2 - y^2 + 1$. Furthermore, the initial condition $y(0) = 2$ gives $x_0 = 0$ and $y_0 = 2$. Using [link] with $n = 0$, we can generate [link].

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	0	2
1	0.1	$y_1 = y_0 + hf(x_0, y_0) = 1.7$
2	0.2	$y_2 = y_1 + hf(x_1, y_1) = 1.514$
3	0.3	$y_3 = y_2 + hf(x_2, y_2) = 1.3968$
4	0.4	$y_4 = y_3 + hf(x_3, y_3) = 1.3287$
5	0.5	$y_5 = y_4 + hf(x_4, y_4) = 1.3001$
6	0.6	$y_6 = y_5 + hf(x_5, y_5) = 1.3061$
7	0.7	$y_7 = y_6 + hf(x_6, y_6) = 1.3435$
8	0.8	$y_8 = y_7 + hf(x_7, y_7) = 1.4100$
9	0.9	$y_9 = y_8 + hf(x_8, y_8) = 1.5032$
10	1.0	$y_{10} = y_9 + hf(x_9, y_9) = 1.6202$

Using Euler's Method to Approximate Solutions to a Differential Equation

With ten calculations, we are able to approximate the values of the solution to the initial-value problem for values of x between 0 and 1.

Note:
For more information on Euler's method use this applet.

Note:
Exercise:

Problem: Consider the initial-value problem
Equation:

$$y' = x^3 + y^2, \quad y(1) = -2.$$

Using a step size of 0.1, generate a table with approximate values for the solution to the initial-value problem for values of x between 1 and 2.

Solution:

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	1	-2
1	1.1	$y_1 = y_0 + hf(x_0, y_0) = -1.5$
2	1.2	$y_2 = y_1 + hf(x_1, y_1) = -1.1419$
3	1.3	$y_3 = y_2 + hf(x_2, y_2) = -0.8387$
4	1.4	$y_4 = y_3 + hf(x_3, y_3) = -0.5487$
5	1.5	$y_5 = y_4 + hf(x_4, y_4) = -0.2442$
6	1.6	$y_6 = y_5 + hf(x_5, y_5) = 0.0993$

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
7	1.7	$y_7 = y_6 + hf(x_6, y_6) = 0.5099$
8	1.8	$y_8 = y_7 + hf(x_7, y_7) = 1.0272$
9	1.9	$y_9 = y_8 + hf(x_8, y_8) = 1.7159$
10	2	$y_{10} = y_9 + hf(x_9, y_9) = 2.6962$

Hint

Start by identifying the value of h, then figure out what $f(x, y)$ is. Then use the formula for Euler's Method to calculate y_1, y_2, and so on.

Note:
Visit this website for a practical application of the material in this section.

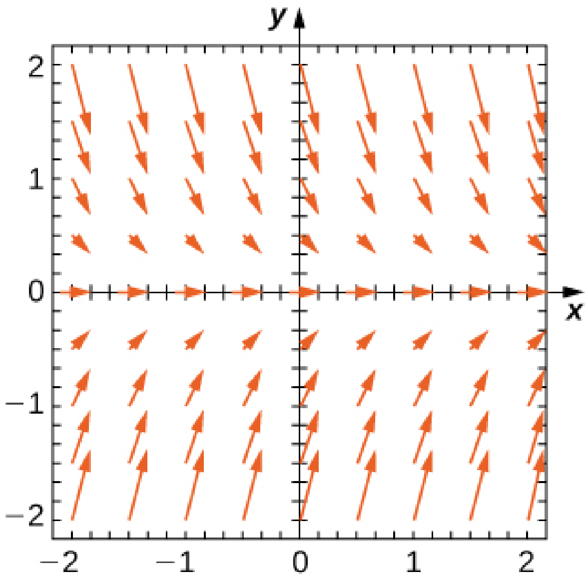
Key Concepts

- A direction field is a mathematical object used to graphically represent solutions to a first-order differential equation.
- Euler's Method is a numerical technique that can be used to approximate solutions to a differential equation.

Key Equations

- **Euler's Method**
  $x_n = x_0 + nh$
  $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, where h is the step size

For the following problems, use the direction field below from the differential equation $y' = -2y$. Sketch the graph of the solution for the given initial conditions.



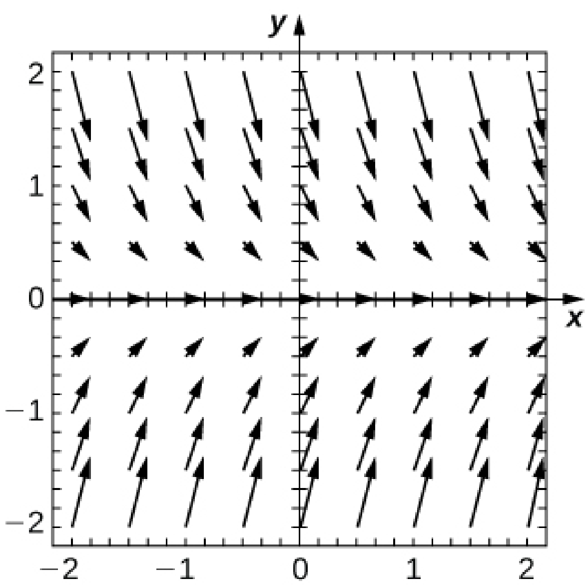
Exercise:

Problem: $y(0) = 1$

Exercise:

Problem: $y(0) = 0$

Solution:



Exercise:

Problem: $y(0) = -1$

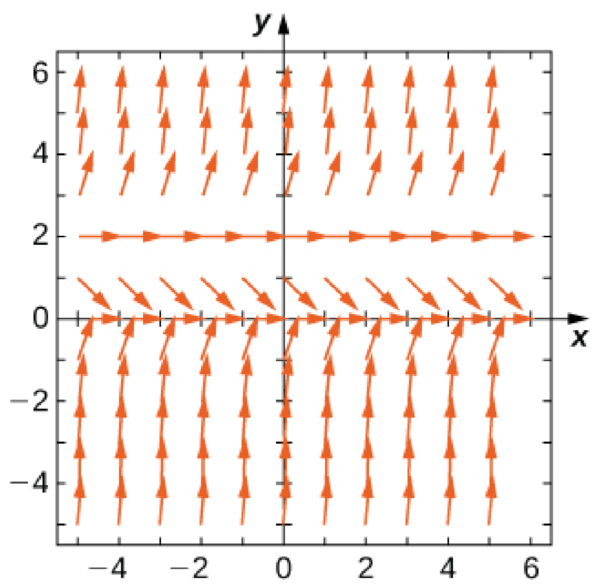
Exercise:

Problem: Are there any equilibria? What are their stabilities?

Solution:

$y = 0$ is a stable equilibrium

For the following problems, use the direction field below from the differential equation $y\prime = y^2 - 2y$. Sketch the graph of the solution for the given initial conditions.



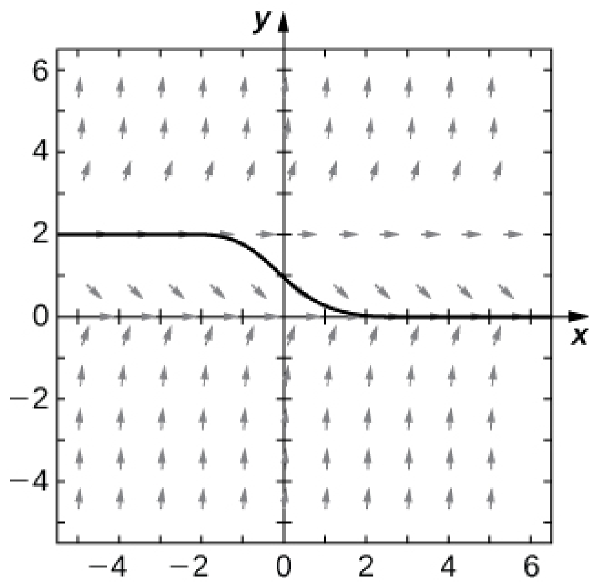
Exercise:

Problem: $y(0) = 3$

Exercise:

Problem: $y(0) = 1$

Solution:



Exercise:

Problem: $y(0) = -1$

Exercise:

Problem: Are there any equilibria? What are their stabilities?

Solution:

$y = 0$ is a stable equilibrium and $y = 2$ is unstable

Draw the direction field for the following differential equations, then solve the differential equation. Draw your solution on top of the direction field. Does your solution follow along the arrows on your direction field?

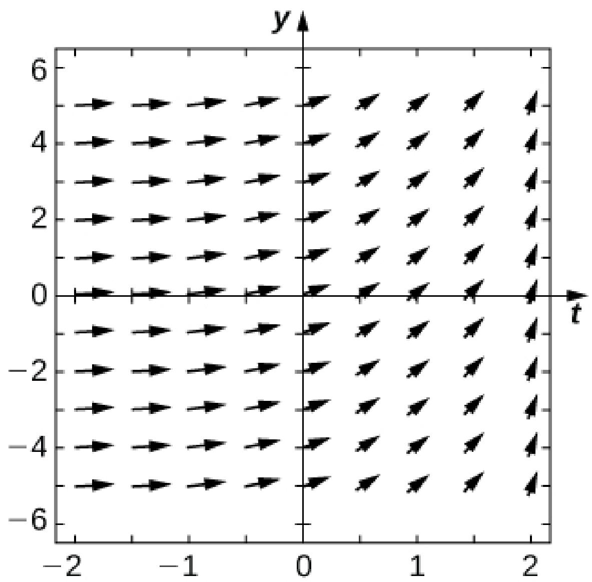
Exercise:

Problem: $y\prime = t^3$

Exercise:

Problem: $y\prime = e^t$

Solution:



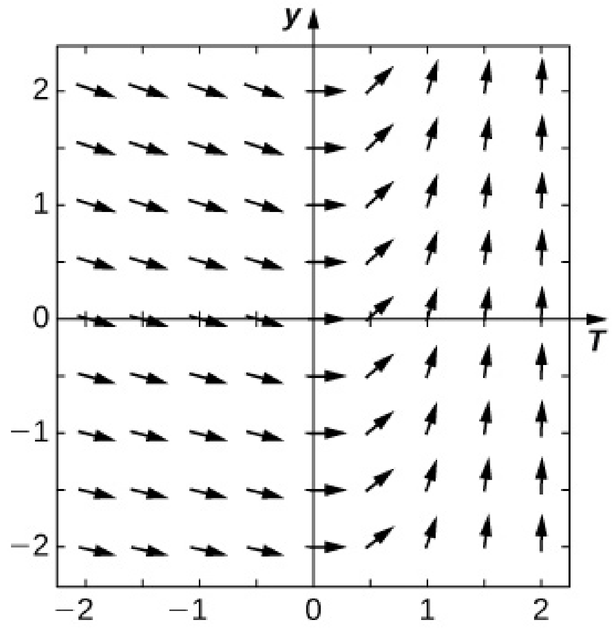
Exercise:

Problem: $\frac{dy}{dx} = x^2 \cos x$

Exercise:

Problem: $\frac{dy}{dt} = te^t$

Solution:



Exercise:

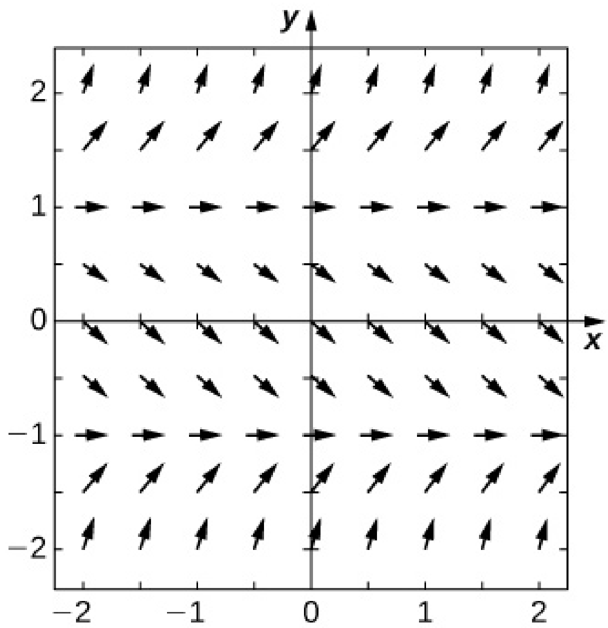
Problem: $\frac{dx}{dt} = \cosh(t)$

Draw the directional field for the following differential equations. What can you say about the behavior of the solution? Are there equilibria? What stability do these equilibria have?

Exercise:

Problem: $y\prime = y^2 - 1$

Solution:



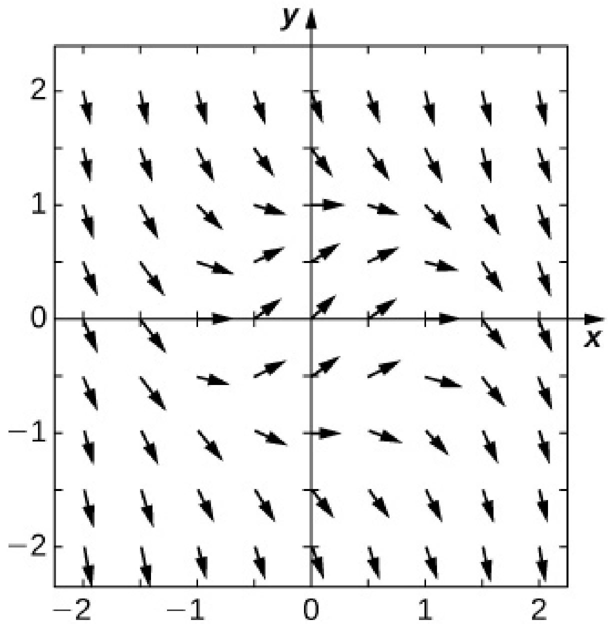
Exercise:

Problem: $y\prime = y - x$

Exercise:

Problem: $y\prime = 1 - y^2 - x^2$

Solution:



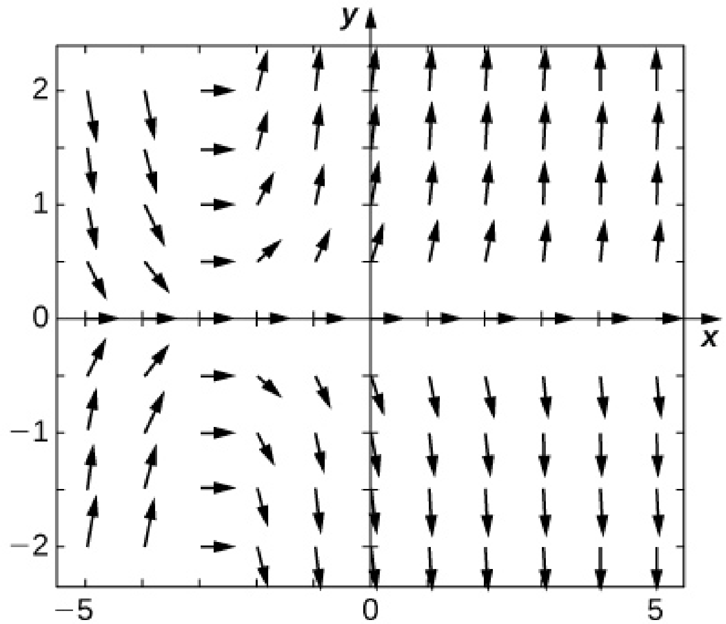
Exercise:

Problem: $y\prime = t^2 \sin y$

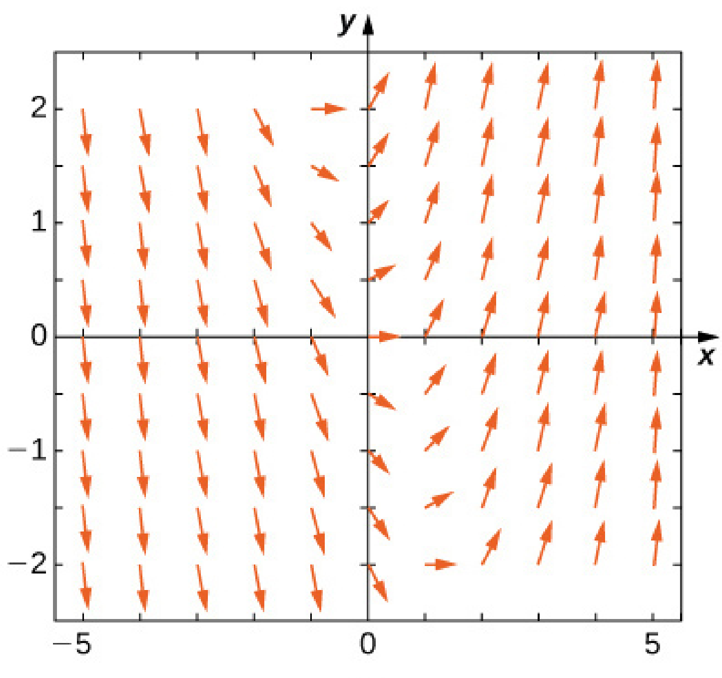
Exercise:

Problem: $y\prime = 3y + xy$

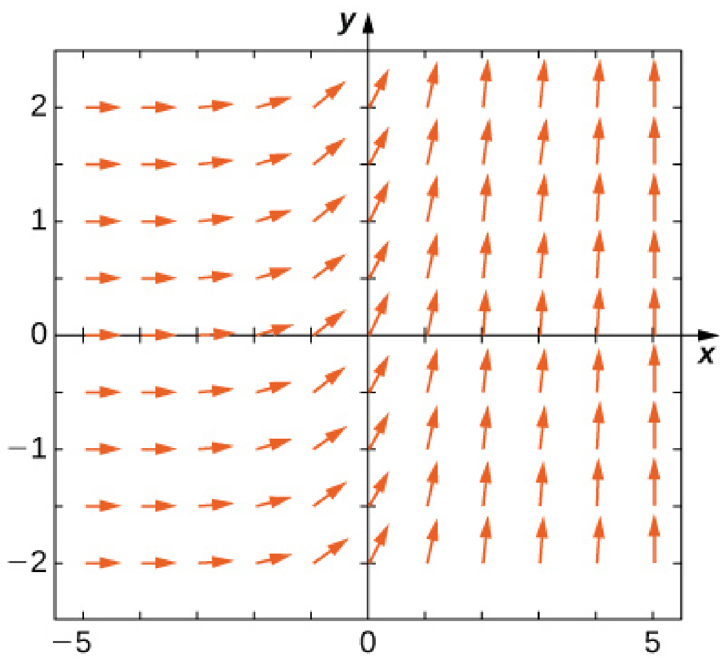
Solution:



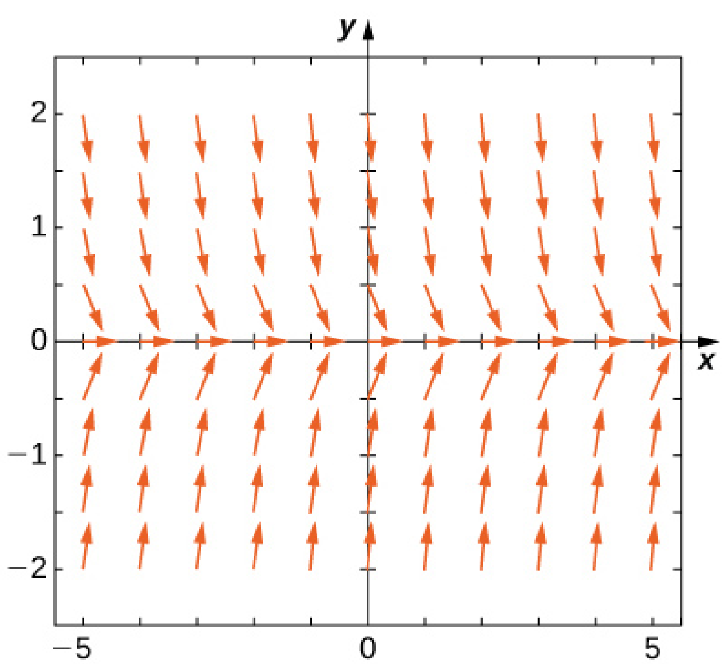
Match the direction field with the given differential equations. Explain your selections.



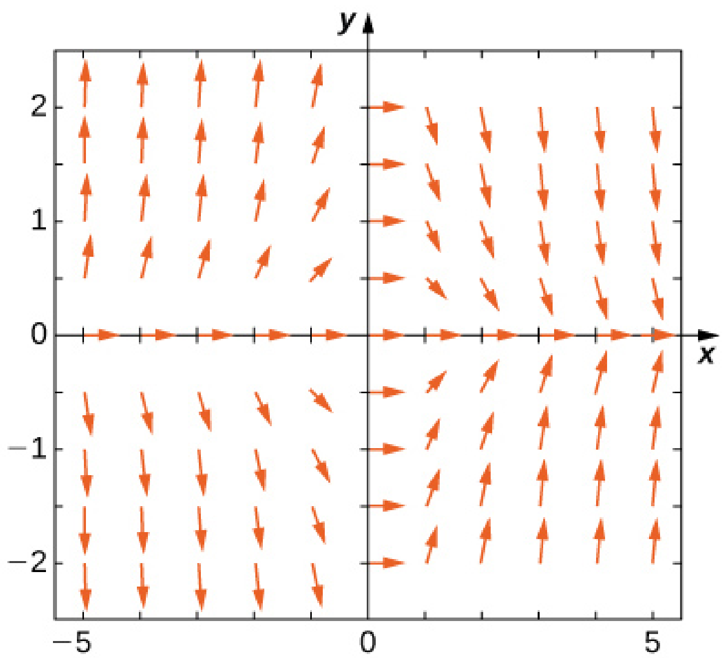
(a)



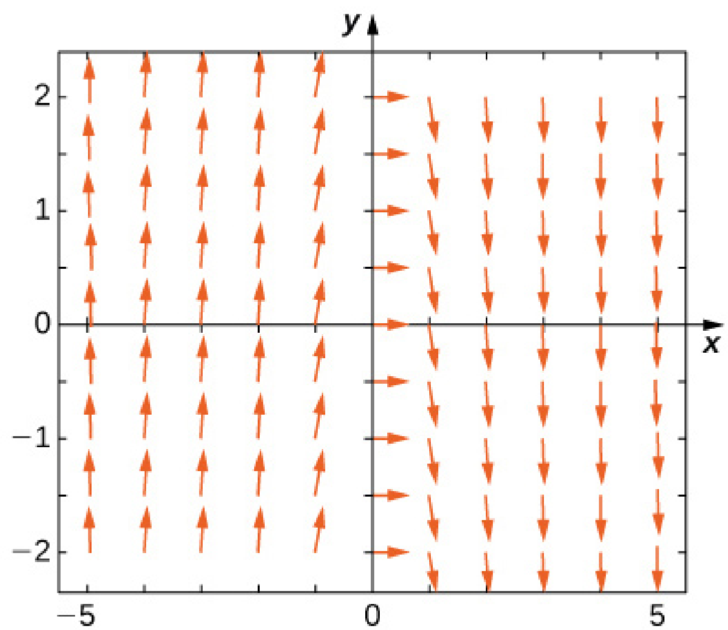
(b)



(c)



(d)



(e)

Exercise:

Problem: $y\prime = -3y$

Exercise:

Problem: $y\prime = -3t$

Solution:

E

Exercise:

Problem: $y\prime = e^t$

Exercise:

Problem: $y\prime = \frac{1}{2}y + t$

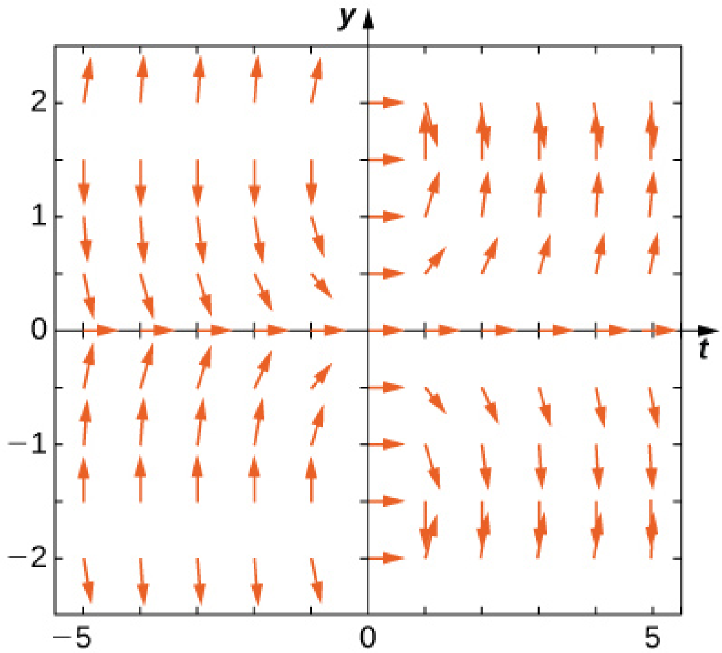
Solution:

A

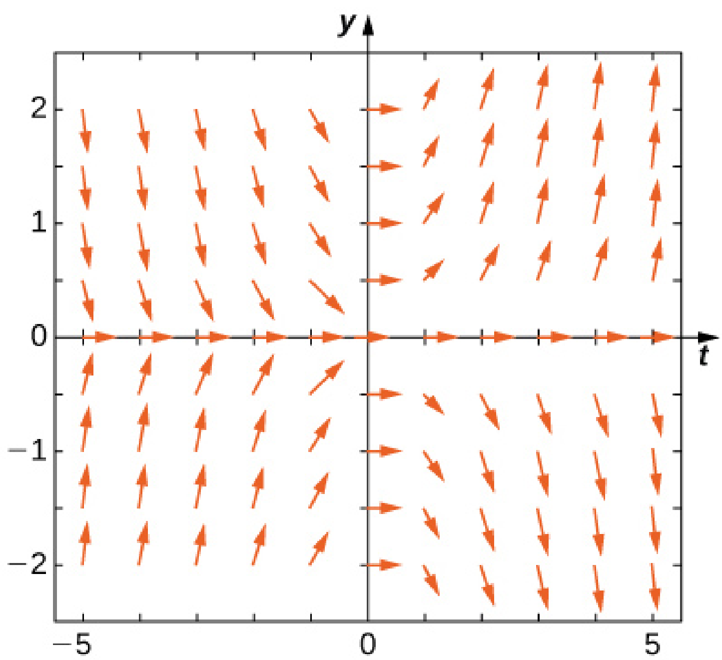
Exercise:

Problem: $y\prime = -ty$

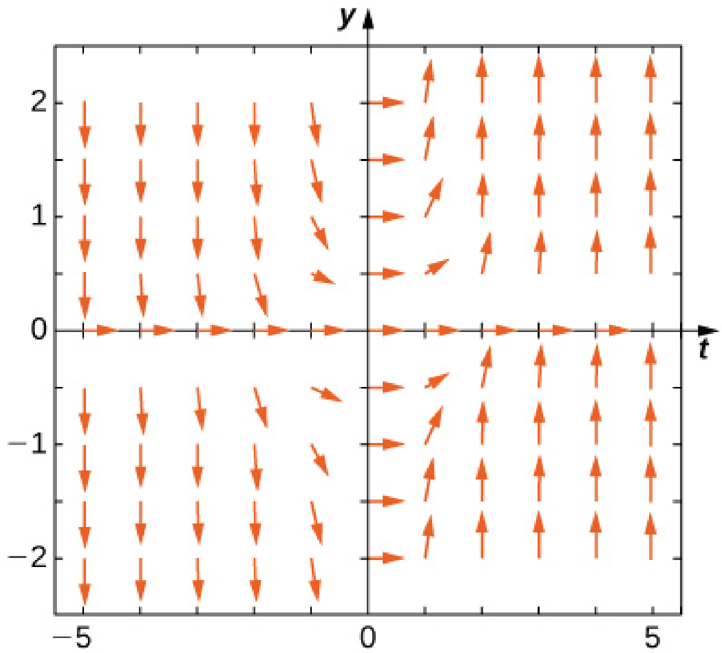
Match the direction field with the given differential equations. Explain your selections.



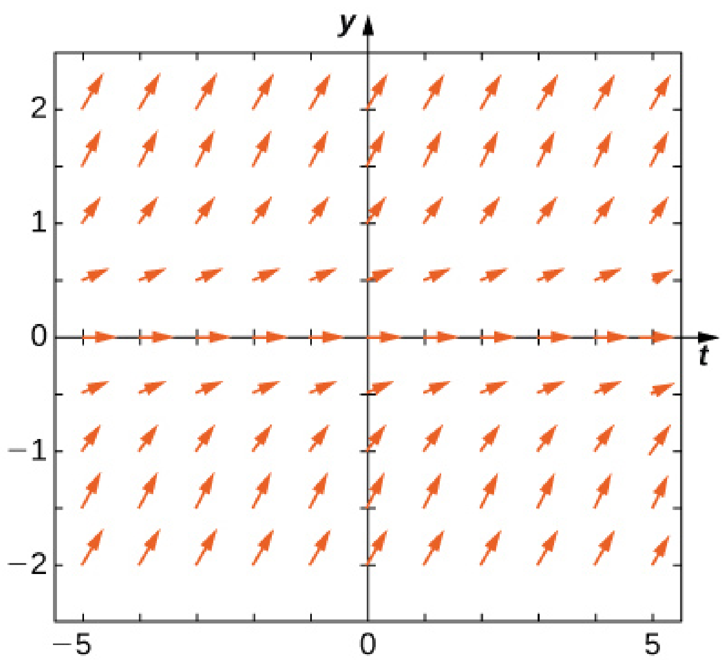
(a)



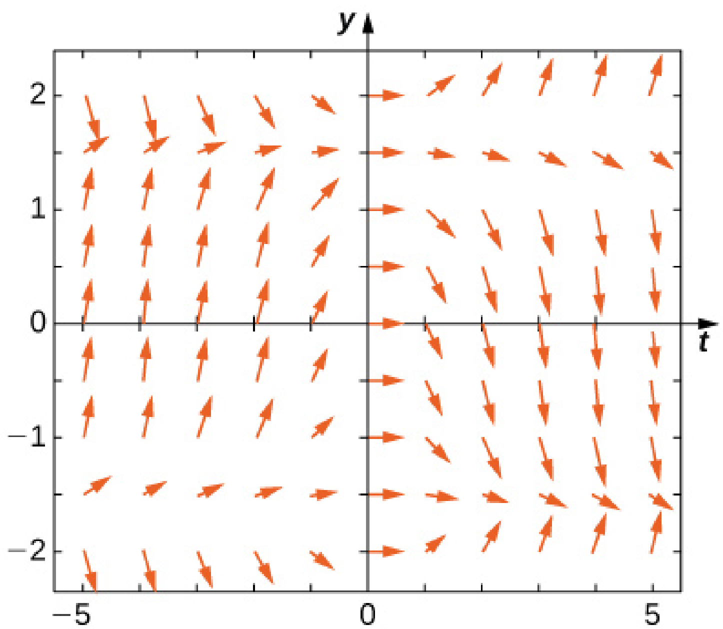
(b)



(c)



(d)



(e)

Exercise:

Problem: $y\prime = t \sin y$

Solution:

B

Exercise:

Problem: $y\prime = -t \cos y$

Exercise:

Problem: $y\prime = t \tan y$

Solution:

A

Exercise:

Problem: $y\prime = \sin^2 y$

Exercise:

Problem: $y\prime = y^2 t^3$

Solution:

C

Estimate the following solutions using Euler's method with $n = 5$ steps over the interval $t = [0, 1]$. If you are able to solve the initial-value problem exactly, compare your solution with the exact solution. If you are unable to solve the initial-value problem, the exact solution will be provided for you to compare with Euler's method. How accurate is Euler's method?

Exercise:

Problem: $y\prime = -3y, \quad y(0) = 1$

Exercise:

Problem: $y\prime = t^2$

Solution:

2.24, exact: 3

Exercise:

Problem: $y' = 3t - y, y(0) = 1$. Exact solution is $y = 3t + 4e^{-t} - 3$

Exercise:

Problem: $y' = y + t^2, y(0) = 3$. Exact solution is $y = 5e^t - 2 - t^2 - 2t$

Solution:

7.739364, exact: $5(e - 1)$

Exercise:

Problem: $y' = 2t, y(0) = 0$

Exercise:

Problem: [T] $y\prime = e^{(x+y)}, y(0) = -1$. Exact solution is $y = -\ln(e + 1 - e^x)$

Solution:

-0.2535 exact: 0

Exercise:

Problem:

$y' = y^2 \ln(x+1), y(0) = 1$. Exact solution is $y = -\frac{1}{(x+1)(\ln(x+1)-1)}$

Exercise:

Problem: $y' = 2^x, y(0) = 0$, Exact solution is $y = \frac{2^x - 1}{\ln(2)}$

Solution:

1.345, exact: $\frac{1}{\ln(2)}$

Exercise:

Problem: $y' = y, y(0) = -1$. Exact solution is $y = -e^x$.

Exercise:

Problem: $y' = -5t, y(0) = -2$. Exact solution is $y = -\frac{5}{2}t^2 - 2$

Solution:

-4, exact: $-1/2$

Differential equations can be used to model disease epidemics. In the next set of problems, we examine the change of size of two sub-populations of people living in a city: individuals who are infected and individuals who are susceptible to infection. S represents the size of the susceptible population, and I represents the size of the infected population. We assume that if a susceptible person interacts with an infected person, there is a probability c that the susceptible person will become infected. Each infected person recovers from the infection at a rate r and becomes susceptible again. We consider the case of influenza, where we assume

that no one dies from the disease, so we assume that the total population size of the two sub-populations is a constant number, N. The differential equations that model these population sizes are

Equation:

$$S\prime = rI - cSI \quad \text{and}$$
$$I\prime = cSI - rI.$$

Here c represents the contact rate and r is the recovery rate.

Exercise:

Problem:

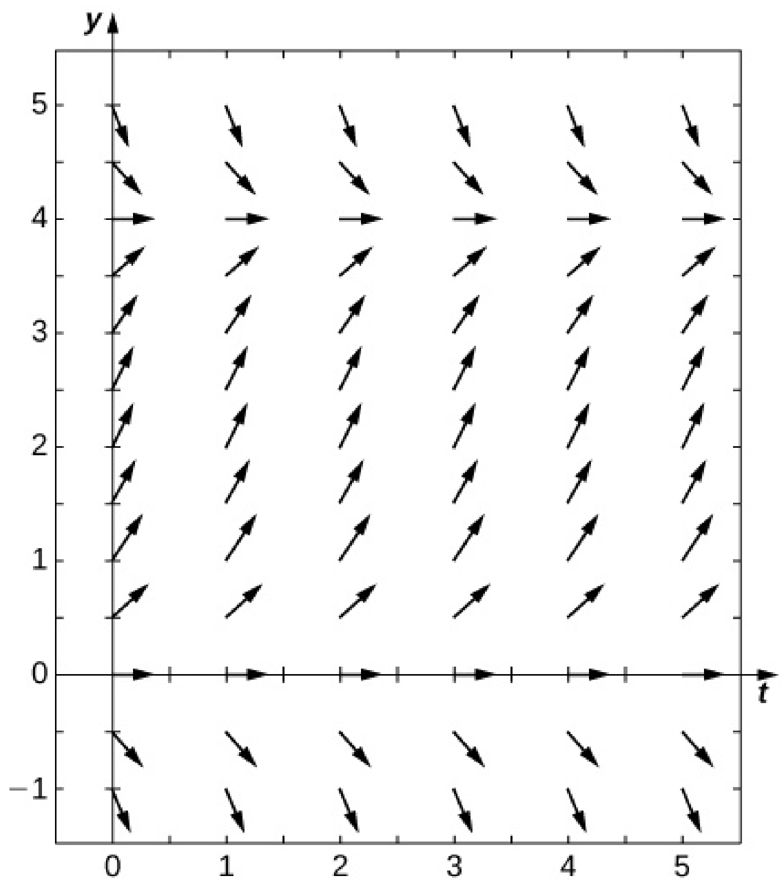
Show that, by our assumption that the total population size is constant $(S + I = N)$, you can reduce the system to a single differential equation in I: $I\prime = c(N - I)I - rI$.

Exercise:

Problem:

Assuming the parameters are $c = 0.5$, $N = 5$, and $r = 0.5$, draw the resulting directional field.

Solution:



Exercise:

Problem:

[T] Use computational software or a calculator to compute the solution to the initial-value problem $y\prime = ty, \quad y(0) = 2$ using Euler's Method with the given step size h. Find the solution at $t = 1$. For a hint, here is "pseudo-code" for how to write a computer program to perform Euler's Method for $y\prime = f(t, y), y(0) = 2$:

Create function $f(t, y)$

Define parameters $y(1) = y_0, \quad t(0) = 0$, step size h, and total number of steps, N

Write a for loop:

$\text{for k} = 1 \text{ to N}$

$\text{fn} = \text{f}(\text{t}(\text{k}), \text{y}(\text{k}))$

$\mathrm{y}\left(\mathrm{k}{+}1\right)=\mathrm{y}\left(\mathrm{k}\right)+\mathrm{h}{*}\mathrm{fn}$

$\mathrm{t}\left(\mathrm{k}{+}1\right)=\mathrm{t}\left(\mathrm{k}\right)+\mathrm{h}$

Exercise:

Problem: Solve the initial-value problem for the exact solution.

Solution:

$y\prime = 2e^{t^2/2}$

Exercise:

Problem: Draw the directional field

Exercise:

Problem: $h=1$

Solution:

2

Exercise:

Problem: **[T]** $h=10$

Exercise:

Problem: **[T]** $h=100$

Solution:

3.2756

Exercise:

Problem: **[T]** $h=1000$

Exercise:

Problem:

[T] Evaluate the exact solution at $t = 1$. Make a table of errors for the relative error between the Euler's method solution and the exact solution. How much does the error change? Can you explain?

Solution:

$2\sqrt{e}$

Step Size	Error
$h = 1$	0.3935
$h = 10$	0.06163
$h = 100$	0.006612
$h = 1000$	0.0006661

Consider the initial-value problem $y\prime = -2y, \quad y(0) = 2.$

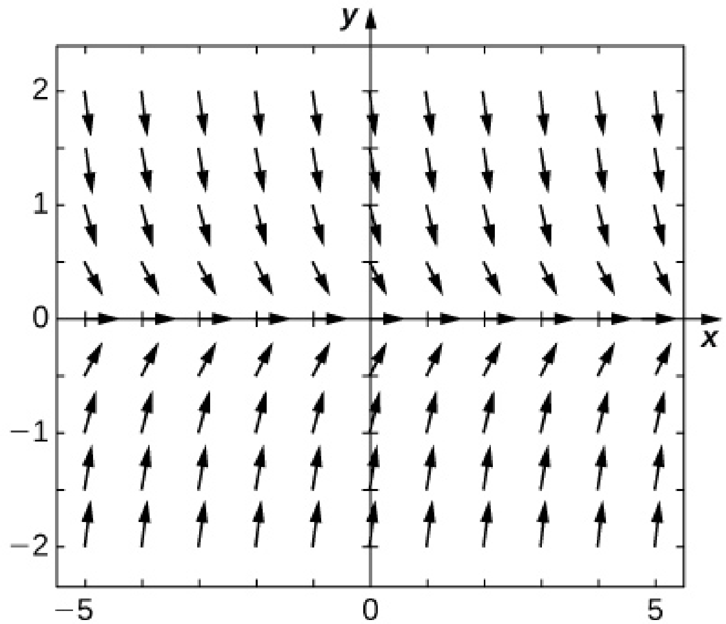
Exercise:

Problem: Show that $y = 2e^{-2x}$ solves this initial-value problem.

Exercise:

Problem: Draw the directional field of this differential equation.

Solution:



Exercise:

Problem:

[T] By hand or by calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 5$.

Exercise:

Problem:

[T] By calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 100$.

Solution:

$4.0741e^{-10}$

Exercise:

Problem:

[T] Plot exact answer and each Euler approximation (for $h = 5$ and $h = 100$) at each h on the directional field. What do you notice?

Glossary

asymptotically semi-stable solution

$y=k$ if it is neither asymptotically stable nor asymptotically unstable

asymptotically stable solution

$y=k$ if there exists $\varepsilon>0$ such that for any value $c\in(k-\varepsilon,k+\varepsilon)$ the solution to the initial-value problem $y'=f(x,y),\quad y(x_0)=c$ approaches k as x approaches infinity

asymptotically unstable solution

$y=k$ if there exists $\varepsilon>0$ such that for any value $c\in(k-\varepsilon,k+\varepsilon)$ the solution to the initial-value problem $y'=f(x,y),\quad y(x_0)=c$ never approaches k as x approaches infinity

direction field (slope field)

a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

equilibrium solution

any solution to the differential equation of the form $y=c$, where c is a constant

Euler’s Method

a numerical technique used to approximate solutions to an initial-value problem

solution curve

a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

step size

the increment h that is added to the x value at each step in Euler’s Method

Separable Equations

- Use separation of variables to solve a differential equation.
- Solve applications using separation of variables.

We now examine a solution technique for finding exact solutions to a class of differential equations known as separable differential equations. These equations are common in a wide variety of disciplines, including physics, chemistry, and engineering. We illustrate a few applications at the end of the section.

Separation of Variables

We start with a definition and some examples.

Note:
Definition
A **separable differential equation** is any equation that can be written in the form
Equation:

$$y' = f(x)g(y).$$

The term ‘separable’ refers to the fact that the right-hand side of the equation can be separated into a function of x times a function of y. Examples of separable differential equations include
Equation:

$$y\prime = \left(x^2 - 4\right)(3y + 2)$$
$$y\prime = 6x^2 + 4x$$
$$y\prime = \sec y + \tan y$$
$$y\prime = xy + 3x - 2y - 6.$$

The second equation is separable with $f(x) = 6x^2 + 4x$ and $g(y) = 1$, the third equation is separable with $f(x) = 1$ and $g(y) = \sec y + \tan y$, and the right-hand side of the fourth equation can be factored as $(x + 3)(y - 2)$, so it is separable as well. The third equation is also called an **autonomous differential equation** because the right-hand side of the equation is a function of y alone. If a differential equation is separable, then it is possible to solve the equation using the method of **separation of variables**.

Note:
Problem-Solving Strategy: Separation of Variables

1. Check for any values of y that make $g(y) = 0$. These correspond to constant solutions.
2. Rewrite the differential equation in the form $\frac{dy}{g(y)} = f(x)dx.$
3. Integrate both sides of the equation.
4. Solve the resulting equation for y if possible.
5. If an initial condition exists, substitute the appropriate values for x and y into the equation and solve for the constant.

Note that Step 4. states "Solve the resulting equation for y if possible." It is not always possible to obtain y as an explicit function of x. Quite often we have to be satisfied with finding y as an implicit function of x.

Example:
Exercise:

Problem:
Using Separation of Variables

Find a general solution to the differential equation $y\prime = \left(x^2 - 4\right)(3y + 2)$ using the method of separation of variables.

Solution:

Follow the five-step method of separation of variables.

1. In this example, $f(x) = x^2 - 4$ and $g(y) = 3y + 2$. Setting $g(y) = 0$ gives $y = -\frac{2}{3}$ as a constant solution.
2. Rewrite the differential equation in the form
   Equation:

$$\frac{dy}{3y + 2} = (x^2 - 4)dx.$$

3. Integrate both sides of the equation:
   Equation:

$$\int \frac{dy}{3y + 2} = \int \left(x^2 - 4\right) dx.$$

   Let $u = 3y + 2$. Then $du = 3\frac{dy}{dx}dx$, so the equation becomes
   Equation:

$$\begin{aligned} \frac{1}{3}\int \frac{1}{u}du &= \frac{1}{3}x^3 - 4x + C \\ \frac{1}{3}\ln|u| &= \frac{1}{3}x^3 - 4x + C \\ \frac{1}{3}\ln|3y + 2| &= \frac{1}{3}x^3 - 4x + C. \end{aligned}$$

4. To solve this equation for y, first multiply both sides of the equation by 3.
   Equation:

$$\ln|3y+2| = x^3 - 12x + 3C$$

Now we use some logic in dealing with the constant C. Since C represents an arbitrary constant, $3C$ also represents an arbitrary constant. If we call the second arbitrary constant C_1, the equation becomes
Equation:

$$\ln|3y+2| = x^3 - 12x + C_1.$$

Now exponentiate both sides of the equation (i.e., make each side of the equation the exponent for the base e).
Equation:

$$\begin{aligned} e^{\ln|3y+2|} &= e^{x^3-12x+C_1} \\ |3y+2| &= e^{C_1}e^{x^3-12x} \end{aligned}$$

Again define a new constant $C_2 = e^{c_1}$ (note that $C_2 > 0$):
Equation:

$$|3y+2| = C_2e^{x^3-12x}.$$

This corresponds to two separate equations: $3y+2 = C_2e^{x^3-12x}$ and $3y+2 = -C_2e^{x^3-12x}$.
The solution to either equation can be written in the form $y = \frac{-2 \pm C_2e^{x^3-12x}}{3}$.
Since $C_2 > 0$, it does not matter whether we use plus or minus, so the constant can actually have either sign. Furthermore, the subscript on the constant C is entirely arbitrary, and can be dropped. Therefore the solution can be written as
Equation:

$$y = \frac{-2 + Ce^{x^3 - 12x}}{3}.$$

5. No initial condition is imposed, so we are finished.

Note:
Exercise:

Problem:

Use the method of separation of variables to find a general solution to the differential equation $y\prime = 2xy + 3y - 4x - 6$.

Solution:

$y = 2 + Ce^{x^2 + 3x}$

Hint

First factor the right-hand side of the equation by grouping, then use the five-step strategy of separation of variables.

Example:
Exercise:

Problem:
Solving an Initial-Value Problem

Using the method of separation of variables, solve the initial-value problem
Equation:

$$y\prime = (2x + 3)(y^2 - 4), \quad y(0) = -3.$$

Solution:

Follow the five-step method of separation of variables.

1. In this example, $f(x) = 2x + 3$ and $g(y) = y^2 - 4$. Setting $g(y) = 0$ gives $y = \pm 2$ as constant solutions.
2. Divide both sides of the equation by $y^2 - 4$ and multiply by dx. This gives the equation
   Equation:
   $$\frac{dy}{y^2 - 4} = (2x + 3)\,dx.$$
3. Next integrate both sides:
   Equation:
   $$\int \frac{1}{y^2 - 4}\,dy = \int (2x + 3)\,dx.$$

   To evaluate the left-hand side, use the method of partial fraction decomposition. This leads to the identity
   Equation:
   $$\frac{1}{y^2 - 4} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right).$$

   Then [link] becomes
   Equation:
   $$\begin{aligned} \tfrac{1}{4}\int\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy &= \int (2x+3)\,dx \\ \tfrac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) &= x^2 + 3x + C. \end{aligned}$$

   Multiplying both sides of this equation by 4 and replacing $4C$

with C_1 gives
Equation:

$$\begin{aligned} \ln|y-2| - \ln|y+2| &= 4x^2 + 12x + C_1 \\ \ln\left|\frac{y-2}{y+2}\right| &= 4x^2 + 12x + C_1. \end{aligned}$$

4. It is possible to solve this equation for *y*. First exponentiate both sides of the equation and define $C_2 = e^{C_1}$:
Equation:

$$\left|\frac{y-2}{y+2}\right| = C_2 e^{4x^2+12x}.$$

Next we can remove the absolute value and let C_2 be either positive or negative. Then multiply both sides by $y+2$.
Equation:

$$\begin{aligned} y - 2 &= C_2\,(y+2)e^{4x^2+12x} \\ y - 2 &= C_2 y e^{4x^2+12x} + 2C_2 e^{4x^2+12x}. \end{aligned}$$

Now collect all terms involving *y* on one side of the equation, and solve for *y*:
Equation:

$$\begin{aligned} y - C_2 y e^{4x^2+12x} &= 2 + 2C_2 e^{4x^2+12x} \\ y(1 - C_2 e^{4x^2+12x}) &= 2 + 2C_2 e^{4x^2+12x} \\ y &= \frac{2+2C_2 e^{4x^2+12x}}{1-C_2 e^{4x^2+12x}}. \end{aligned}$$

5. To determine the value of C_2, substitute $x = 0$ and $y = -1$ into the general solution. Alternatively, we can put the same values into an earlier equation, namely the equation $\frac{y-2}{y+2} = C_2 e^{4x^2+12}$. This is much easier to solve for C_2:

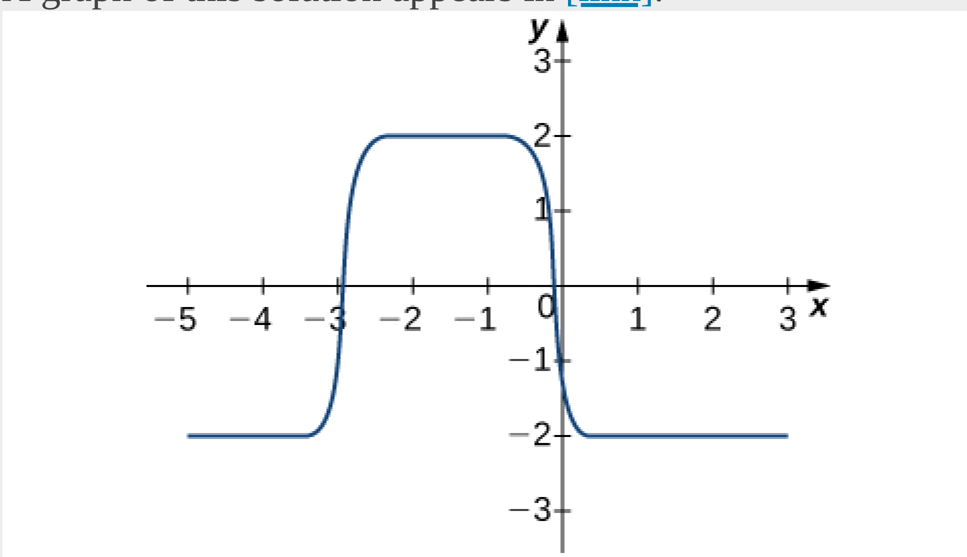
Equation:

$$\begin{aligned} \frac{y-2}{y+2} &= C_2 e^{4x^2+12x} \\ \frac{-1-2}{-1+2} &= C_2 e^{4(0)^2+12(0)} \\ C_2 &= -3. \end{aligned}$$

Therefore the solution to the initial-value problem is
Equation:

$$y = \frac{2-6e^{4x^2+12x}}{1+3e^{4x^2+12x}}.$$

A graph of this solution appears in [link].



Graph of the solution to the initial-value problem $y\prime = (2x+3)\left(y^2-4\right), \quad y(0) = -3.$

Note:
Exercise:

Problem: Find the solution to the initial-value problem
Equation:

$$6y\prime = (2x+1)\left(y^2 - 2y - 8\right), \quad y(0) = -3$$

using the method of separation of variables.

Solution:

$y = \frac{4+14e^{x^2+x}}{1-7e^{x^2+x}}$

Hint

Follow the steps for separation of variables to solve the initial-value problem.

Applications of Separation of Variables

Many interesting problems can be described by separable equations. We illustrate two types of problems: solution concentrations and Newton's law of cooling.

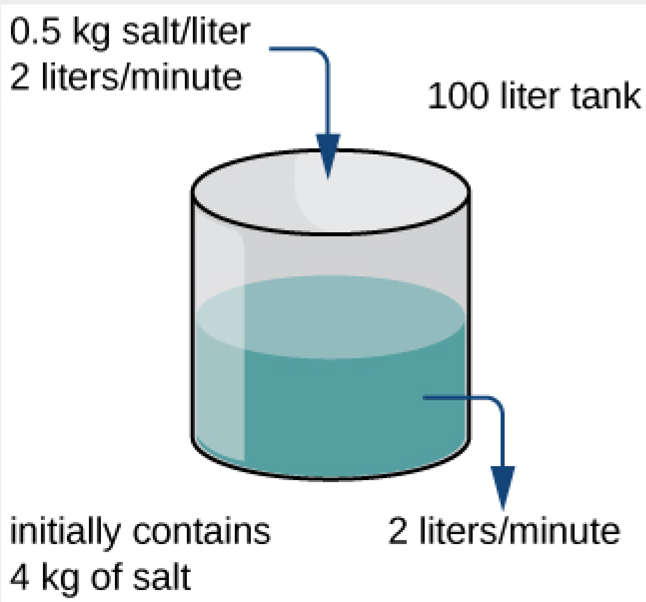
Solution concentrations

Consider a tank being filled with a salt solution. We would like to determine the amount of salt present in the tank as a function of time. We can apply the process of separation of variables to solve this problem and similar problems involving solution concentrations.

Example:
Exercise:

Problem:
Determining Salt Concentration over Time

A tank containing $100\ \text{L}$ of a brine solution initially has $4\ \text{kg}$ of salt dissolved in the solution. At time $t = 0$, another brine solution flows into the tank at a rate of $2\ \text{L/min}$. This brine solution contains a concentration of $0.5\ \text{kg/L}$ of salt. At the same time, a stopcock is opened at the bottom of the tank, allowing the combined solution to flow out at a rate of $2\ \text{L/min}$, so that the level of liquid in the tank remains constant ([link]). Find the amount of salt in the tank as a function of time (measured in minutes), and find the limiting amount of salt in the tank, assuming that the solution in the tank is well mixed at all times.



A brine tank with an initial amount of salt solution accepts an input flow and delivers an output flow. How does the amount of salt change with time?

Solution:

First we define a function $u(t)$ that represents the amount of salt in kilograms in the tank as a function of time. Then $\frac{du}{dt}$ represents the rate at which the amount of salt in the tank changes as a function of time. Also, $u(0)$ represents the amount of salt in the tank at time $t = 0$, which is 4 kilograms.

The general setup for the differential equation we will solve is of the form

Equation:

$$\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}.$$

INFLOW RATE represents the rate at which salt enters the tank, and OUTFLOW RATE represents the rate at which salt leaves the tank. Because solution enters the tank at a rate of 2 L/min, and each liter of solution contains 0.5 kilogram of salt, every minute $2(0.5) = 1$ kilogram of salt enters the tank. Therefore INFLOW RATE = 1.

To calculate the rate at which salt leaves the tank, we need the concentration of salt in the tank at any point in time. Since the actual amount of salt varies over time, so does the concentration of salt. However, the volume of the solution remains fixed at 100 liters. The number of kilograms of salt in the tank at time t is equal to $u(t)$. Thus, the concentration of salt is $\frac{u(t)}{100}$ kg/L, and the solution leaves the tank at a rate of 2 L/min. Therefore salt leaves the tank at a rate of $\frac{u(t)}{100} \cdot 2 = \frac{u(t)}{50}$ kg/min, and OUTFLOW RATE is equal to $\frac{u(t)}{50}$. Therefore the differential equation becomes $\frac{du}{dt} = 1 - \frac{u}{50}$, and the initial condition is $u(0) = 4$. The initial-value problem to be solved is

Equation:

$$\frac{du}{dt} = 1 - \frac{u}{50}, \quad u(0) = 4.$$

The differential equation is a separable equation, so we can apply the five-step strategy for solution.

Step 1. Setting $1 - \frac{u}{50} = 0$ gives $u = 50$ as a constant solution. Since the initial amount of salt in the tank is 4 kilograms, this solution does not apply.

Step 2. Rewrite the equation as
Equation:

$$\frac{du}{dt} = \frac{50 - u}{50}.$$

Then multiply both sides by dt and divide both sides by $50 - u$:
Equation:

$$\frac{du}{50 - u} = \frac{dt}{50}.$$

Step 3. Integrate both sides:
Equation:

$$\begin{aligned} \int \frac{du}{50 - u} &= \int \frac{dt}{50} \\ -\ln|50 - u| &= \tfrac{t}{50} + C. \end{aligned}$$

Step 4. Solve for $u(t)$:
Equation:

$$\begin{aligned} \ln|50-u| &= -\frac{t}{50} - C \\ e^{\ln|50-u|} &= e^{-(t/50)-C} \\ |50-u| &= C_1 e^{-t/50}. \end{aligned}$$

Eliminate the absolute value by allowing the constant to be either positive or negative:
Equation:

$$50-u = C_1 e^{-t/50}.$$

Finally, solve for $u(t)$:
Equation:

$$u(t) = 50 - C_1 e^{-t/50}.$$

Step 5. Solve for C_1:
Equation:

$$\begin{aligned} u(0) &= 50 - C_1 e^{-0/50} \\ 4 &= 50 - C_1 \\ C_1 &= 46. \end{aligned}$$

The solution to the initial value problem is $u(t) = 50 - 46e^{-t/50}$. To find the limiting amount of salt in the tank, take the limit as t approaches infinity:
Equation:

$$\begin{aligned} \lim_{t\to\infty} u(t) &= 50 - 46e^{-t/50} \\ &= 50 - 46(0) \\ &= 50. \end{aligned}$$

Note that this was the constant solution to the differential equation. If the initial amount of salt in the tank is 50 kilograms, then it remains constant. If it starts at less than 50 kilograms, then it approaches 50 kilograms over time.

Note:
Exercise:

Problem:

A tank contains 3 kilograms of salt dissolved in 75 liters of water. A salt solution of $0.4\,\text{kg salt/L}$ is pumped into the tank at a rate of $6\,\text{L/min}$ and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed at all times.

Solution:

Initial value problem:

$$\frac{du}{dt} = 2.4 - \frac{2u}{25}, \quad u(0) = 3$$

Solution: $u(t) = 30 - 27e^{-t/50}$

Hint

Follow the steps in [link] and determine an expression for INFLOW and OUTFLOW. Formulate an initial-value problem, and then solve it.

Newton’s law of cooling

Newton’s law of cooling states that the rate of change of an object’s temperature is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). If we let $T(t)$ represent the temperature of an object as a function of time,

then $\frac{dT}{dt}$ represents the rate at which that temperature changes. The temperature of the object's surroundings can be represented by T_s. Then Newton's law of cooling can be written in the form

Equation:

$$\frac{dT}{dt} = k\left(T(t) - T_s\right)$$

or simply

Equation:

$$\frac{dT}{dt} = k\left(T - T_s\right).$$

The temperature of the object at the beginning of any experiment is the initial value for the initial-value problem. We call this temperature T_0. Therefore the initial-value problem that needs to be solved takes the form

Equation:

$$\frac{dT}{dt} = k\left(T - T_s\right), \quad T(0) = T_0,$$

where k is a constant that needs to be either given or determined in the context of the problem. We use these equations in [link].

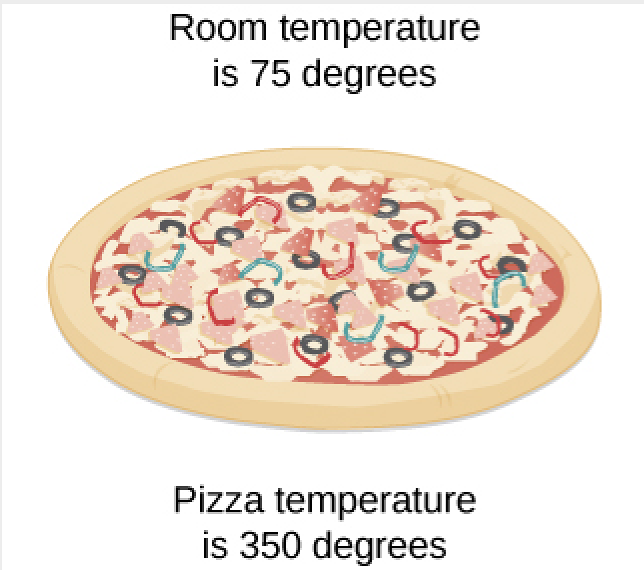
Example:

Exercise:

Problem:

Waiting for a Pizza to Cool

A pizza is removed from the oven after baking thoroughly, and the temperature of the oven is 350°F. The temperature of the kitchen is 75°F, and after 5 minutes the temperature of the pizza is 340°F. We would like to wait until the temperature of the pizza reaches 300°F before cutting and serving it ([link]). How much longer will we have to wait?



From Newton's law of cooling, if the pizza cools 10°F in 5 minutes, how long before it cools to 300°F?

Solution:

The ambient temperature (surrounding temperature) is 75°F, so $T_s = 75$. The temperature of the pizza when it comes out of the oven is 350°F, which is the initial temperature (i.e., initial value), so $T_0 = 350$. Therefore [link] becomes

Equation:

$$\frac{dT}{dt} = k\left(T - 75\right), \quad T\left(0\right) = 350.$$

To solve the differential equation, we use the five-step technique for solving separable equations.

1. Setting the right-hand side equal to zero gives $T = 75$ as a constant solution. Since the pizza starts at $350^{\circ}\mathrm{F}$, this is not the solution we are seeking.
2. Rewrite the differential equation by multiplying both sides by dt and dividing both sides by $T - 75$:
   Equation:
   $$\frac{dT}{T-75} = kdt.$$
3. Integrate both sides:
   Equation:
   $$\begin{aligned}\int \frac{dT}{T-75} &= \int kdt \\ \ln|T-75| &= kt + C.\end{aligned}$$
4. Solve for T by first exponentiating both sides:
   Equation:
   $$\begin{aligned}e^{\ln|T-75|} &= e^{kt+C} \\ |T-75| &= C_1 e^{kt} \\ T-75 &= C_1 e^{kt} \\ T(t) &= 75 + C_1 e^{kt}.\end{aligned}$$
5. Solve for C_1 by using the initial condition $T(0) = 350$:
   Equation:
   $$\begin{aligned}T(t) &= 75 + C_1 e^{kt} \\ T(0) &= 75 + C_1 e^{k(0)} \\ 350 &= 75 + C_1 \\ C_1 &= 275.\end{aligned}$$

Therefore the solution to the initial-value problem is
Equation:

$$T(t) = 75 + 275e^{kt}.$$

To determine the value of k, we need to use the fact that after 5 minutes the temperature of the pizza is $340^\circ\mathrm{F}$. Therefore $T\left(5\right) = 340$. Substituting this information into the solution to the initial-value problem, we have
Equation:

$$\begin{aligned}
T(t) &= 75 + 275e^{kt} \\
T\left(5\right) &= 340 = 75 + 275e^{5k} \\
265 &= 275e^{5k} \\
e^{5k} &= \tfrac{53}{55} \\
\ln e^{5k} &= \ln\left(\tfrac{53}{55}\right) \\
5k &= \ln\left(\tfrac{53}{55}\right) \\
k &= \tfrac{1}{5}\ln\left(\tfrac{53}{55}\right) \approx -0.007408.
\end{aligned}$$

So now we have $T(t) = 75 + 275e^{-0.007048t}$. When is the temperature $300^\circ\mathrm{F}$? Solving for t, we find
Equation:

$$\begin{aligned} T(t) &= 75+275e^{-0.007048t} \\ 300 &= 75+275e^{-0.007048t} \\ 225 &= 275e^{-0.007048t} \\ e^{-0.007048t} &= \frac{9}{11} \\ \ln e^{-0.007048t} &= \ln\frac{9}{11} \\ -0.007048t &= \ln\frac{9}{11} \\ t &= -\frac{1}{0.007048}\ln\frac{9}{11} \approx 28.5. \end{aligned}$$

Therefore we need to wait an additional 23.5 minutes (after the temperature of the pizza reached 340°F). That should be just enough time to finish this calculation.

Note:
Exercise:

Problem:

A cake is removed from the oven after baking thoroughly, and the temperature of the oven is 450°F. The temperature of the kitchen is 70°F, and after 10 minutes the temperature of the cake is 430°F.

a. Write the appropriate initial-value problem to describe this situation.
b. Solve the initial-value problem for $T(t)$.
c. How long will it take until the temperature of the cake is within 5°F of room temperature?

Solution:

a. Initial-value problem
$\frac{dT}{dt} = k(T - 70), \quad T(0) = 450$
b. $T(t) = 70 + 380e^{kt}$
c. Approximately 114 minutes.

Hint

Determine the values of T_s and T_0 then use [link].

Key Concepts

- A separable differential equation is any equation that can be written in the form $y\prime = f(x)g(y)$.
- The method of separation of variables is used to find the general solution to a separable differential equation.

Key Equations

- **Separable differential equation**
  $y' = f(x)g(y)$
- **Solution concentration**
  $\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}$
- **Newton's law of cooling**
  $\frac{dT}{dt} = k(T - T_s)$

Solve the following initial-value problems with the initial condition $y_0 = 0$ and graph the solution.
Exercise:

Problem: $\frac{dy}{dt} = y + 1$

Solution:

$y = e^t - 1$

Exercise:

Problem: $\frac{dy}{dt} = y - 1$

Exercise:

Problem: $\frac{dy}{dt} = y + 1$

Solution:

$y = 1 - e^{-t}$

Exercise:

Problem: $\frac{dy}{dt} = -y - 1$

Find the general solution to the differential equation.

Exercise:

Problem: $x^2 y\prime = (x+1)y$

Solution:

$y = Cxe^{-1/x}$

Exercise:

Problem: $y\prime = \tan(y)x$

Exercise:

Problem: $y\prime = 2xy^2$

Solution:

$y = \frac{1}{C - x^2}$

Exercise:

Problem: $\frac{dy}{dt} = y\cos\left(3t+2\right)$

Exercise:

Problem: $2x\frac{dy}{dx} = y^2$

Solution:

$y = -\frac{2}{C+\ln x}$

Exercise:

Problem: $y\prime = e^y x^2$

Exercise:

Problem: $(1+x)y\prime = \left(x+2\right)\left(y-1\right)$

Solution:

$y = Ce^x\left(x+1\right)+1$

Exercise:

Problem: $\frac{dx}{dt} = 3t^2\left(x^2+4\right)$

Exercise:

Problem: $t\frac{dy}{dt} = \sqrt{1-y^2}$

Solution:

$y = \sin\left(\ln t + C\right)$

Exercise:

Problem: $y\prime = e^x e^y$

Find the solution to the initial-value problem.

Exercise:

Problem: $y\prime = e^{y-x}, y(0) = 0$

Solution:

$y = -\ln(e^{-x})$

Exercise:

Problem: $y\prime = y^2(x+1), y(0) = 2$

Exercise:

Problem: $\frac{dy}{dx} = y^3 x e^{x^2}, y(0) = 1$

Solution:

$y = \frac{1}{\sqrt{2-e^{x^2}}}$

Exercise:

Problem: $\frac{dy}{dt} = y^2 e^x \sin(3x), y(0) = 1$

Exercise:

Problem: $y\prime = \frac{x}{\operatorname{sech}^2 y}, y(0) = 0$

Solution:

$y = \tanh^{-1}\left(\frac{x^2}{2}\right)$

Exercise:

Problem: $y\prime = 2xy(1+2y), y(0) = -1$

Exercise:

Problem: $\frac{dx}{dt} = \ln(t)\sqrt{1-x^2}, x(0) = 0$

Solution:

$x = -\sin\left(t - t\ln t\right)$

Exercise:

Problem: $y\prime = 3x^2(y^2+4), y(0) = 0$

Exercise:

Problem: $y\prime = e^y 5^x, y(0) = \ln(\ln(5))$

Solution:

$y = \ln(\ln(5)) - \ln(2 - 5^x)$

Exercise:

Problem: $y\prime = -2x\tan(y), y(0) = \frac{\pi}{2}$

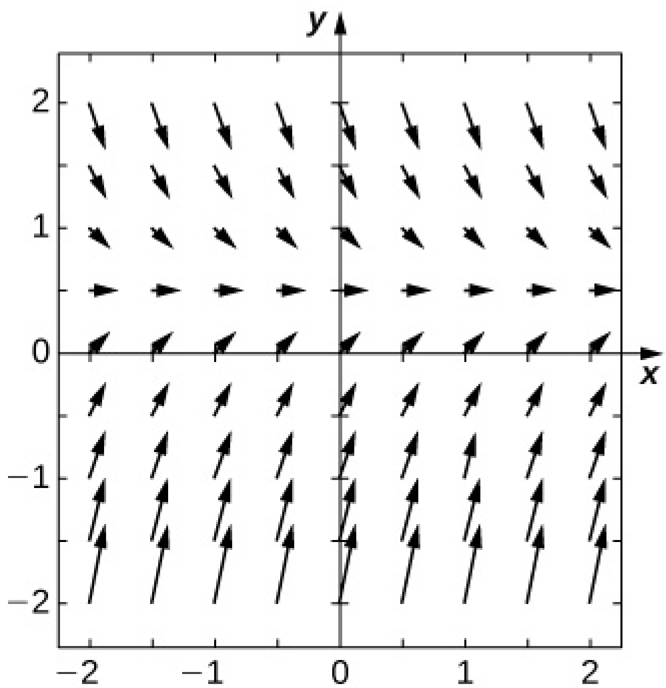
For the following problems, use a software program or your calculator to generate the directional fields. Solve explicitly and draw solution curves for several initial conditions. Are there some critical initial conditions that change the behavior of the solution?

Exercise:

Problem: **[T]** $y\prime = 1 - 2y$

Solution:

$y = Ce^{-2x} + \frac{1}{2}$



Exercise:

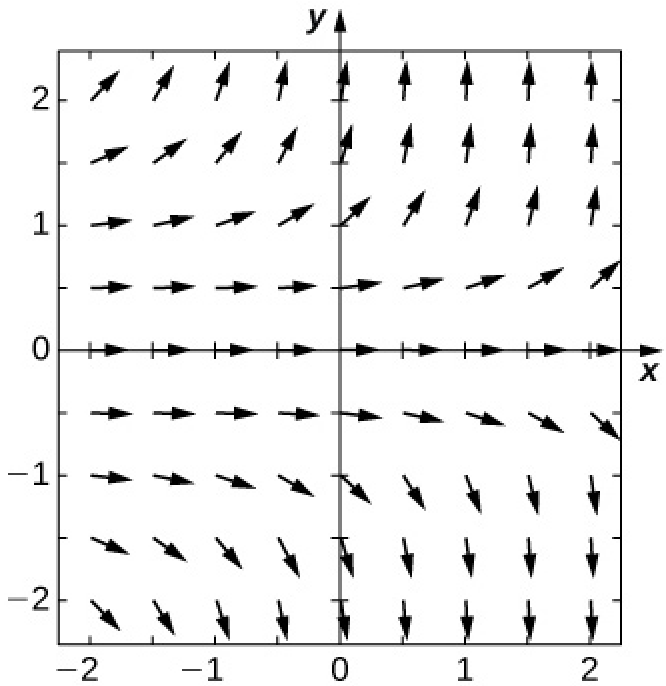
Problem: **[T]** $y\prime = y^2 x^3$

Exercise:

Problem: **[T]** $y\prime = y^3 e^x$

Solution:

$y = \frac{1}{\sqrt{2}\sqrt{C - e^x}}$



Exercise:

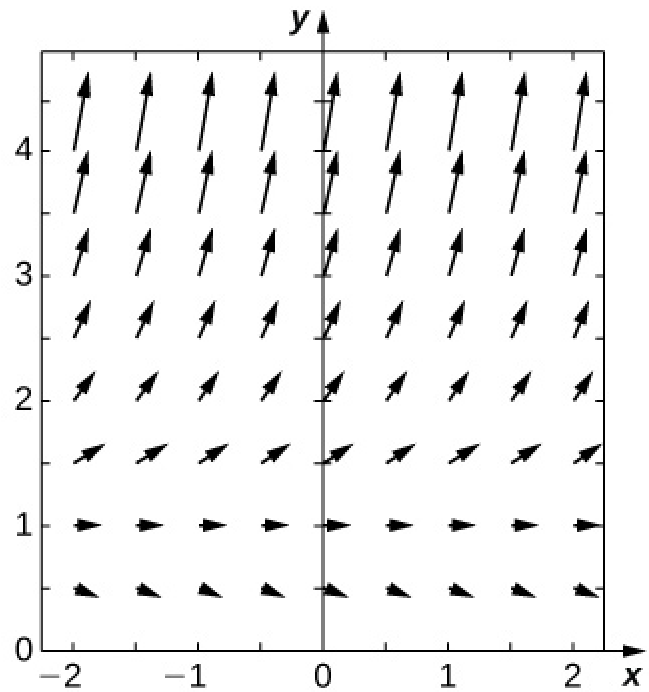
Problem: **[T]** $y\prime = e^y$

Exercise:

Problem: **[T]** $y\prime = y\ln(x)$

Solution:

$y = Ce^{-x}x^x$



Exercise:

Problem:

Most drugs in the bloodstream decay according to the equation $y\prime = cy$, where y is the concentration of the drug in the bloodstream. If the half-life of a drug is 2 hours, what fraction of the initial dose remains after 6 hours?

Exercise:

Problem:

A drug is administered intravenously to a patient at a rate r mg/h and is cleared from the body at a rate proportional to the amount of drug still present in the body, d Set up and solve the differential equation, assuming there is no drug initially present in the body.

Solution:

$y = \frac{r}{d}\left(1 - e^{-dt}\right)$

Exercise:

Problem:

[T] How often should a drug be taken if its dose is 3 mg, it is cleared at a rate $c = 0.1$ mg/h, and 1 mg is required to be in the bloodstream at all times?

Exercise:

Problem:

A tank contains 1 kilogram of salt dissolved in 100 liters of water. A salt solution of 0.1 kg salt/L is pumped into the tank at a rate of 2 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed.

Solution:

$y(t) = 10 - 9e^{-x/50}$

Exercise:

Problem:

A tank containing 10 kilograms of salt dissolved in 1000 liters of water has two salt solutions pumped in. The first solution of 0.2 kg salt/L is pumped in at a rate of 20 L/min and the second solution of 0.05 kg salt/L is pumped in at a rate of 5 L/min. The tank drains at 25 L/min. Assume the tank is well mixed. Solve for the salt concentration at time t.

Exercise:

Problem:

[T] For the preceding problem, find how much salt is in the tank 1 hour after the process begins.

Solution:

134.3 kilograms

Exercise:

Problem:

Torricelli's law states that for a water tank with a hole in the bottom that has a cross-section of A and with a height of water h above the bottom of the tank, the rate of change of volume of water flowing from the tank is proportional to the square root of the height of water, according to $\frac{dV}{dt} = -A\sqrt{2gh}$, where g is the acceleration due to gravity. Note that $\frac{dV}{dt} = A\frac{dh}{dt}$. Solve the resulting initial-value problem for the height of the water, assuming a tank with a hole of radius 2 ft. The initial height of water is 100 ft.

Exercise:

Problem:

For the preceding problem, determine how long it takes the tank to drain.

Solution:

720 seconds

For the following problems, use Newton's law of cooling.

Exercise:

Problem:

The liquid base of an ice cream has an initial temperature of 200°F before it is placed in a freezer with a constant temperature of 0°F. After 1 hour, the temperature of the ice-cream base has decreased to 140°F. Formulate and solve the initial-value problem to determine the temperature of the ice cream.

Exercise:

Problem:

[T] The liquid base of an ice cream has an initial temperature of $210^{\circ}\mathrm{F}$ before it is placed in a freezer with a constant temperature of $20^{\circ}\mathrm{F}$. After 2 hours, the temperature of the ice-cream base has decreased to $170^{\circ}\mathrm{F}$. At what time will the ice cream be ready to eat? (Assume $30^{\circ}\mathrm{F}$ is the optimal eating temperature.)

Solution:

12 hours 14 minutes

Exercise:

Problem:

[T] You are organizing an ice cream social. The outside temperature is $80^{\circ}\mathrm{F}$ and the ice cream is at $10^{\circ}\mathrm{F}$. After 10 minutes, the ice cream temperature has risen by $10^{\circ}\mathrm{F}$. How much longer can you wait before the ice cream melts at $40^{\circ}\mathrm{F}$?

Exercise:

Problem:

You have a cup of coffee at temperature $70^{\circ}\mathrm{C}$ and the ambient temperature in the room is $20^{\circ}\mathrm{C}$. Assuming a cooling rate k of 0.125, write and solve the differential equation to describe the temperature of the coffee with respect to time.

Solution:

$T(t)=20+50e^{-0.125t}$

Exercise:

Problem:

[T] You have a cup of coffee at temperature $70^{\circ}\mathrm{C}$ that you put outside, where the ambient temperature is $0^{\circ}\mathrm{C}$. After 5 minutes, how much colder is the coffee?

Exercise:

Problem:

You have a cup of coffee at temperature $70^{\circ}\mathrm{C}$ and you immediately pour in 1 part milk to 5 parts coffee. The milk is initially at temperature $1^{\circ}\mathrm{C}$. Write and solve the differential equation that governs the temperature of this coffee.

Solution:

$T(t) = 20 + 38.5e^{-0.125t}$

Exercise:

Problem:

You have a cup of coffee at temperature $70^{\circ}\mathrm{C}$, which you let cool 10 minutes before you pour in the same amount of milk at $1^{\circ}\mathrm{C}$ as in the preceding problem. How does the temperature compare to the previous cup after 10 minutes?

Exercise:

Problem:

Solve the generic problem $y\prime = ay + b$ with initial condition $y(0) = c$.

Solution:

$y = \left(c + \frac{b}{a}\right)e^{ax} - \frac{b}{a}$

Exercise:

Problem:

Prove the basic continual compounded interest equation. Assuming an initial deposit of P_0 and an interest rate of r, set up and solve an equation for continually compounded interest.

Exercise:

Problem:

Assume an initial nutrient amount of I kilograms in a tank with L liters. Assume a concentration of c kg/L being pumped in at a rate of r L/min. The tank is well mixed and is drained at a rate of r L/min. Find the equation describing the amount of nutrient in the tank.

Solution:

$y(t) = cL + (I - cL)e^{-rt/L}$

Exercise:

Problem:

Leaves accumulate on the forest floor at a rate of 2 g/cm^2/yr and also decompose at a rate of 90% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor, assuming at time 0 there is no leaf litter on the ground. Does this amount approach a steady value? What is that value?

Exercise:

Problem:

Leaves accumulate on the forest floor at a rate of 4 g/cm^2/yr. These leaves decompose at a rate of 10% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor. Does this amount approach a steady value? What is that value?

Solution:

$y = 40\left(1 - e^{-0.1t}\right), 40 \text{ g/cm}^2$

Glossary

autonomous differential equation

an equation in which the right-hand side is a function of y alone

separable differential equation
any equation that can be written in the form $y' = f(x)g(y)$

separation of variables
a method used to solve a separable differential equation

The Logistic Equation

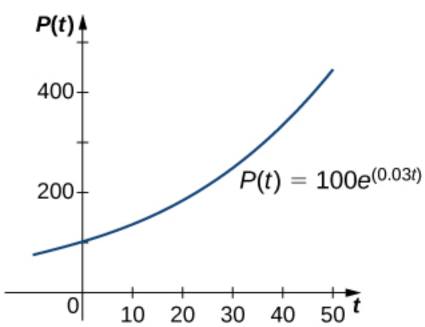
- Describe the concept of environmental carrying capacity in the logistic model of population growth.
- Draw a direction field for a logistic equation and interpret the solution curves.
- Solve a logistic equation and interpret the results.

Differential equations can be used to represent the size of a population as it varies over time. We saw this in an earlier chapter in the section on exponential growth and decay, which is the simplest model. A more realistic model includes other factors that affect the growth of the population. In this section, we study the logistic differential equation and see how it applies to the study of population dynamics in the context of biology.

Population Growth and Carrying Capacity

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable $t.$ will represent time. The units of time can be hours, days, weeks, months, or even years. Any given problem must specify the units used in that particular problem. The variable P will represent population. Since the population varies over time, it is understood to be a function of time. Therefore we use the notation $P\left(t\right)$ for the population as a function of time. If $P\left(t\right)$ is a differentiable function, then the first derivative $\frac{dP}{dt}$ represents the instantaneous rate of change of the population as a function of time.

In Exponential Growth and Decay, we studied the exponential growth and decay of populations and radioactive substances. An example of an exponential growth function is $P\left(t\right)=P_0e^{rt}.$ In this function, $P\left(t\right)$ represents the population at time $t,$ P_0 represents the **initial population** (population at time $t=0),$ and the constant $r>0$ is called the **growth rate**. [link] shows a graph of $P(t)=100e^{0.03t}$. Here $P_0=100$ and $r=0.03.$



An exponential growth model of population.

We can verify that the function $P\left(t\right)=P_0e^{rt}$ satisfies the initial-value problem

Equation:

$$\frac{dP}{dt}=rP,\quad P\left(0\right)=P_0.$$

This differential equation has an interesting interpretation. The left-hand side represents the rate at which the population increases (or decreases). The right-hand side is equal to a positive constant multiplied by the current population. Therefore the differential equation states that the rate at which the population increases is proportional to the population at that point in time. Furthermore, it states that the constant of proportionality never changes.

One problem with this function is its prediction that as time goes on, the population grows without bound. This is unrealistic in a real-world setting. Various factors limit the rate of growth of a particular population, including birth rate, death rate, food supply, predators, and so on. The growth constant r usually takes into consideration the birth and death rates but none of the other factors, and it can be interpreted as a net (birth minus death) percent growth rate per unit time. A natural question to ask is whether the population growth rate stays constant, or whether it changes over time. Biologists have found that in many biological systems, the population grows until a certain steady-state population is reached. This possibility is not taken into account with exponential growth. However, the concept of carrying capacity allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

Note:
Definition
The **carrying capacity** of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.

We use the variable K to denote the carrying capacity. The growth rate is represented by the variable r. Using these variables, we can define the logistic differential equation.

Note:
Definition
Let K represent the carrying capacity for a particular organism in a given environment, and let r be a real number that represents the growth rate. The function $P(t)$ represents the population of this organism as a function of time t, and the constant P_0 represents the initial population (population of the organism at time $t = 0$). Then the **logistic differential equation** is
Equation:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Note:
See this website for more information on the logistic equation.

The logistic equation was first published by Pierre Verhulst in 1845. This differential equation can be coupled with the initial condition $P(0) = P_0$ to form an initial-value problem for $P(t)$.

Suppose that the initial population is small relative to the carrying capacity. Then $\frac{P}{K}$ is small, possibly close to zero. Thus, the quantity in parentheses on the right-hand side of [link] is close to 1, and the right-hand side of this equation is close to rP. If $r > 0$, then the population grows rapidly, resembling exponential growth.

However, as the population grows, the ratio $\frac{P}{K}$ also grows, because K is constant. If the population remains below the carrying capacity, then $\frac{P}{K}$ is less than 1, so $1 - \frac{P}{K} > 0$. Therefore the right-hand side of [link] is still positive, but the quantity in parentheses gets smaller, and the growth rate decreases as a result. If $P = K$ then the right-hand side is equal to zero, and the population does not change.

Now suppose that the population starts at a value higher than the carrying capacity. Then $\frac{P}{K} > 1$, and $1 - \frac{P}{K} < 0$. Then the right-hand side of [link] is negative, and the population decreases. As long as $P > K$, the population decreases. It never actually reaches K because $\frac{dP}{dt}$ will get smaller and smaller, but the population approaches the carrying capacity as t approaches infinity. This analysis can be represented visually by way of a phase line. A **phase line** describes the general behavior of a solution to an autonomous differential equation, depending on the initial condition. For the case of a carrying capacity in the logistic equation, the phase line is as shown in [link].

A phase line for the differential equation $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$.

This phase line shows that when P is less than zero or greater than K, the population decreases over time. When P is between 0 and K, the population increases over time.

Example:
Exercise:

Problem:
Chapter Opener: Examining the Carrying Capacity of a Deer Population

(credit: modification of work by Rachel Kramer, Flickr)

Let's consider the population of white-tailed deer (*Odocoileus virginianus*) in the state of Kentucky. The Kentucky Department of Fish and Wildlife Resources (KDFWR) sets guidelines for hunting and fishing in the state. Before the hunting season of 2004, it estimated a population of $900{,}000$ deer. Johnson notes: "A deer population that has plenty to eat and is not hunted by humans or other predators will double every three years." (George Johnson, "The Problem of Exploding Deer Populations Has No Attractive Solutions," January $12, 2001$, accessed April 9, 2015, http://www.txtwriter.com/onscience/Articles/deerpops.html.) This observation corresponds to a rate of increase $r = \frac{\ln(2)}{3} = 0.2311$, so the approximate growth rate is 23.11% per year. (This assumes that the population grows exponentially, which is reasonable—at least in the short term—with plentiful food supply and no predators.) The KDFWR also reports deer population densities for 32 counties in Kentucky, the average of which is approximately 27 deer per square mile. Suppose this is the deer density for the whole state $(39{,}732$ square miles). The carrying capacity K is $39{,}732$ square miles times 27 deer per square mile, or $1{,}072{,}764$ deer.

a. For this application, we have $P_0 = 900{,}000$, $K = 1{,}072{,}764$, and $r = 0.2311$. Substitute these values into [link] and form the initial-value problem.
b. Solve the initial-value problem from part a.
c. According to this model, what will be the population in 3 years? Recall that the doubling time predicted by Johnson for the deer population was 3 years. How do these values compare?
d. Suppose the population managed to reach $1{,}200{,}000$ deer. What does the logistic equation predict will happen to the population in this scenario?

Solution:

a. The initial value problem is
$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1{,}072{,}764}\right), \quad P(0) = 900{,}000.$
b. The logistic equation is an autonomous differential equation, so we can use the method of separation of variables.
Step 1: Setting the right-hand side equal to zero gives $P = 0$ and $P = 1{,}072{,}764$. This means that if the population starts at zero it will never change, and if it starts at the carrying capacity, it will never change.
Step 2: Rewrite the differential equation and multiply both sides by:
Equation:

$$\begin{aligned} \frac{dP}{dt} &= 0.2311P\left(\frac{1{,}072{,}764 - P}{1{,}072{,}764}\right) \\ dP &= 0.2311P\left(\frac{1{,}072{,}764 - P}{1{,}072{,}764}\right)dt. \end{aligned}$$

Divide both sides by $P(1{,}072{,}764 - P)$:
Equation:

$$\frac{dP}{P(1{,}072{,}764 - P)} = \frac{0.2311}{1{,}072{,}764}dt.$$

Step 3: Integrate both sides of the equation using partial fraction decomposition:
Equation:

$$\begin{aligned}
\int \frac{dP}{P\left(1{,}072{,}764 - P\right)} &= \int \frac{0.2311}{1{,}072{,}764} dt \\
\tfrac{1}{1{,}072{,}764} \int \left(\frac{1}{P} + \frac{1}{1{,}072{,}764 - P} \right) dP &= \tfrac{0.2311t}{1{,}072{,}764} + C \\
\tfrac{1}{1{,}072{,}764} \left(\ln |P| - \ln |1{,}072{,}764 - P| \right) &= \tfrac{0.2311t}{1{,}072{,}764} + C.
\end{aligned}$$

Step 4: Multiply both sides by $1{,}072{,}764$ and use the quotient rule for logarithms:
Equation:

$$\ln \left| \frac{P}{1{,}072{,}764 - P} \right| = 0.2311t + C_1.$$

Here $C_1 = 1{,}072{,}764C$. Next exponentiate both sides and eliminate the absolute value:
Equation:

$$\begin{aligned}
e^{\ln \left| \frac{P}{1{,}072{,}764 - P} \right|} &= e^{0.2311t + C_1} \\
\left| \tfrac{P}{1{,}072{,}764 - P} \right| &= C_2 e^{0.2311t} \\
\tfrac{P}{1{,}072{,}764 - P} &= C_2 e^{0.2311t}.
\end{aligned}$$

Here $C_2 = e^{C_1}$ but after eliminating the absolute value, it can be negative as well. Now solve for:
Equation:

$$\begin{aligned}
P &= C_2 e^{0.2311t} \left(1{,}072{,}764 - P\right). \\
P &= 1{,}072{,}764 C_2 e^{0.2311t} - C_2 P e^{0.2311t} \\
P + C_2 P e^{0.2311t} &= 1{,}072{,}764 C_2 e^{0.2311t} \\
P \left(1 + C_2 e^{0.2311t}\right) &= 1{,}072{,}764 C_2 e^{0.2311t} \\
P\left(t\right) &= \tfrac{1{,}072{,}764 C_2 e^{0.2311t}}{1 + C_2 e^{0.2311t}}.
\end{aligned}$$

Step 5: To determine the value of C_2, it is actually easier to go back a couple of steps to where C_2 was defined. In particular, use the equation
Equation:

$$\frac{P}{1{,}072{,}764 - P} = C_2 e^{0.2311t}.$$

The initial condition is $P\left(0\right) = 900{,}000$. Replace P with $900{,}000$ and t with zero:
Equation:

$$\begin{aligned}
\tfrac{P}{1{,}072{,}764 - P} &= C_2 e^{0.2311t} \\
\tfrac{900{,}000}{1{,}072{,}764 - 900{,}000} &= C_2 e^{0.2311(0)} \\
\tfrac{900{,}000}{172{,}764} &= C_2 \\
C_2 &= \tfrac{25{,}000}{4{,}799} \approx 5.209.
\end{aligned}$$

Therefore

Equation:

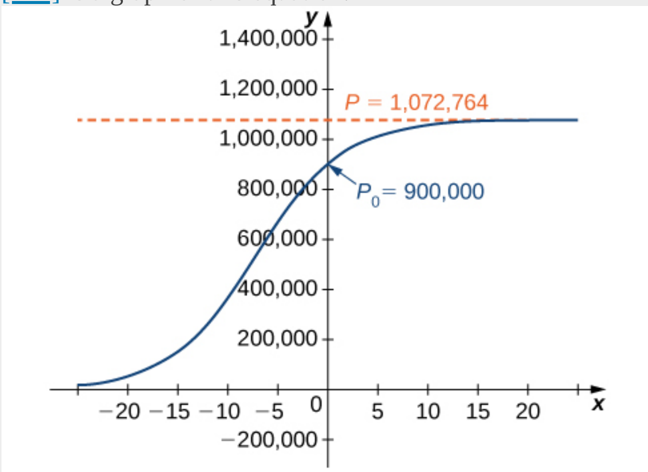
$$\begin{aligned} P(t) &= \frac{1{,}072{,}764\left(\frac{25000}{4799}\right)e^{0.2311t}}{1+\left(\frac{25000}{4799}\right)e^{0.2311t}} \\ &= \frac{1{,}072{,}764(25000)e^{0.2311t}}{4799+25000e^{0.2311t}}. \end{aligned}$$

Dividing the numerator and denominator by $25{,}000$ gives

Equation:

$$P(t) = \frac{1{,}072{,}764e^{0.2311t}}{0.19196 + e^{0.2311t}}.$$

[link] is a graph of this equation.



Logistic curve for the deer population with an initial population of 900,000 deer.

c. Using this model we can predict the population in 3 years.

Equation:

$$P(3) = \frac{1{,}072{,}764e^{0.2311(3)}}{0.19196 + e^{0.2311(3)}} \approx 978{,}830 \text{ deer}$$

This is far short of twice the initial population of $900{,}000$. Remember that the doubling time is based on the assumption that the growth rate never changes, but the logistic model takes this possibility into account.

d. If the population reached $1{,}200{,}000$ deer, then the new initial-value problem would be

Equation:

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1{,}072{,}764}\right), \quad P(0) = 1{,}200{,}000.$$

The general solution to the differential equation would remain the same.

Equation:

$$P\left(t\right)=\frac{1{,}072{,}764C_2e^{0.2311t}}{1+C_2e^{0.2311t}}$$

To determine the value of the constant, return to the equation
Equation:

$$\frac{P}{1{,}072{,}764-P}=C_2e^{0.2311t}.$$

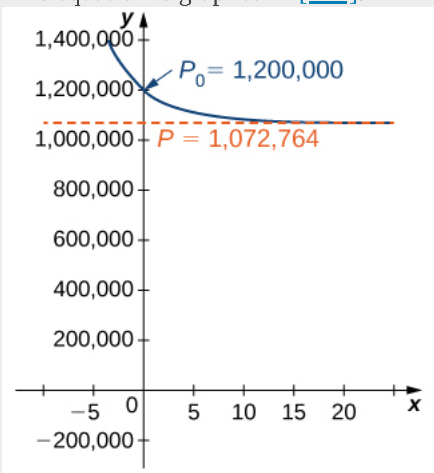
Substituting the values $t=0$ and $P=1{,}200{,}000$, you get
Equation:

$$\begin{aligned} C_2e^{0.2311(0)} &= \frac{1{,}200{,}000}{1{,}072{,}764-1{,}200{,}000} \\ C_2 &= -\frac{100{,}000}{10{,}603}\approx -9.431. \end{aligned}$$

Therefore
Equation:

$$\begin{aligned} P\left(t\right) &= \frac{1{,}072{,}764C_2e^{0.2311t}}{1+C_2e^{0.2311t}} \\ &= \frac{1{,}072{,}764\left(-\frac{100{,}000}{10{,}603}\right)e^{0.2311t}}{1+\left(-\frac{100{,}000}{10{,}603}\right)e^{0.2311t}} \\ &= -\frac{107{,}276{,}400{,}000e^{0.2311t}}{100{,}000e^{0.2311t}-10{,}603} \\ &\approx \frac{10{,}117{,}551e^{0.2311t}}{9.43129e^{0.2311t}-1}. \end{aligned}$$

This equation is graphed in [link].



Logistic curve for the deer population with an initial population of 1,200,000 deer.

Solving the Logistic Differential Equation

The logistic differential equation is an autonomous differential equation, so we can use separation of variables to find the general solution, as we just did in [link].

Step 1: Setting the right-hand side equal to zero leads to $P = 0$ and $P = K$ as constant solutions. The first solution indicates that when there are no organisms present, the population will never grow. The second solution indicates that when the population starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation in the form
Equation:

$$\frac{dP}{dt} = \frac{rP\left(K - P\right)}{K}.$$

Then multiply both sides by dt and divide both sides by $P\left(K - P\right)$. This leads to
Equation:

$$\frac{dP}{P\left(K - P\right)} = \frac{r}{K}dt.$$

Multiply both sides of the equation by K and integrate:
Equation:

$$\int \frac{K}{P\left(K - P\right)}dP = \int rdt.$$

The left-hand side of this equation can be integrated using partial fraction decomposition. We leave it to you to verify that
Equation:

$$\frac{K}{P\left(K - P\right)} = \frac{1}{P} + \frac{1}{K - P}.$$

Then the equation becomes
Equation:

$$\begin{aligned} \int \frac{1}{P} + \frac{1}{K - P}dP &= \int rdt \\ \ln\left|P\right| - \ln\left|K - P\right| &= rt + C \\ \ln\left|\tfrac{P}{K - P}\right| &= rt + C. \end{aligned}$$

Now exponentiate both sides of the equation to eliminate the natural logarithm:
Equation:

$$\begin{aligned} e^{\ln\left|\frac{P}{K - P}\right|} &= e^{rt + C} \\ \left|\tfrac{P}{K - P}\right| &= e^{C}e^{rt}. \end{aligned}$$

We define $C_1 = e^{c}$ so that the equation becomes
Equation:

$$\frac{P}{K-P} = C_1 e^{rt}.$$

To solve this equation for $P(t)$, first multiply both sides by $K - P$ and collect the terms containing P on the left-hand side of the equation:

Equation:

$$\begin{aligned} P &= C_1 e^{rt} (K-P) \\ P &= C_1 K e^{rt} - C_1 P e^{rt} \\ P + C_1 P e^{rt} &= C_1 K e^{rt}. \end{aligned}$$

Next, factor P from the left-hand side and divide both sides by the other factor:

Equation:

$$\begin{aligned} P\left(1 + C_1 e^{rt}\right) &= C_1 K e^{rt} \\ P(t) &= \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}. \end{aligned}$$

The last step is to determine the value of C_1. The easiest way to do this is to substitute $t = 0$ and P_0 in place of P in [link] and solve for C_1:

Equation:

$$\begin{aligned} \frac{P}{K-P} &= C_1 e^{rt} \\ \frac{P_0}{K-P_0} &= C_1 e^{r(0)} \\ C_1 &= \frac{P_0}{K-P_0}. \end{aligned}$$

Finally, substitute the expression for C_1 into [link]:

Equation:

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}} = \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}}$$

Now multiply the numerator and denominator of the right-hand side by $(K - P_0)$ and simplify:

Equation:

$$\begin{aligned} P(t) &= \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}} \\ &= \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}} \cdot \frac{K-P_0}{K-P_0} \\ &= \frac{P_0 K e^{rt}}{(K-P_0) + P_0 e^{rt}}. \end{aligned}$$

We state this result as a theorem.

Note:
Solution of the Logistic Differential Equation

Consider the logistic differential equation subject to an initial population of P_0 with carrying capacity K and growth rate r. The solution to the corresponding initial-value problem is given by

Equation:

$$P\left(t\right)=\frac{P_0Ke^{rt}}{\left(K-P_0\right)+P_0e^{rt}}.$$

Now that we have the solution to the initial-value problem, we can choose values for P_0, r, and K and study the solution curve. For example, in [link] we used the values $r=0.2311, K=1{,}072{,}764$, and an initial population of 900,000 deer. This leads to the solution

Equation:

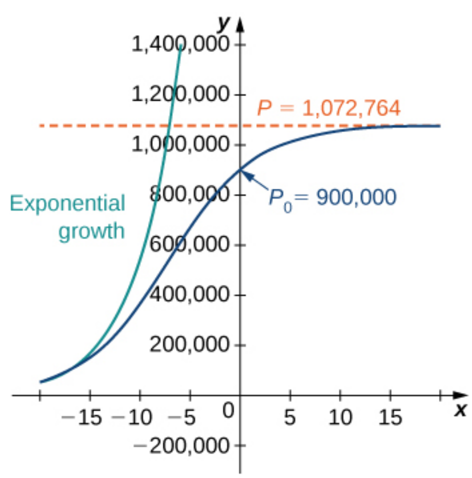
$$\begin{aligned}P\left(t\right)&=\frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}\\&=\frac{900{,}000(1{,}072{,}764)e^{0.2311t}}{(1{,}072{,}764-900{,}000)+900{,}000e^{0.2311t}}\\&=\frac{900{,}000(1{,}072{,}764)e^{0.2311t}}{172{,}764+900{,}000e^{0.2311t}}.\end{aligned}$$

Dividing top and bottom by 900,000 gives

Equation:

$$P\left(t\right)=\frac{1{,}072{,}764e^{0.2311t}}{0.19196+e^{0.2311t}}.$$

This is the same as the original solution. The graph of this solution is shown again in blue in [link], superimposed over the graph of the exponential growth model with initial population $900{,}000$ and growth rate 0.2311 (appearing in green). The red dashed line represents the carrying capacity, and is a horizontal asymptote for the solution to the logistic equation.



A comparison of exponential versus logistic growth for the same initial population of $900{,}000$ organisms and growth rate of 23.11%.

Working under the assumption that the population grows according to the logistic differential equation, this graph predicts that approximately 20 years earlier (1984), the growth of the population was very close to exponential. The net growth rate at that time would have been around 23.1% per year. As time goes on, the two graphs separate. This happens because the population increases, and the logistic differential equation states that the growth rate decreases as the population increases. At the time the population was measured (2004), it was close to carrying capacity, and the population was starting to level off.

The solution to the logistic differential equation has a point of inflection. To find this point, set the second derivative equal to zero:

Equation:

$$\begin{aligned} P(t) &= \frac{P_0 K e^{rt}}{(K-P_0)+P_0 e^{rt}} \\ P'(t) &= \frac{r P_0 K (K-P_0) e^{rt}}{((K-P_0)+P_0 e^{rt})^2} \\ P\prime\prime(t) &= \frac{r^2 P_0 K (K-P_0)^2 e^{rt} - r^2 P_0{}^2 K (K-P_0) e^{2rt}}{((K-P_0)+P_0 e^{rt})^3} \\ &= \frac{r^2 P_0 K (K-P_0) e^{rt} \left((K-P_0)-P_0 e^{rt}\right)}{((K-P_0)+P_0 e^{rt})^3}. \end{aligned}$$

Setting the numerator equal to zero,

Equation:

$$r^2 P_0 K\,(K-P_0) e^{rt} \left((K-P_0) - P_0 e^{rt}\right) = 0.$$

As long as $P_0 \neq K$, the entire quantity before and including e^{rt} is nonzero, so we can divide it out:

Equation:

$$(K-P_0) - P_0 e^{rt} = 0.$$

Solving for t,

Equation:

$$\begin{aligned} P_0 e^{rt} &= K - P_0 \\ e^{rt} &= \frac{K-P_0}{P_0} \\ \ln e^{rt} &= \ln \frac{K-P_0}{P_0} \\ rt &= \ln \frac{K-P_0}{P_0} \\ t &= \frac{1}{r} \ln \frac{K-P_0}{P_0}. \end{aligned}$$

Notice that if $P_0 > K$, then this quantity is undefined, and the graph does not have a point of inflection. In the logistic graph, the point of inflection can be seen as the point where the graph changes from concave up to concave down. This is where the “leveling off” starts to occur, because the net growth rate becomes slower as the population starts to approach the carrying capacity.

Note:
Exercise:

Problem:

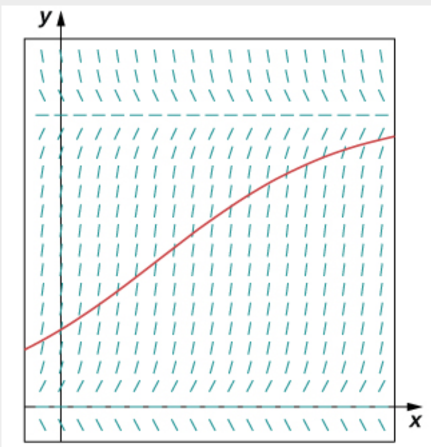
A population of rabbits in a meadow is observed to be 200 rabbits at time $t = 0$. After a month, the rabbit population is observed to have increased by 4%. Using an initial population of 200 and a growth rate of 0.04, with a carrying capacity of 750 rabbits,

a. Write the logistic differential equation and initial condition for this model.
b. Draw a slope field for this logistic differential equation, and sketch the solution corresponding to an initial population of 200 rabbits.
c. Solve the initial-value problem for $P\left(t\right)$.
d. Use the solution to predict the population after 1 year.

Solution:

a. $\frac{dP}{dt} = 0.04\left(1 - \frac{P}{750}\right), \quad P(0) = 200$

b.



c. $P\left(t\right) = \frac{3000e^{.04t}}{11+4e^{.04t}}$

d. After 12 months, the population will be $P(12) \approx 278$ rabbits.

Hint

First determine the values of $r, \quad K$, and P_0. Then create the initial-value problem, draw the direction field, and solve the problem.

Note:
Student Project: Logistic Equation with a Threshold Population
An improvement to the logistic model includes a **threshold population**. The threshold population is defined to be the minimum population that is necessary for the species to survive. We use the variable T to represent the threshold population. A differential equation that incorporates both the threshold population T and carrying capacity K is
Equation:

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

where r represents the growth rate, as before.

1. The threshold population is useful to biologists and can be utilized to determine whether a given species should be placed on the endangered list. A group of Australian researchers say they have determined the threshold population for any species to survive: 5000 adults. (Catherine Clabby, "A Magic Number," *American Scientist* 98(1): 24, doi:10.1511/2010.82.24. accessed April 9, 2015, http://www.americanscientist.org/issues/pub/a-magic-number). Therefore we use $T = 5000$ as the threshold population in this project. Suppose that the environmental carrying capacity in Montana for elk is $25{,}000$. Set up [link] using the carrying capacity of $25{,}000$ and threshold population of 5000. Assume an annual net growth rate of 18%.
2. Draw the direction field for the differential equation from step 1, along with several solutions for different initial populations. What are the constant solutions of the differential equation? What do these solutions correspond to in the original population model (i.e., in a biological context)?
3. What is the limiting population for each initial population you chose in step 2? (Hint: use the slope field to see what happens for various initial populations, i.e., look for the horizontal asymptotes of your solutions.)
4. This equation can be solved using the method of separation of variables. However, it is very difficult to get the solution as an explicit function of t. Using an initial population of $18{,}000$ elk, solve the initial-value problem and express the solution as an implicit function of t, or solve the general initial-value problem, finding a solution in terms of $r, K, T,$ and P_0.

Key Concepts

- When studying population functions, different assumptions—such as exponential growth, logistic growth, or threshold population—lead to different rates of growth.
- The logistic differential equation incorporates the concept of a carrying capacity. This value is a limiting value on the population for any given environment.
- The logistic differential equation can be solved for any positive growth rate, initial population, and carrying capacity.

Key Equations

- **Logistic differential equation and initial-value problem**
  $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P\left(0\right) = P_0$
- **Solution to the logistic differential equation/initial-value problem**
  $P\left(t\right) = \frac{P_0 K e^{rt}}{\left(K - P_0\right) + P_0 e^{rt}}$
- **Threshold population model**
  $\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$

For the following problems, consider the logistic equation in the form $P\prime = CP - P^2$. Draw the directional field and find the stability of the equilibria.

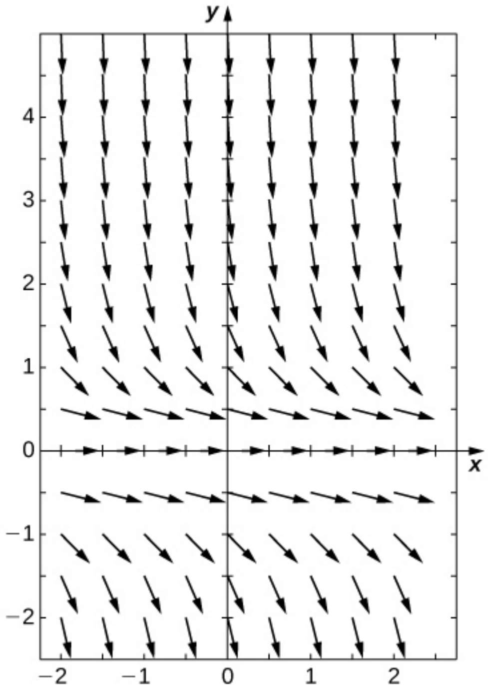
Exercise:

Problem: $C = 3$

Exercise:

Problem: $C = 0$

Solution:



$P = 0$ semi-stable

Exercise:

Problem: $C = -3$

Exercise:

Problem: Solve the logistic equation for $C = 10$ and an initial condition of $P(0) = 2$.

Solution:

$P = \frac{10e^{10x}}{e^{10x}+4}$

Exercise:

Problem: Solve the logistic equation for $C = -10$ and an initial condition of $P(0) = 2$.

Exercise:

Problem:

A population of deer inside a park has a carrying capacity of 200 and a growth rate of 2%. If the initial population is 50 deer, what is the population of deer at any given time?

Solution:

$P\left(t\right) = \frac{10000e^{0.02t}}{150+50e^{0.02t}}$

Exercise:

Problem:

A population of frogs in a pond has a growth rate of 5%. If the initial population is 1000 frogs and the carrying capacity is 6000, what is the population of frogs at any given time?

Exercise:

Problem:

[T] Bacteria grow at a rate of 20% per hour in a petri dish. If there is initially one bacterium and a carrying capacity of 1 million cells, how long does it take to reach $500,000$ cells?

Solution:

69 hours 5 minutes

Exercise:

Problem:

[T] Rabbits in a park have an initial population of 10 and grow at a rate of 4% per year. If the carrying capacity is 500, at what time does the population reach 100 rabbits?

Exercise:

Problem:

[T] Two monkeys are placed on an island. After 5 years, there are 8 monkeys, and the estimated carrying capacity is 25 monkeys. When does the population of monkeys reach 16 monkeys?

Solution:

7 years 2 months

Exercise:

Problem:

[T] A butterfly sanctuary is built that can hold 2000 butterflies, and 400 butterflies are initially moved in. If after 2 months there are now 800 butterflies, when does the population get to 1500 butterflies?

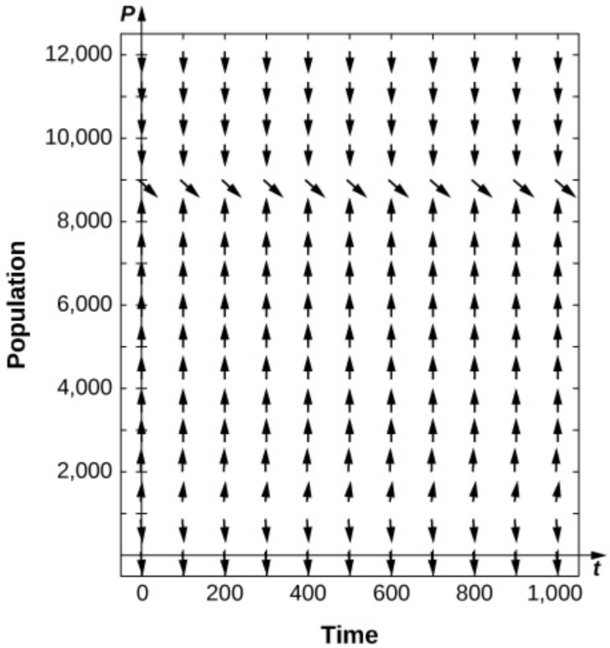
The following problems consider the logistic equation with an added term for depletion, either through death or emigration.

Exercise:

Problem:

[T] The population of trout in a pond is given by $P\prime = 0.4P\left(1 - \frac{P}{10000}\right) - 400$, where 400 trout are caught per year. Use your calculator or computer software to draw a directional field and draw a few sample solutions. What do you expect for the behavior?

Solution:



Exercise:

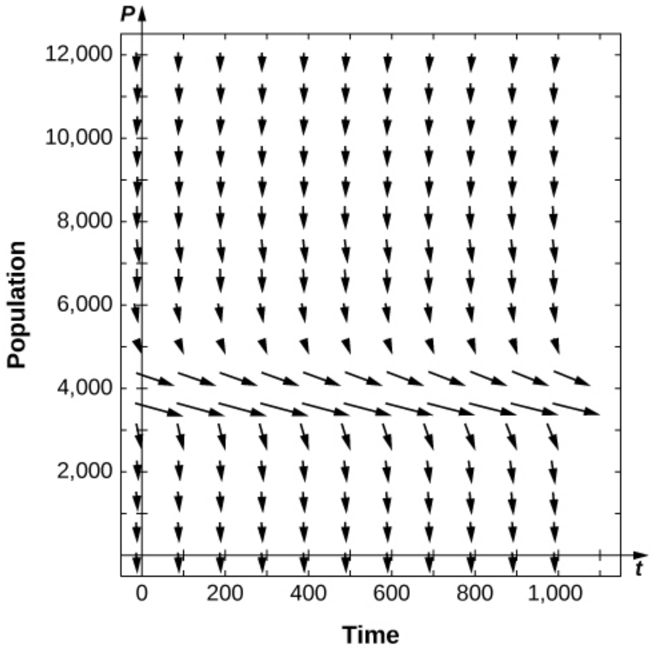
Problem: In the preceding problem, what are the stabilities of the equilibria $0 < P_1 < P_2$?

Exercise:

Problem:

[T] For the preceding problem, use software to generate a directional field for the value $f = 400$. What are the stabilities of the equilibria?

Solution:



P_1 semi-stable

Exercise:

Problem:

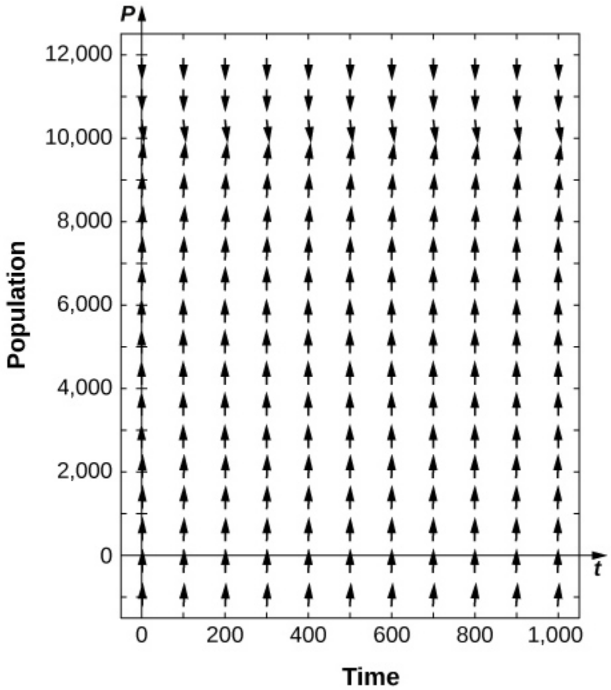
[T] For the preceding problems, use software to generate a directional field for the value $f = 600$. What are the stabilities of the equilibria?

Exercise:

Problem:

[T] For the preceding problems, consider the case where a certain number of fish are added to the pond, or $f = -200$. What are the nonnegative equilibria and their stabilities?

Solution:



$P_2 > 0$ stable

It is more likely that the amount of fishing is governed by the current number of fish present, so instead of a constant number of fish being caught, the rate is proportional to the current number of fish present, with proportionality constant k, as

$P\prime = 0.4P\left(1 - \frac{P}{10000}\right) - kP.$

Exercise:

Problem:

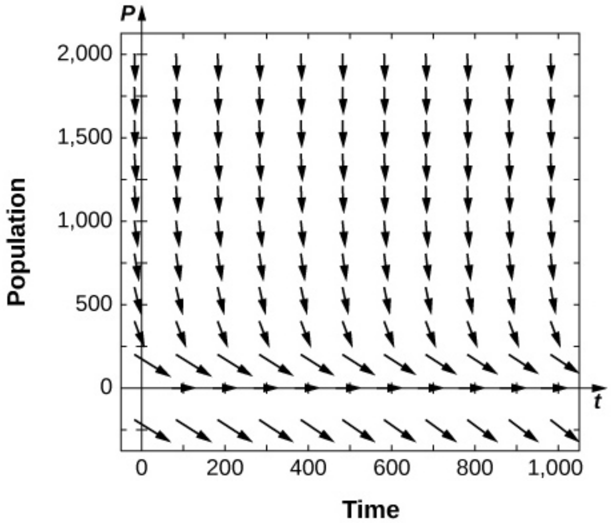
[T] For the previous fishing problem, draw a directional field assuming $k = 0.1$. Draw some solutions that exhibit this behavior. What are the equilibria and what are their stabilities?

Exercise:

Problem:

[T] Use software or a calculator to draw directional fields for $k = 0.4$. What are the nonnegative equilibria and their stabilities?

Solution:



$P_1 = 0$ is semi-stable

Exercise:

Problem:

[T] Use software or a calculator to draw directional fields for $k = 0.6$. What are the equilibria and their stabilities?

Exercise:

Problem: Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 2000 fish.

Solution:

$y = \frac{-20}{4 \times 10^{-6} - 0.002e^{0.01t}}$

Exercise:

Problem: Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 5000 fish.

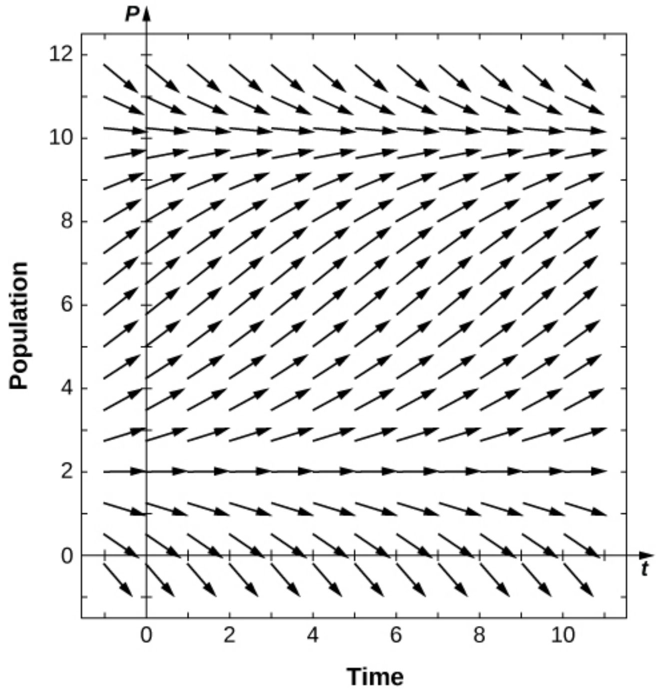
The following problems add in a minimal threshold value for the species to survive, T, which changes the differential equation to $P\prime(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$.

Exercise:

Problem:

Draw the directional field of the threshold logistic equation, assuming $K = 10, r = 0.1, T = 2$. When does the population survive? When does it go extinct?

Solution:



Exercise:

Problem:

For the preceding problem, solve the logistic threshold equation, assuming the initial condition $P(0) = P_0$.

Exercise:

Problem:

Bengal tigers in a conservation park have a carrying capacity of 100 and need a minimum of 10 to survive. If they grow in population at a rate of 1% per year, with an initial population of 15 tigers, solve for the number of tigers present.

Solution:

$P(t) = \frac{850+500e^{0.009t}}{85+5e^{0.009t}}$

Exercise:

Problem:

A forest containing ring-tailed lemurs in Madagascar has the potential to support 5000 individuals, and the lemur population grows at a rate of 5% per year. A minimum of 500 individuals is needed for the lemurs to survive. Given an initial population of 600 lemurs, solve for the population of lemurs.

Exercise:

Problem:

The population of mountain lions in Northern Arizona has an estimated carrying capacity of 250 and grows at a rate of 0.25% per year and there must be 25 for the population to survive. With an initial population of 30 mountain lions, how many years will it take to get the mountain lions off the endangered species list (at least 100)?

Solution:

13 years months

The following questions consider the Gompertz equation, a modification for logistic growth, which is often used for modeling cancer growth, specifically the number of tumor cells.

Exercise:

Problem:

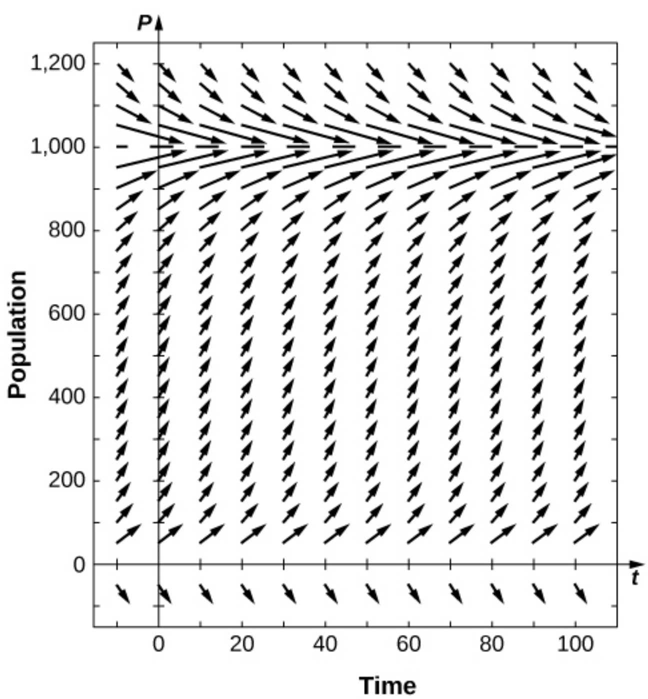
The Gompertz equation is given by $P\left(t\right)\prime=\alpha \ln\left(\frac{K}{P(t)}\right)P\left(t\right)$. Draw the directional fields for this equation assuming all parameters are positive, and given that $K=1$.

Exercise:

Problem:

Assume that for a population, $K=1000$ and $\alpha =0.05$. Draw the directional field associated with this differential equation and draw a few solutions. What is the behavior of the population?

Solution:



Exercise:

Problem: Solve the Gompertz equation for generic α and K and $P\left(0\right)={P}_{0}$.

Exercise:

Problem:

[T] The Gompertz equation has been used to model tumor growth in the human body. Starting from one tumor cell on day 1 and assuming $\alpha =0.1$ and a carrying capacity of 10 million cells, how long does it take to reach “detection” stage at 5 million cells?

Solution:

31.465 days

Exercise:

Problem:

[T] It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for logistic growth, and determine what year the population reached 7 billion.

Exercise:

Problem:

[T] It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for Gompertz growth, and determine what year the population reached 7 billion. Was logistic growth or Gompertz growth more accurate, considering world population reached 7 billion on October $31, 2011$?

Solution:

September 2008

Exercise:

Problem:

Show that the population grows fastest when it reaches half the carrying capacity for the logistic equation $P\prime = rP\left(1 - \frac{P}{K}\right)$.

Exercise:

Problem:

When does population increase the fastest in the threshold logistic equation $P\prime(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$?

Solution:

$\frac{K+T}{2}$

Exercise:

Problem: When does population increase the fastest for the Gompertz equation $P\left(t\right)\prime = \alpha \ln\left(\frac{K}{P(t)}\right)P\left(t\right)$?

Below is a table of the populations of whooping cranes in the wild from 1940 to 2000. The population rebounded from near extinction after conservation efforts began. The following problems consider applying population models to fit the data. Assume a carrying capacity of $10,000$ cranes. Fit the data assuming years since 1940 (so your initial population at time 0 would be 22 cranes).

Year (years since conservation began)	Whooping Crane Population
$1940\left(0\right)$	22

Year (years since conservation began)	Whooping Crane Population
$1950\,(10)$	31
$1960\,(20)$	36
$1970\,(30)$	57
$1980\,(40)$	91
$1990\,(50)$	159
$2000\,(60)$	256

Source: https://www.savingcranes.org/images/stories/site_images/conservation/whooping_crane/pdfs/historic_wc_numbers.pdf

Exercise:

Problem: Find the equation and parameter r that best fit the data for the logistic equation.

Solution:

$r = 0.0405$

Exercise:

Problem: Find the equation and parameters r and T that best fit the data for the threshold logistic equation.

Exercise:

Problem: Find the equation and parameter α that best fit the data for the Gompertz equation.

Solution:

$\alpha = 0.0081$

Exercise:

Problem:

Graph all three solutions and the data on the same graph. Which model appears to be most accurate?

Exercise:

Problem:

Using the three equations found in the previous problems, estimate the population in 2010 (year 70 after conservation). The real population measured at that time was 437. Which model is most accurate?

Solution:

Logistic: 361, Threshold: 436, Gompertz: 309.

Glossary

carrying capacity

the maximum population of an organism that the environment can sustain indefinitely

growth rate
: the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

initial population
: the population at time $t = 0$

logistic differential equation
: a differential equation that incorporates the carrying capacity K and growth rate r into a population model

phase line
: a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

threshold population
: the minimum population that is necessary for a species to survive

First-order Linear Equations

- Write a first-order linear differential equation in standard form.
- Find an integrating factor and use it to solve a first-order linear differential equation.
- Solve applied problems involving first-order linear differential equations.

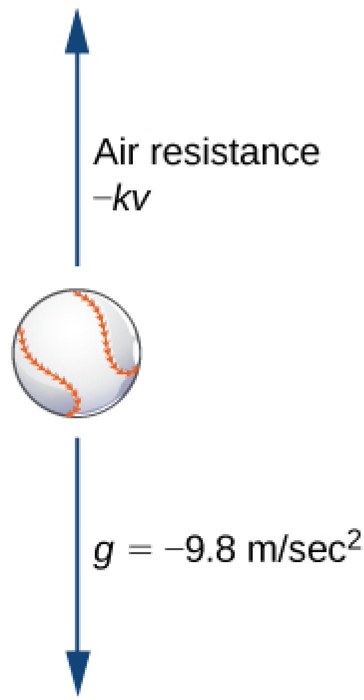
Earlier, we studied an application of a first-order differential equation that involved solving for the velocity of an object. In particular, if a ball is thrown upward with an initial velocity of v_0 ft/s, then an initial-value problem that describes the velocity of the ball after t seconds is given by

Equation:

$$\frac{dv}{dt} = -32, \quad v(0) = v_0.$$

This model assumes that the only force acting on the ball is gravity. Now we add to the problem by allowing for the possibility of air resistance acting on the ball.

Air resistance always acts in the direction opposite to motion. Therefore if an object is rising, air resistance acts in a downward direction. If the object is falling, air resistance acts in an upward direction ([link]). There is no exact relationship between the velocity of an object and the air resistance acting on it. For very small objects, air resistance is proportional to velocity; that is, the force due to air resistance is numerically equal to some constant k times v. For larger (e.g., baseball-sized) objects, depending on the shape, air resistance can be approximately proportional to the square of the velocity. In fact, air resistance may be proportional to $v^{1.5}$, or $v^{0.9}$, or some other power of v.



Forces acting on a moving baseball: gravity acts in a downward direction and air resistance acts in a direction opposite to the direction of motion.

We will work with the linear approximation for air resistance. If we assume $k > 0$, then the expression for the force F_A due to air resistance is given by $F_A = -kv$. Therefore the sum of the forces acting on the object is equal to the sum of the gravitational force and the force due to air resistance. This, in turn, is equal to the mass of the object multiplied by its acceleration at time t (Newton's second law). This gives us the differential equation

Equation:

$$m\frac{dv}{dt} = -kv - mg.$$

Finally, we impose an initial condition $v(0) = v_0$, where v_0 is the initial velocity measured in meters per second. This makes $g = 9.8\,\text{m/s}^2$. The initial-value problem becomes

Equation:

$$m\frac{dv}{dt} = -kv - mg, \quad v(0) = v_0.$$

The differential equation in this initial-value problem is an example of a first-order linear differential equation. (Recall that a differential equation is first-order if the highest-order derivative that appears in the equation is 1.) In this section, we study first-order linear equations and examine a method for finding a general solution to these types of equations, as well as solving initial-value problems involving them.

Note:
Definition
A first-order differential equation is **linear** if it can be written in the form

Equation:

$$a(x)y' + b(x)y = c(x),$$

where $a(x), b(x)$, and $c(x)$ are arbitrary functions of x.

Remember that the unknown function y depends on the variable x; that is, x is the independent variable and y is the dependent variable. Some examples of first-order linear differential equations are

Equation:

$$\begin{aligned}(3x^2 - 4)y' + (x-3)y &= \sin x \\ (\sin x)y' - (\cos x)y &= \cot x \\ 4xy' + (3\ln x)y &= x^3 - 4x.\end{aligned}$$

Examples of first-order nonlinear differential equations include
Equation:

$$\begin{aligned}(y\prime)^4-(y\prime)^3&=(3x-2)(y+4)\\4y\prime+3y^3&=4x-5\\(y\prime)^2&=\sin y+\cos x.\end{aligned}$$

These equations are nonlinear because of terms like $(y')^4, y^3$, etc. Due to these terms, it is impossible to put these equations into the same form as [link].

Standard Form

Consider the differential equation
Equation:

$$\left(3x^2-4\right)y'+(x-3)y=\sin x.$$

Our main goal in this section is to derive a solution method for equations of this form. It is useful to have the coefficient of y' be equal to 1. To make this happen, we divide both sides by $3x^2-4$.
Equation:

$$y'+\left(\frac{x-3}{3x^2-4}\right)y=\frac{\sin x}{3x^2-4}$$

This is called the **standard form** of the differential equation. We will use it later when finding the solution to a general first-order linear differential equation. Returning to [link], we can divide both sides of the equation by $a(x)$. This leads to the equation
Equation:

$$y' + \frac{b(x)}{a(x)} y = \frac{c(x)}{a(x)}.$$

Now define $p(x) = \frac{b(x)}{a(x)}$ and $q(x) = \frac{c(x)}{a(x)}$. Then [link] becomes

Equation:

$$y' + p(x)y = q(x).$$

We can write any first-order linear differential equation in this form, and this is referred to as the standard form for a first-order linear differential equation.

Example:

Exercise:

Problem:

Writing First-Order Linear Equations in Standard Form

Put each of the following first-order linear differential equations into standard form. Identify $p(x)$ and $q(x)$ for each equation.

a. $y\prime = 3x - 4y$
b. $\frac{3xy\prime}{4y-3} = 2$ (here $x > 0$)
c. $y = 3y\prime - 4x^2 + 5$

Solution:

a. Add $4y$ to both sides:

Equation:

$$y\prime + 4y = 3x.$$

In this equation, $p(x) = 4$ and $q(x) = 3x$.

b. Multiply both sides by $4y - 3$, then subtract $8y$ from each side:

Equation:

$$\begin{aligned} \frac{3xy\prime}{4y-3} &= 2 \\ 3xy\prime &= 2(4y-3) \\ 3xy\prime &= 8y-6 \\ 3xy\prime-8y &= -6. \end{aligned}$$

Finally, divide both sides by $3x$ to make the coefficient of $y\prime$ equal to 1:

Equation:

$$y\prime - \frac{8}{3x}y = -\frac{2}{3x}.$$

This is allowable because in the original statement of this problem we assumed that $x > 0$. (If $x = 0$ then the original equation becomes $0 = 2$, which is clearly a false statement.) In this equation, $p(x) = -\frac{8}{3x}$ and $q(x) = -\frac{2}{3x}$.

c. Subtract y from each side and add $4x^2 - 5$:

Equation:

$$3y\prime - y = 4x^2 - 5.$$

Next divide both sides by 3:

Equation:

$$y\prime - \frac{1}{3}y = \frac{4}{3}x^2 - \frac{5}{3}.$$

In this equation, $p(x) = -\frac{1}{3}$ and $q(x) = \frac{4}{3}x^2 - \frac{5}{3}$.

Note:
Exercise:

Problem:

Put the equation $\frac{(x+3)y\prime}{2x-3y-4} = 5$ into standard form and identify $p(x)$ and $q(x)$.

Solution:

$y\prime + \frac{15}{x+3}y = \frac{10x-20}{x+3}; p(x) = \frac{15}{x+3}$ and $q(x) = \frac{10x-20}{x+3}$

Hint

Multiply both sides by the common denominator, then collect all terms involving y on one side.

Integrating Factors

We now develop a solution technique for any first-order linear differential equation. We start with the standard form of a first-order linear differential equation:

Equation:

$$y\prime + p(x)y = q(x).$$

The first term on the left-hand side of [link] is the derivative of the unknown function, and the second term is the product of a known function with the unknown function. This is somewhat reminiscent of the power rule from the Differentiation Rules section. If we multiply [link] by a yet-to-be-determined function $\mu(x)$, then the equation becomes

Equation:

$$\mu (x)y' + \mu (x)p (x)y = \mu (x)q (x).$$

The left-hand side [link] can be matched perfectly to the product rule:
Equation:

$$\frac{d}{dx}[f (x)g (x)] = f' (x)g (x) + f (x)g' (x).$$

Matching term by term gives $y = f (x)$, $g (x) = \mu (x)$, and $g' (x) = \mu (x)p (x)$. Taking the derivative of $g (x) = \mu (x)$ and setting it equal to the right-hand side of $g' (x) = \mu (x)p (x)$ leads to
Equation:

$$\mu' (x) = \mu (x)p (x).$$

This is a first-order, separable differential equation for $\mu (x)$. We know $p(x)$ because it appears in the differential equation we are solving. Separating variables and integrating yields
Equation:

$$\begin{aligned}
\tfrac{\mu'(x)}{\mu(x)} &= p (x) \\
\int \frac{\mu' (x)}{\mu (x)} dx &= \int p (x) dx \\
\ln |\mu (x)| &= \int p (x) dx + C \\
e^{\ln|\mu(x)|} &= e^{\int p(x) dx + C} \\
|\mu (x)| &= C_1 e^{\int p(x) dx} \\
\mu (x) &= C_2 e^{\int p(x) dx}.
\end{aligned}$$

Here C_2 can be an arbitrary (positive or negative) constant. This leads to a general method for solving a first-order linear differential equation. We first multiply both sides of [link] by the **integrating factor** $\mu(x)$. This gives

Equation:

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x).$$

The left-hand side of [link] can be rewritten as $\frac{d}{dx}(\mu(x)y)$.

Equation:

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x).$$

Next integrate both sides of [link] with respect to x.

Equation:

$$\begin{aligned} \int \frac{d}{dx}(\mu(x)y)\,dx &= \int \mu(x)q(x)\,dx \\ \mu(x)y &= \int \mu(x)q(x)\,dx. \end{aligned}$$

Divide both sides of [link] by $\mu(x)$:

Equation:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)q(x)\,dx + C\right].$$

Since $\mu(x)$ was previously calculated, we are now finished. An important note about the integrating constant C: It may seem that we are inconsistent in the usage of the integrating constant. However, the integral involving $p(x)$ is necessary in order to find an integrating factor for [link]. Only one integrating factor is needed in order to solve the equation; therefore, it is safe to assign a value for C for this integral. We chose $C = 0$. When calculating the integral inside the brackets in [link], it is necessary to keep

our options open for the value of the integrating constant, because our goal is to find a general family of solutions to [link]. This integrating factor guarantees just that.

Note:
Problem-Solving Strategy: Solving a First-order Linear Differential Equation

1. Put the equation into standard form and identify $p\left(x\right)$ and $q\left(x\right)$.
2. Calculate the integrating factor $\mu\left(x\right)=e^{\int p(x)\,dx}$.
3. Multiply both sides of the differential equation by $\mu\left(x\right)$.
4. Integrate both sides of the equation obtained in step 3, and divide both sides by $\mu\left(x\right)$.
5. If there is an initial condition, determine the value of C.

Example:
Exercise:

Problem:
Solving a First-order Linear Equation

Find a general solution for the differential equation $xy\prime+3y=4x^2-3x$. Assume $x>0$.

Solution:

1. To put this differential equation into standard form, divide both sides by x:
   Equation:

$$y\prime+\frac{3}{x}y=4x-3.$$

Therefore $p\left(x\right)=\frac{3}{x}$ and $q\left(x\right)=4x-3.$

2. The integrating factor is $\mu\left(x\right)=e^{\int \left(3/x\right)dx}=e^{3\ln x}=x^{3}.$
3. Multiplying both sides of the differential equation by $\mu\left(x\right)$ gives us
   Equation:

$$\begin{array}{rcl} x^{3}y'+x^{3}\left(\frac{3}{x}\right)y & = & x^{3}\left(4x-3\right) \\ x^{3}y'+3x^{2}y & = & 4x^{4}-3x^{3} \\ \frac{d}{dx}\left(x^{3}y\right) & = & 4x^{4}-3x^{3}. \end{array}$$

4. Integrate both sides of the equation.
   Equation:

$$\begin{array}{rcl} \int \frac{d}{dx}\left(x^{3}y\right)dx & = & \int 4x^{4}-3x^{3}dx \\ x^{3}y & = & \frac{4x^{5}}{5}-\frac{3x^{4}}{4}+C \\ y & = & \frac{4x^{2}}{5}-\frac{3x}{4}+Cx^{-3}. \end{array}$$

5. There is no initial value, so the problem is complete.

Analysis

You may have noticed the condition that was imposed on the differential equation; namely, $x>0$. For any nonzero value of C, the general solution is not defined at $x=0$. Furthermore, when $x<0$, the integrating factor changes. The integrating factor is given by [link] as $f\left(x\right)=e^{\int p\left(x\right)dx}$. For this $p\left(x\right)$ we get

Equation:

$$e^{\int p\left(x\right)dx=}e^{\int \left(3/x\right)dx}=e^{3\ln \left|x\right|}=\left|x\right|^{3},$$

since $x < 0$. The behavior of the general solution changes at $x = 0$ largely due to the fact that $p\left(x\right)$ is not defined there.

Note:
Exercise:

Problem:

Find the general solution to the differential equation $(x-2)y\prime+y=3x^2+2x$. Assume $x > 2$.

Solution:

$y=\frac{x^3+x^2+C}{x-2}$

Hint

Use the method outlined in the problem-solving strategy for first-order linear differential equations.

Now we use the same strategy to find the solution to an initial-value problem.

Example:
Exercise:

Problem:
A First-order Linear Initial-Value Problem

Solve the initial-value problem
Equation:

$$y' + 3y = 2x - 1, \quad y(0) = 3.$$

Solution:

1. This differential equation is already in standard form with $p(x) = 3$ and $q(x) = 2x - 1$.
2. The integrating factor is $\mu(x) = e^{\int 3dx} = e^{3x}$.
3. Multiplying both sides of the differential equation by $\mu(x)$ gives
   Equation:

$$\begin{aligned} e^{3x}y' + 3e^{3x}y &= (2x-1)e^{3x} \\ \tfrac{d}{dx}\left[ye^{3x}\right] &= (2x-1)e^{3x}. \end{aligned}$$

   Integrate both sides of the equation:
   Equation:

$$\begin{aligned} \int \frac{d}{dx}\left[ye^{3x}\right]dx &= \int (2x-1)e^{3x}dx \\ ye^{3x} &= \tfrac{e^{3x}}{3}(2x-1) - \int \frac{2}{3}e^{3x}dx \\ ye^{3x} &= \tfrac{e^{3x}(2x-1)}{3} - \tfrac{2e^{3x}}{9} + C \\ y &= \tfrac{2x-1}{3} - \tfrac{2}{9} + Ce^{-3x} \\ y &= \tfrac{2x}{3} - \tfrac{5}{9} + Ce^{-3x}. \end{aligned}$$

4. Now substitute $x = 0$ and $y = 3$ into the general solution and solve for C:
   Equation:

$$\begin{aligned} y &= \tfrac{2}{3}x - \tfrac{5}{9} + Ce^{-3x} \\ 3 &= \tfrac{2}{3}(0) - \tfrac{5}{9} + Ce^{-3(0)} \\ 3 &= -\tfrac{5}{9} + C \\ C &= \tfrac{32}{9}. \end{aligned}$$

Therefore the solution to the initial-value problem is
Equation:

$$y = \frac{2}{3}x - \frac{5}{9} + \frac{32}{9}e^{-3x}.$$

Note:
Exercise:

Problem:

Solve the initial-value problem $y\prime - 2y = 4x + 3 \quad y(0) = -2.$

Solution:

$y = -2x - 4 + 2e^{2x}$

Applications of First-order Linear Differential Equations

We look at two different applications of first-order linear differential equations. The first involves air resistance as it relates to objects that are rising or falling; the second involves an electrical circuit. Other applications are numerous, but most are solved in a similar fashion.

Free fall with air resistance

We discussed air resistance at the beginning of this section. The next example shows how to apply this concept for a ball in vertical motion.

Other factors can affect the force of air resistance, such as the size and shape of the object, but we ignore them here.

Example:
Exercise:

Problem:
A Ball with Air Resistance

A racquetball is hit straight upward with an initial velocity of 2 m/s. The mass of a racquetball is approximately 0.0427 kg. Air resistance acts on the ball with a force numerically equal to $0.5v$, where v represents the velocity of the ball at time t.

a. Find the velocity of the ball as a function of time.
b. How long does it take for the ball to reach its maximum height?
c. If the ball is hit from an initial height of 1 meter, how high will it reach?

Solution:

a. The mass $m = 0.0427\,\text{kg}$, $k = 0.5$, and $g = 9.8\,\text{m/s}^2$. The initial velocity is $v_0 = 2$ m/s. Therefore the initial-value problem is
Equation:

$$0.0427\frac{dv}{dt} = -0.5v - 0.0427\,(9.8), \quad v_0 = 2.$$

Dividing the differential equation by 0.0427 gives
Equation:

$$\frac{dv}{dt} = -11.7096v - 9.8, \quad v_0 = 2.$$

The differential equation is linear. Using the problem-solving strategy for linear differential equations:
Step 1. Rewrite the differential equation as $\frac{dv}{dt} + 11.7096v = -9.8$. This gives $p(t) = 11.7096$ and $q(t) = -9.8$

Step 2. The integrating factor is $\mu(t) = e^{\int 11.7096dt} = e^{11.7096t}$.
Step 3. Multiply the differential equation by $\mu(t)$:
Equation:

$$\begin{aligned} e^{11.7096t}\frac{dv}{dt} + 11.7096ve^{11.7096t} &= -9.8e^{11.7096t} \\ \frac{d}{dt}\left[ve^{11.7096t}\right] &= -9.8e^{11.7096t}. \end{aligned}$$

Step 4. Integrate both sides:
Equation:

$$\begin{aligned} \int \frac{d}{dt}\left[ve^{11.7096t}\right]dt &= \int -9.8e^{11.7096t}dt \\ ve^{11.7096t} &= \frac{-9.8}{11.7096}e^{11.7096t} + C \\ v(t) &= -0.8369 + Ce^{-11.7096t}. \end{aligned}$$

Step 5. Solve for C using the initial condition $v_0 = v(0) = 2$:
Equation:

$$\begin{aligned} v(t) &= -0.8369 + Ce^{-11.7096t} \\ v(0) &= -0.8369 + Ce^{-11.7096(0)} \\ 2 &= -0.8369 + C \\ C &= 2.8369. \end{aligned}$$

Therefore the solution to the initial-value problem is $v(t) = 2.8369e^{-11.7096t} - 0.8369$.

b. The ball reaches its maximum height when the velocity is equal to zero. The reason is that when the velocity is positive, it is rising, and when it is negative, it is falling. Therefore when it is zero, it is neither rising nor falling, and is at its maximum height:
Equation:

$$\begin{aligned} 2.8369e^{-11.7096t} - 0.8369 &= 0 \\ 2.8369e^{-11.7096t} &= 0.8369 \\ e^{-11.7096t} &= \tfrac{0.8369}{2.8369} \approx 0.295 \\ \ln e^{-11.7096t} &= \ln 0.295 \approx -1.221 \\ -11.7096t &= -1.221 \\ t &\approx 0.104. \end{aligned}$$

Therefore it takes approximately 0.104 second to reach maximum height.

c. To find the height of the ball as a function of time, use the fact that the derivative of position is velocity, i.e., if $h(t)$ represents the height at time t, then $h'(t) = v(t)$. Because we know $v(t)$ and the initial height, we can form an initial-value problem:
Equation:

$$h'(t) = 2.8369e^{-11.7096t} - 0.8369, \quad h(0) = 1.$$

Integrating both sides of the differential equation with respect to t gives
Equation:

$$\begin{aligned} \int h'(t)\,dt &= \int 2.8369e^{-11.7096t} - 0.8369dt \\ h(t) &= -\tfrac{2.8369}{11.7096}e^{-11.7096t} - 0.8369t + C \\ h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C. \end{aligned}$$

Solve for C by using the initial condition:
Equation:

$$\begin{aligned} h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C \\ h(0) &= -0.2423e^{-11.7096(0)} - 0.8369\,(0) + C \\ 1 &= -0.2423 + C \\ C &= 1.2423. \end{aligned}$$

Therefore
Equation:

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423.$$

After 0.104 second, the height is given by
$h(0.2) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423 \approx 1.0836$
meter.

Note:
Exercise:

Problem:

The weight of a penny is 2.5 grams (United States Mint, "Coin Specifications," accessed April 9, 2015, http://www.usmint.gov/about_the_mint/?action=coin_specifications), and the upper observation deck of the Empire State Building is 369 meters above the street. Since the penny is a small and relatively smooth object, air resistance acting on the penny is actually quite small. We assume the air resistance is numerically equal to $0.0025v$. Furthermore, the penny is dropped with no initial velocity imparted to it.

a. Set up an initial-value problem that represents the falling penny.
b. Solve the problem for $v\left(t\right)$.
c. What is the terminal velocity of the penny (i.e., calculate the limit of the velocity as t approaches infinity)?

Solution:

a. $$\begin{aligned}\frac{dv}{dt} &= -v-9.8\\ v\left(0\right) &= 0\end{aligned}$$

b. $v\left(t\right)=9.8\left(e^{-t}-1\right)$

c. $$\lim_{t\to\infty}v\left(t\right)=\lim_{t\to\infty}\left(9.8\left(e^{-t}-1\right)\right)=-9.8\,\text{m/s}\approx-21.922\,\text{mph}$$

Hint

Set up the differential equation the same way as [link]. Remember to convert from grams to kilograms.

Electrical Circuits

A source of electromotive force (e.g., a battery or generator) produces a flow of current in a closed circuit, and this current produces a voltage drop across each resistor, inductor, and capacitor in the circuit. Kirchhoff’s Loop Rule states that the sum of the voltage drops across resistors, inductors, and capacitors is equal to the total electromotive force in a closed circuit. We have the following three results:

1. The voltage drop across a resistor is given by
   Equation:

   $$E_R=Ri,$$

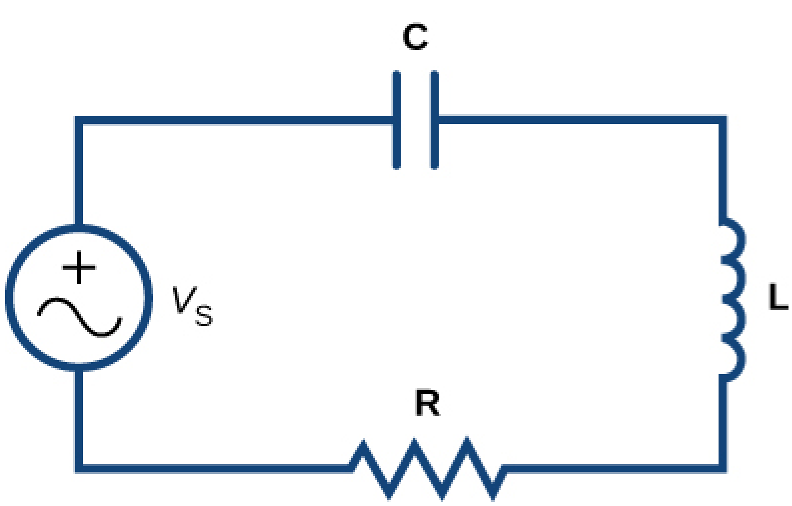
   where R is a constant of proportionality called the *resistance,* and i is

the current.

2. The voltage drop across an inductor is given by
Equation:
$$E_L = Li',$$
where L is a constant of proportionality called the *inductance*, and i again denotes the current.
3. The voltage drop across a capacitor is given by
Equation:
$$E_C = \frac{1}{C}q,$$

where C is a constant of proportionality called the *capacitance*, and q is the instantaneous charge on the capacitor. The relationship between i and q is $i = q'$.

We use units of volts (V) to measure voltage E, amperes (A) to measure current i, coulombs (C) to measure charge q, ohms (Ω) to measure resistance R, henrys (H) to measure inductance L, and farads (F) to measure capacitance C. Consider the circuit in [link].



A typical electric circuit, containing a voltage generator (V_S), capacitor (C), inductor (L), and resistor (R).

Applying Kirchhoff’s Loop Rule to this circuit, we let E denote the electromotive force supplied by the voltage generator. Then

Equation:

$$E_L + E_R + E_C = E.$$

Substituting the expressions for E_L, E_R, and E_C into this equation, we obtain

Equation:

$$Li' + Ri + \frac{1}{C}q = E.$$

If there is no capacitor in the circuit, then the equation becomes

Equation:

$$Li' + Ri = E.$$

This is a first-order differential equation in i. The circuit is referred to as an LR circuit.

Next, suppose there is no inductor in the circuit, but there is a capacitor and a resistor, so $L = 0$, $R \neq 0$, and $C \neq 0$. Then [link] can be rewritten as

Equation:

$$Rq' + \frac{1}{C}q = E,$$

which is a first-order linear differential equation. This is referred to as an *RC* circuit. In either case, we can set up and solve an initial-value problem.

Example:
Exercise:

Problem:
Finding Current in an *RL* Electric Circuit

A circuit has in series an electromotive force given by $E = 50 \sin 20t \text{ V}$, a resistor of 5Ω, and an inductor of 0.4 H. If the initial current is 0, find the current at time $t > 0$.

Solution:

We have a resistor and an inductor in the circuit, so we use [link]. The voltage drop across the resistor is given by $E_R = Ri = 5i$. The voltage drop across the inductor is given by $E_L = Li' = 0.4i'$. The electromotive force becomes the right-hand side of [link]. Therefore [link] becomes
Equation:

$$0.4i' + 5i = 50 \sin 20t.$$

Dividing both sides by 0.4 gives the equation
Equation:

$$i' + 12.5i = 125 \sin 20t.$$

Since the initial current is 0, this result gives an initial condition of $i(0) = 0$. We can solve this initial-value problem using the five-step strategy for solving first-order differential equations.

Step 1. Rewrite the differential equation as $i' + 12.5i = 125 \sin 20t$. This gives $p(t) = 12.5$ and $q(t) = 125 \sin 20t$.

Step 2. The integrating factor is $\mu(t) = e^{\int 12.5dt} = e^{12.5t}$.

Step 3. Multiply the differential equation by $\mu(t)$:
Equation:

$$\begin{aligned} e^{12.5t}i' + 12.5e^{12.5t}i &= 125e^{12.5t}\sin 20t \\ \frac{d}{dt}\left[ie^{12.5t}\right] &= 125e^{12.5t}\sin 20t. \end{aligned}$$

Step 4. Integrate both sides:
Equation:

$$\begin{aligned} \int \frac{d}{dt}\left[ie^{12.5t}\right] dt &= \int 125e^{12.5t}\sin 20t\, dt \\ ie^{12.5t} &= \left(\frac{250\sin 20t - 400\cos 20t}{89}\right)e^{12.5t} + C \\ i(t) &= \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t}. \end{aligned}$$

Step 5. Solve for C using the initial condition $v(0) = 2$:
Equation:

$$\begin{aligned} i(t) &= \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t} \\ i(0) &= \frac{250\sin 20(0) - 400\cos 20(0)}{89} + Ce^{-12.5(0)} \\ 0 &= -\frac{400}{89} + C \\ C &= \frac{400}{89}. \end{aligned}$$

Therefore the solution to the initial-value problem is
$i(t) = \frac{250\sin 20t - 400\cos 20t + 400e^{-12.5t}}{89} = \frac{250\sin 20t - 400\cos 20t}{89} + \frac{400e^{-12.5t}}{89}$.

The first term can be rewritten as a single cosine function. First, multiply and divide by $\sqrt{250^2 + 400^2} = 50\sqrt{89}$:
Equation:

$$\begin{aligned}\frac{250\sin 20t-400\cos 20t}{89} &= \frac{50\sqrt{89}}{89}\left(\frac{250\sin 20t-400\cos 20t}{50\sqrt{89}}\right)\\ &= -\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}}-\frac{5\sin 20t}{\sqrt{89}}\right).\end{aligned}$$

Next, define φ to be an acute angle such that $\cos\varphi=\frac{8}{\sqrt{89}}$. Then $\sin\varphi=\frac{5}{\sqrt{89}}$ and

Equation:

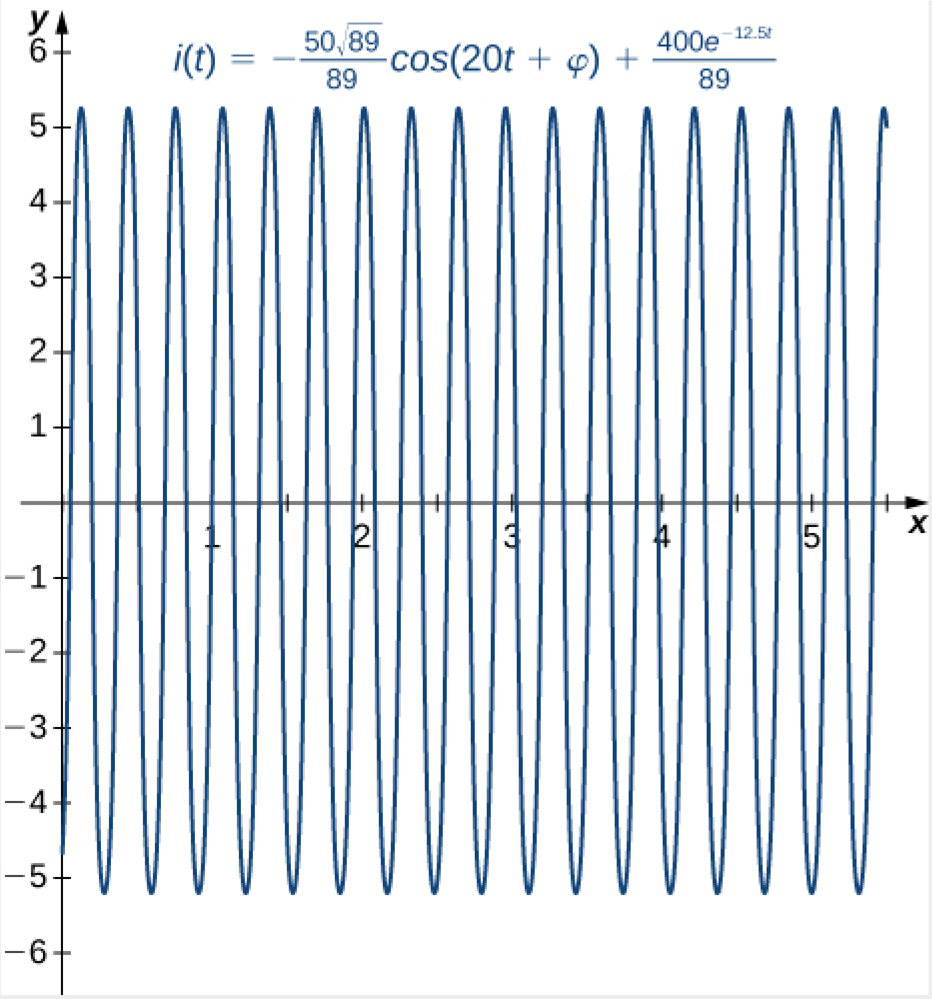
$$\begin{aligned}-\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}}-\frac{5\sin 20t}{\sqrt{89}}\right) &= -\frac{50\sqrt{89}}{89}\left(\cos\varphi\cos 20t-\sin\varphi\sin 20t\right)\\ &= -\frac{50\sqrt{89}}{89}\cos\left(20t+\varphi\right).\end{aligned}$$

Therefore the solution can be written as

Equation:

$$i\left(t\right)=-\frac{50\sqrt{89}}{89}\cos\left(20t+\varphi\right)+\frac{400e^{-12.5t}}{89}.$$

The second term is called the *attenuation* term, because it disappears rapidly as *t* grows larger. The phase shift is given by φ, and the amplitude of the steady-state current is given by $\frac{50\sqrt{89}}{89}$. The graph of this solution appears in [link]:



Note:
Exercise:

Problem:

A circuit has in series an electromotive force given by $E = 20\sin 5t$ V, a capacitor with capacitance $0.02\,\text{F}$, and a resistor of $8\,\Omega$. If the initial charge is $4\,\text{C}$, find the charge at time $t > 0$.

Solution:

Initial-value problem:

$$8q' + \frac{1}{0.02}q = 20\sin 5t, \quad q(0) = 4$$

$$q(t) = \frac{10\sin 5t - 8\cos 5t + 172e^{-6.25t}}{41}$$

Hint

Use [link] for an RC circuit to set up an initial-value problem.

Key Concepts

- Any first-order linear differential equation can be written in the form $y\prime + p(x)y = q(x)$.
- We can use a five-step problem-solving strategy for solving a first-order linear differential equation that may or may not include an initial value.
- Applications of first-order linear differential equations include determining motion of a rising or falling object with air resistance and finding current in an electrical circuit.

Key Equations

- **standard form**
  $y\prime + p(x)y = q(x)$
- **integrating factor**
  $\mu(x) = e^{\int p(x)\,dx}$

Are the following differential equations linear? Explain your reasoning.

Exercise:

Problem: $\frac{dy}{dx} = x^2y + \sin x$

Exercise:

Problem: $\frac{dy}{dt} = ty$

Solution:

Yes

Exercise:

Problem: $\frac{dy}{dt} + y^2 = x$

Exercise:

Problem: $y\prime = x^3 + e^x$

Solution:

Yes

Exercise:

Problem: $y\prime = y + e^y$

Write the following first-order differential equations in standard form.

Exercise:

Problem: $y\prime = x^3 y + \sin x$

Solution:

$y\prime - x^3 y = \sin x$

Exercise:

Problem: $y\prime + 3y - \ln x = 0$

Exercise:

Problem: $-xy\prime = (3x + 2)y + xe^x$

Solution:

$y\prime + \frac{(3x+2)}{x} y = -e^x$

Exercise:

Problem: $\frac{dy}{dt} = 4y + ty + \tan t$

Exercise:

Problem: $\frac{dy}{dt} = yx\left(x+1\right)$

Solution:

$\frac{dy}{dt} - yx\left(x+1\right) = 0$

What are the integrating factors for the following differential equations?

Exercise:

Problem: $y\prime = xy + 3$

Exercise:

Problem: $y\prime + e^x y = \sin x$

Solution:

e^x

Exercise:

Problem: $y\prime = x\ln\left(x\right)y + 3x$

Exercise:

Problem: $\frac{dy}{dx} = \tanh\left(x\right)y + 1$

Solution:

$-\ln(\cosh x)$

Exercise:

Problem: $\frac{dy}{dt} + 3ty = e^t y$

Solve the following differential equations by using integrating factors.

Exercise:

Problem: $y\prime = 3y + 2$

Solution:

$y = Ce^{3x} - \frac{2}{3}$

Exercise:

Problem: $y\prime = 2y - x^2$

Exercise:

Problem: $xy\prime = 3y - 6x^2$

Solution:

$y = Cx^3 + 6x^2$

Exercise:

Problem: $(x+2)y\prime = 3x + y$

Exercise:

Problem: $y\prime = 3x + xy$

Solution:

$y = Ce^{x^2/2} - 3$

Exercise:

Problem: $xy\prime = x + y$

Exercise:

Problem: $\sin(x)y\prime = y + 2x$

Solution:

$y = C\tan\left(\frac{x}{2}\right) - 2x + 4\tan\left(\frac{x}{2}\right)\ln\left(\sin\left(\frac{x}{2}\right)\right)$

Exercise:

Problem: $y\prime = y + e^x$

Exercise:

Problem: $xy\prime = 3y + x^2$

Solution:

$y = Cx^3 - x^2$

Exercise:

Problem: $y\prime + \ln x = \frac{y}{x}$

Solve the following differential equations. Use your calculator to draw a family of solutions. Are there certain initial conditions that change the behavior of the solution?

Exercise:

Problem: **[T]** $(x+2)y\prime = 2y-1$

Solution:

$y = C(x+2)^2 + \frac{1}{2}$

Exercise:

Problem: **[T]** $y\prime = 3e^{t/3} - 2y$

Exercise:

Problem: **[T]** $xy\prime + \frac{y}{2} = \sin(3t)$

Solution:

$y = \frac{C}{\sqrt{x}} + 2\sin(3t)$

Exercise:

Problem: **[T]** $xy\prime = 2\frac{\cos x}{x} - 3y$

Exercise:

Problem: **[T]** $(x+1)y\prime = 3y + x^2 + 2x + 1$

Solution:

$y = C(x+1)^3 - x^2 - 2x - 1$

Exercise:

Problem: **[T]** $\sin(x)y\prime + \cos(x)y = 2x$

Exercise:

Problem: **[T]** $\sqrt{x^2+1}y\prime = y+2$

Solution:

$y = Ce^{\sinh^{-1}x} - 2$

Exercise:

Problem: **[T]** $x^3y\prime + 2x^2y = x+1$

Solve the following initial-value problems by using integrating factors.

Exercise:

Problem: $y\prime + y = x, y(0) = 3$

Solution:

$y = x + 4e^x - 1$

Exercise:

Problem: $y\prime = y + 2x^2, y(0) = 0$

Exercise:

Problem: $xy\prime = y - 3x^3, y(1) = 0$

Solution:

$y = -\frac{3x}{2}\left(x^2 - 1\right)$

Exercise:

Problem: $x^2y\prime = xy - \ln x, y(1) = 1$

Exercise:

Problem: $\left(1+x^2\right)y\prime=y-1, y(0)=0$

Solution:

$y=1-e^{\tan^{-1}x}$

Exercise:

Problem: $xy\prime=y+2x\ln x, y(1)=5$

Exercise:

Problem: $(2+x)y\prime=y+2+x, y(0)=0$

Solution:

$y=(x+2)\ln\left(\frac{x+2}{2}\right)$

Exercise:

Problem: $y\prime=xy+2xe^x, y(0)=2$

Exercise:

Problem: $\sqrt{x}y\prime=y+2x, y(0)=1$

Solution:

$y=2e^{2\sqrt{x}}-2x-2\sqrt{x}-1$

Exercise:

Problem: $y\prime=2y+xe^x, y(0)=-1$

Exercise:

Problem:

A falling object of mass m can reach terminal velocity when the drag force is proportional to its velocity, with proportionality constant k. Set up the differential equation and solve for the velocity given an initial velocity of 0.

Solution:

$v(t) = \frac{gm}{k}\left(1 - e^{-kt/m}\right)$

Exercise:

Problem:

Using your expression from the preceding problem, what is the terminal velocity? (*Hint:* Examine the limiting behavior; does the velocity approach a value?)

Exercise:

Problem:

[T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is 9.8 m/s^2 and the proportionality constant is 4?

Solution:

40.451 seconds

Exercise:

Problem:

A more accurate way to describe terminal velocity is that the drag force is proportional to the square of velocity, with a proportionality constant k. Set up the differential equation and solve for the velocity.

Exercise:

Problem:

Using your expression from the preceding problem, what is the terminal velocity? (*Hint:* Examine the limiting behavior: Does the velocity approach a value?)

Solution:

$\sqrt{\frac{gm}{k}}$

Exercise:

Problem:

[T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is $9.8\ \mathrm{m/s}^2$ and the proportionality constant is 4? Does it take more or less time than your initial estimate?

For the following problems, determine how parameter a affects the solution.

Exercise:

Problem:

Solve the generic equation $y\prime = ax + y$. How does varying a change the behavior?

Solution:

$y = Ce^x - a(x + 1)$

Exercise:

Problem:

Solve the generic equation $y\prime = ax + y$. How does varying a change the behavior?

Exercise:

Problem:

Solve the generic equation $y\prime = ax + xy$. How does varying a change the behavior?

Solution:

$y = Ce^{x^2/2} - a$

Exercise:

Problem:

Solve the generic equation $y\prime = x + axy$. How does varying a change the behavior?

Exercise:

Problem:

Solve $y\prime - y = e^{kt}$ with the initial condition $y(0) = 0$. As k approaches 1, what happens to your formula?

Solution:

$y = \frac{e^{kt} - e^{t}}{k-1}$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem: The differential equation $y\prime = 3x^2 y - \cos(x)y\prime\prime$ is linear.

Exercise:

Problem: The differential equation $y\prime = x - y$ is separable.

Solution:

F

Exercise:

Problem:

You can explicitly solve all first-order differential equations by separation or by the method of integrating factors.

Exercise:

Problem:

You can determine the behavior of all first-order differential equations using directional fields or Euler's method.

Solution:

T

For the following problems, find the general solution to the differential equations.

Exercise:

Problem: $y' = x^2 + 3e^x - 2x$

Exercise:

Problem: $y\prime = 2^x + \cos^{-1}x$

Solution:

$y\left(x\right) = \frac{2^x}{\ln(2)} + x\cos^{-1}x - \sqrt{1 - x^2} + C$

Exercise:

Problem: $y\prime = y\left(x^2 + 1\right)$

Exercise:

Problem: $y\prime = e^{-y}\sin x$

Solution:

$y(x) = \ln\left(C - \cos x\right)$

Exercise:

Problem: $y\prime = 3x - 2y$

Exercise:

Problem: $y\prime = y\ln y$

Solution:

$y(x) = e^{e^{C+x}}$

For the following problems, find the solution to the initial value problem.

Exercise:

Problem: $y\prime = 8x - \ln x - 3x^4, y(1) = 5$

Exercise:

Problem: $y\prime = 3x - \cos x + 2, y(0) = 4$

Solution:

$y(x) = 4 + \frac{3}{2}x^2 + 2x - \sin x$

Exercise:

Problem: $xy\prime = y\left(x - 2\right), y(1) = 3$

Exercise:

Problem: $y\prime = 3y^2\ (x + \cos x), y(0) = -2$

Solution:

$y(x) = -\frac{2}{1+3(x^2+2\sin x)}$

Exercise:

Problem: $(x - 1)y\prime = y - 2, y(0) = 0$

Exercise:

Problem: $y\prime = 3y - x + 6x^2, y(0) = -1$

Solution:

$y(x) = -2x^2 - 2x - \frac{1}{3} - \frac{2}{3}e^{3x}$

For the following problems, draw the directional field associated with the differential equation, then solve the differential equation. Draw a sample solution on the directional field.

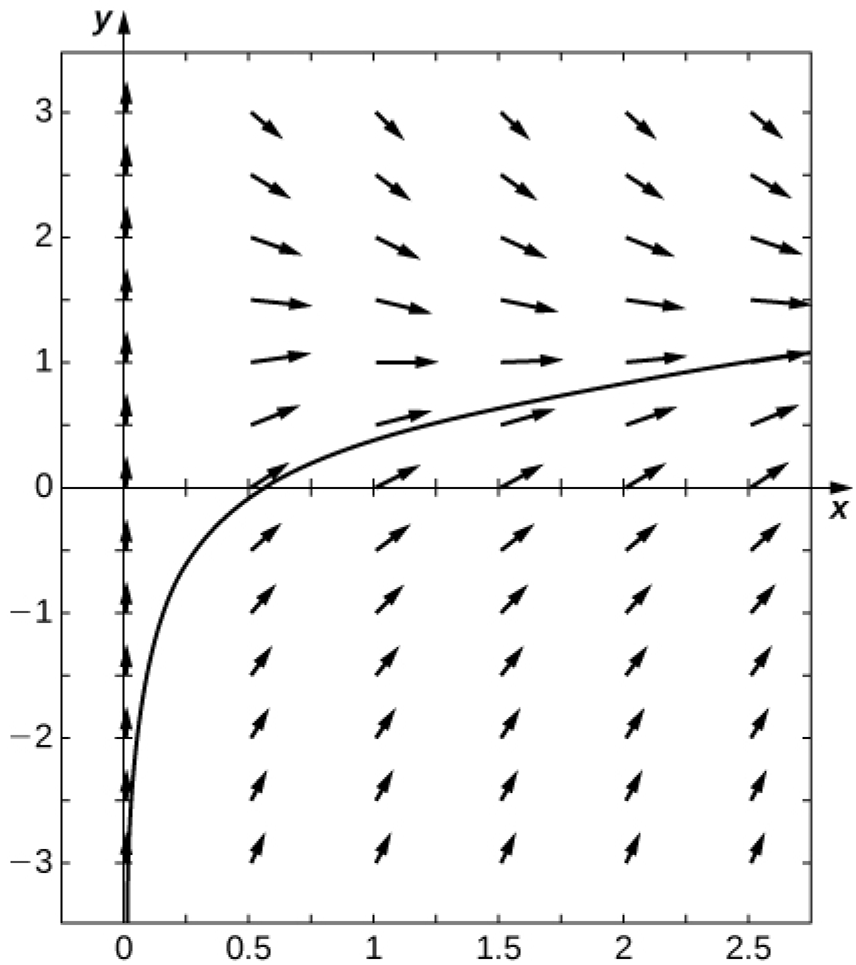
Exercise:

Problem: $y\prime = 2y - y^2$

Exercise:

Problem: $y\prime = \frac{1}{x} + \ln x - y$, for $x > 0$

Solution:



$$y(x) = Ce^{-x} + \ln x$$

For the following problems, use Euler's Method with $n = 5$ steps over the interval $t = [0, 1]$. Then solve the initial-value problem exactly. How close is your Euler's Method estimate?

Exercise:

Problem: $y\prime = -4yx, y(0) = 1$

Exercise:

Problem: $y\prime = 3^x - 2y, y(0) = 0$

Solution:

Euler: 0.6939, exact solution: $y(x) = \frac{3^x - e^{-2x}}{2+\ln(3)}$

For the following problems, set up and solve the differential equations.

Exercise:

Problem:

A car drives along a freeway, accelerating according to $a = 5\sin(\pi t)$, where t represents time in minutes. Find the velocity at any time t, assuming the car starts with an initial speed of 60 mph.

Exercise:

Problem:

You throw a ball of mass 2 kilograms into the air with an upward velocity of 8 m/s. Find exactly the time the ball will remain in the air, assuming that gravity is given by $g = 9.8\,\mathrm{m/s}^2$.

Solution:

$\frac{40}{49}$ second

Exercise:

Problem:

You drop a ball with a mass of 5 kilograms out an airplane window at a height of 5000 m. How long does it take for the ball to reach the ground?

Exercise:

Problem:

You drop the same ball of mass 5 kilograms out of the same airplane window at the same height, except this time you assume a drag force proportional to the ball's velocity, using a proportionality constant of 3 and the ball reaches terminal velocity. Solve for the distance fallen as a function of time. How long does it take the ball to reach the ground?

Solution:

$x(t) = 5000 + \frac{245}{9} - \frac{49}{3}t - \frac{245}{9}e^{-5/3t}, t = 307.8$ seconds

Exercise:

Problem:

A drug is administered to a patient every 24 hours and is cleared at a rate proportional to the amount of drug left in the body, with proportionality constant 0.2. If the patient needs a baseline level of 5 mg to be in the bloodstream at all times, how large should the dose be?

Exercise:

Problem:

A 1000-liter tank contains pure water and a solution of 0.2 kg salt/L is pumped into the tank at a rate of 1 L/min and is drained at the same rate. Solve for total amount of salt in the tank at time t.

Solution:

$T(t) = 200\left(1 - e^{-t/1000}\right)$

Exercise:

Problem:

You boil water to make tea. When you pour the water into your teapot, the temperature is $100^\circ\mathrm{C}$. After 5 minutes in your $15^\circ\mathrm{C}$ room, the temperature of the tea is $85^\circ\mathrm{C}$. Solve the equation to determine the temperatures of the tea at time t. How long must you wait until the tea is at a drinkable temperature $(72^\circ\mathrm{C})$?

Exercise:

Problem:

The human population (in thousands) of Nevada in 1950 was roughly 160. If the carrying capacity is estimated at 10 million individuals, and assuming a growth rate of 2% per year, develop a logistic growth model and solve for the population in Nevada at any time (use 1950 as time = 0). What population does your model predict for 2000? How close is your prediction to the true value of $1{,}998{,}257$?

Solution:

$P(t) = \frac{1600000e^{0.02t}}{9840+160e^{0.02t}}$

Exercise:

Problem:

Repeat the previous problem but use Gompertz growth model. Which is more accurate?

Glossary

integrating factor
: any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

linear
: description of a first-order differential equation that can be written in the form $a(x)y' + b(x)y = c(x)$

standard form
: the form of a first-order linear differential equation obtained by writing the differential equation in the form $y\prime+p(x)y = q(x)$

Introduction
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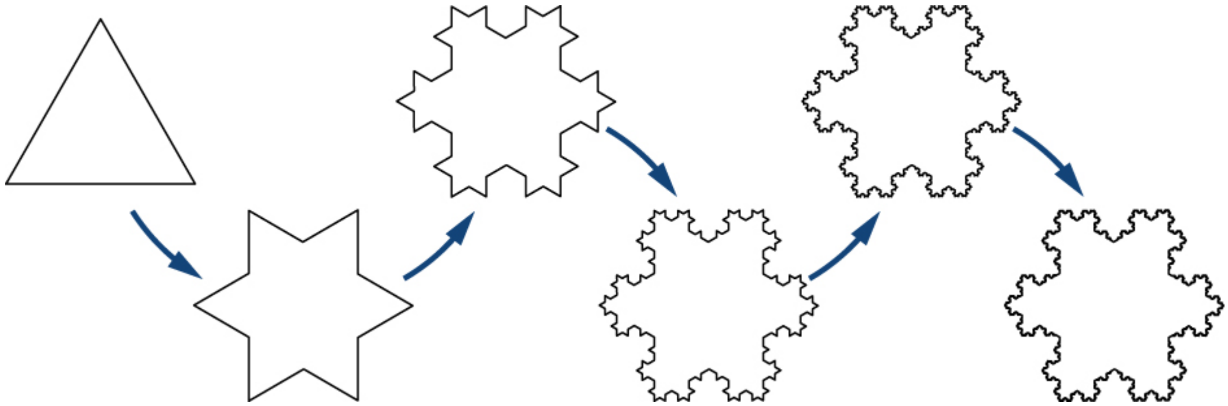
The Koch snowflake is constructed by using an iterative process. Starting with an equilateral triangle, at each step of the process the middle third of each line segment is removed and replaced with an equilateral triangle pointing outward.



The Koch snowflake is constructed from an infinite number of nonoverlapping equilateral triangles. Consequently, we can express its area as a sum of infinitely many terms. How do we add an infinite number of terms? Can a sum of an infinite number of terms be finite? To answer these questions, we need to introduce the concept of an infinite series, a sum with infinitely many terms. Having defined the necessary tools, we will be able to calculate the area of the Koch snowflake (see [link]).

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

Sequences

- Find the formula for the general term of a sequence.
- Calculate the limit of a sequence if it exists.
- Determine the convergence or divergence of a given sequence.

In this section, we introduce sequences and define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge, often by using the properties of limits for functions discussed earlier. We close this section with the Monotone Convergence Theorem, a tool we can use to prove that certain types of sequences converge.

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form
Equation:

$$a_1, a_2, a_3, \ldots, a_n, \ldots .$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation
Equation:

$$\{a_n\}_{n=1}^{\infty}, \text{ or simply } \{a_n\},$$

to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list
Equation:

$$2, 4, 8, 16, 32, \ldots .$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2$, $a_2 = 4$, and $a_3 = 8$. You can probably see that the terms in this sequence have the following pattern:
Equation:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4, \text{ and } a_5 = 2^5.$$

Assuming this pattern continues, we can write the $n\text{th}$ term in the sequence by the explicit formula $a_n = 2^n$. Using this notation, we can write this sequence as
Equation:

$$\{2^n\}_{n=1}^{\infty} \text{ or } \{2^n\}.$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the $n\text{th}$ term a_n in terms of the previous term a_{n-1}. In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and for all $n \geq 2$, each term a_n is defined by the **recurrence relation**$a_n = 2a_{n-1}$.

Note:
Definition
An **infinite sequence**$\{a_n\}$ is an ordered list of numbers of the form
Equation:

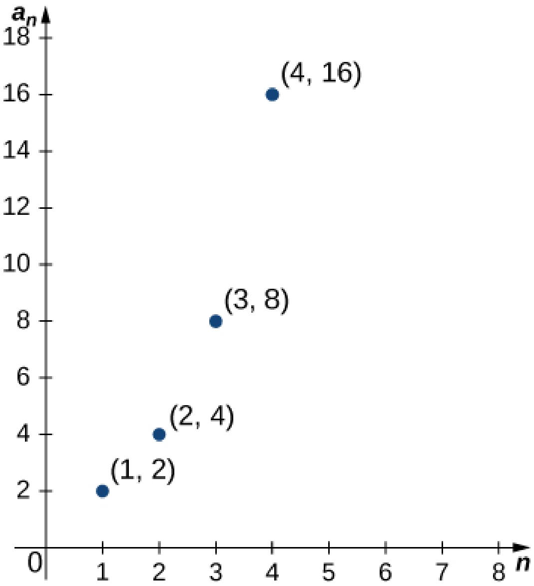
$$a_1, a_2, \ldots, a_n, \ldots .$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a **term** of the sequence. Sometimes sequences are defined by **explicit formulas**, in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$, in which case the sequence would be
Equation:

$$a_0, a_1, a_2, \ldots .$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of a_{n-1}. Since a sequence $\{a_n\}$ has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n. [link] shows the graph of $\{2^n\}$.



The plotted points are a graph of the sequence $\{2^n\}$.

Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences. In an **arithmetic sequence**, the *difference* between every pair of consecutive terms is the same. For example, consider the sequence

Equation:

$$3, 7, 11, 15, 19, \ldots .$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

Equation:

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4 \text{ for } n \geq 2. \end{cases}$$

Note that

Equation:

$$\begin{aligned} a_2 &= 3 + 4 \\ a_3 &= 3 + 4 + 4 = 3 + 2 \cdot 4 \\ a_4 &= 3 + 4 + 4 + 4 = 3 + 3 \cdot 4. \end{aligned}$$

Thus the sequence can also be described using the explicit formula

Equation:

$$\begin{aligned} a_n &= 3 + 4(n-1) \\ &= 4n - 1. \end{aligned}$$

In general, an arithmetic sequence is any sequence of the form $a_n = cn + b$.

In a **geometric sequence**, the *ratio* of every pair of consecutive terms is the same. For example, consider the sequence
Equation:

$$2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81}, \ldots.$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is a geometric sequence. It can be defined recursively as
Equation:

$$\begin{aligned} &a_1 = 2 \\ &a_n = -\tfrac{1}{3} \cdot a_{n-1} \text{ for } n \geq 2. \end{aligned}$$

Alternatively, since
Equation:

$$\begin{aligned} &a_2 = -\tfrac{1}{3} \cdot 2 \\ &a_3 = \left(-\tfrac{1}{3}\right)\left(-\tfrac{1}{3}\right)(2) = \left(-\tfrac{1}{3}\right)^2 \cdot 2 \\ &a_4 = \left(-\tfrac{1}{3}\right)\left(-\tfrac{1}{3}\right)\left(-\tfrac{1}{3}\right)(2) = \left(-\tfrac{1}{3}\right)^3 \cdot 2, \end{aligned}$$

we see that the sequence can be described by using the explicit formula
Equation:

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1}.$$

The sequence $\{2^n\}$ that we discussed earlier is a geometric sequence, where the ratio of any term to the previous term is 2. In general, a geometric sequence is any sequence of the form $a_n = cr^n$.

Example:
Exercise:

Problem:
Finding Explicit Formulas

For each of the following sequences, find an explicit formula for the $n\text{th}$ term of the sequence.

a. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6},\ldots$
b. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16},\ldots$

Solution:

a. First, note that the sequence is alternating from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the $n\text{th}$ term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $\{1, 2, 3,\ldots\}$ and the sequence of denominators $\{2, 3, 4,\ldots\}$. We can see that both of these sequences are arithmetic sequences. The $n\text{th}$ term in the sequence of numerators is n, and the $n\text{th}$ term in the sequence of denominators is $n + 1$. Therefore, the sequence can be described by the explicit formula
Equation:

$$a_n = \frac{(-1)^n n}{n + 1}.$$

b. The sequence of numerators $3, 9, 27, 81, 243,\ldots$ is a geometric sequence. The numerator of the $n\text{th}$ term is 3^n The sequence of denominators $4, 7, 10, 13, 16,\ldots$ is an arithmetic sequence. The denominator of the $n\text{th}$ term is $4 + 3(n - 1) = 3n + 1$. Therefore, we can describe the sequence by the explicit formula $a_n = \frac{3^n}{3n+1}$.

Note:
Exercise:

Problem: Find an explicit formula for the $n\text{th}$ term of the sequence $\left\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11},\ldots\right\}$.

Solution:

$a_n = \frac{(-1)^{n+1}}{3+2n}$

Hint

The denominators form an arithmetic sequence.

Example:
Exercise:

Problem:
Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

a. $a_1 = 2$, $a_n = -3a_{n-1}$ for $n \geq 2$
b. $a_1 = \frac{1}{2}$, $a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geq 2$

Solution:

a. Writing out the first few terms, we have
Equation:

$$\begin{aligned} a_1 &= 2 \\ a_2 &= -3a_1 = -3\left(2\right) \\ a_3 &= -3a_2 = \left(-3\right)^2 2 \\ a_4 &= -3a_3 = \left(-3\right)^3 2. \end{aligned}$$

In general,
Equation:

$$a_n = 2(-3)^{n-1}.$$

b. Write out the first few terms:
Equation:

$$\begin{aligned} a_1 &= \tfrac{1}{2} \\ a_2 &= a_1 + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4} \\ a_3 &= a_2 + \left(\tfrac{1}{2}\right)^3 = \tfrac{3}{4} + \tfrac{1}{8} = \tfrac{7}{8} \\ a_4 &= a_3 + \left(\tfrac{1}{2}\right)^4 = \tfrac{7}{8} + \tfrac{1}{16} = \tfrac{15}{16}. \end{aligned}$$

From this pattern, we derive the explicit formula
Equation:

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Note:
Exercise:

Problem:

Find an explicit formula for the sequence defined recursively such that $a_1 = -4$ and $a_n = a_{n-1} + 6.$

Solution:

$a_n = 6n - 10$

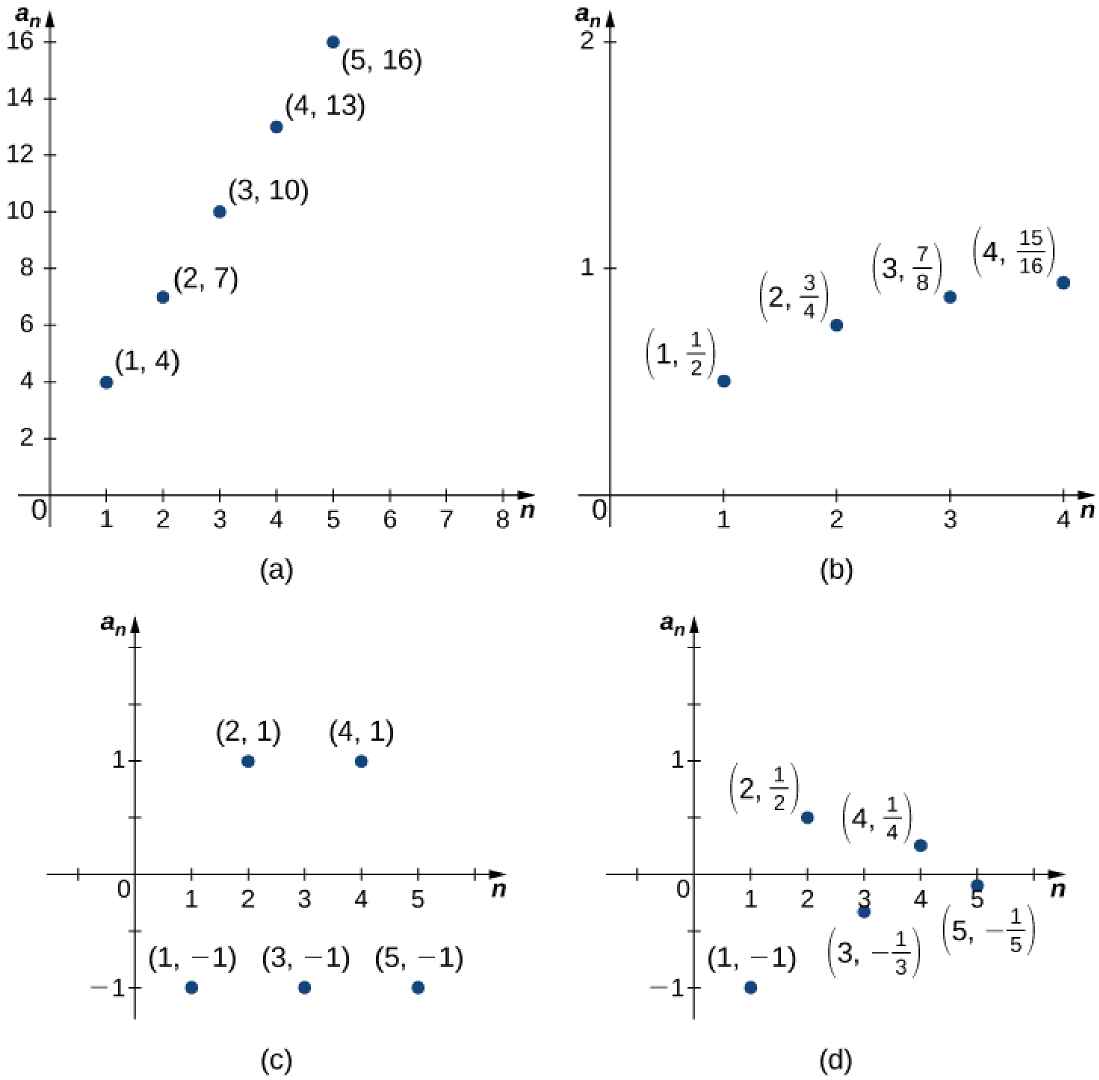
Hint

This is an arithmetic sequence.

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (see [link]):

a. $\{1 + 3n\} = \{4, 7, 10, 13, \ldots\}$. The terms $1 + 3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$ as $n \to \infty$.
b. $\left\{1 - \left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\}$. The terms $1 - \left(\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.
c. $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$. The terms alternate but do not approach one single value as $n \to \infty$.
d. $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right\}$. The terms alternate for this sequence as well, but $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.



(a) The terms in the sequence become arbitrarily large as $n \to \infty$. (b) The terms in the sequence approach 1 as $n \to \infty$. (c) The terms in the sequence alternate between 1 and -1 as $n \to \infty$. (d) The terms in the sequence alternate between positive and negative values but approach 0 as $n \to \infty$.

From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. In two of the sequences, the terms approach a finite number as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a convergent sequence and the real number L is the limit of the sequence. We can give an informal definition here.

Note:
Definition

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. In this case, we write

Equation:

$$\lim_{n\to\infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence**.

From [link], we see that the terms in the sequence $\left\{1-\left(\frac{1}{2}\right)^n\right\}$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $\left\{1-\left(\frac{1}{2}\right)^n\right\}$ is a convergent sequence and its limit is 1. In contrast, from [link], we see that the terms in the sequence $1+3n$ are not approaching a finite number as n becomes larger. We say that $\{1+3n\}$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms "arbitrarily close" and "sufficiently large." Although these phrases help illustrate the meaning of a converging sequence, they are somewhat vague. To be more precise, we now present the more formal definition of limit for a sequence and show these ideas graphically in [link].

Note:

Definition

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ if $n \geq N$. The number L is the limit of the sequence and we write

Equation:

$$\lim_{n\to\infty} a_n = L \mathit{\ or\ } a_n \to L.$$

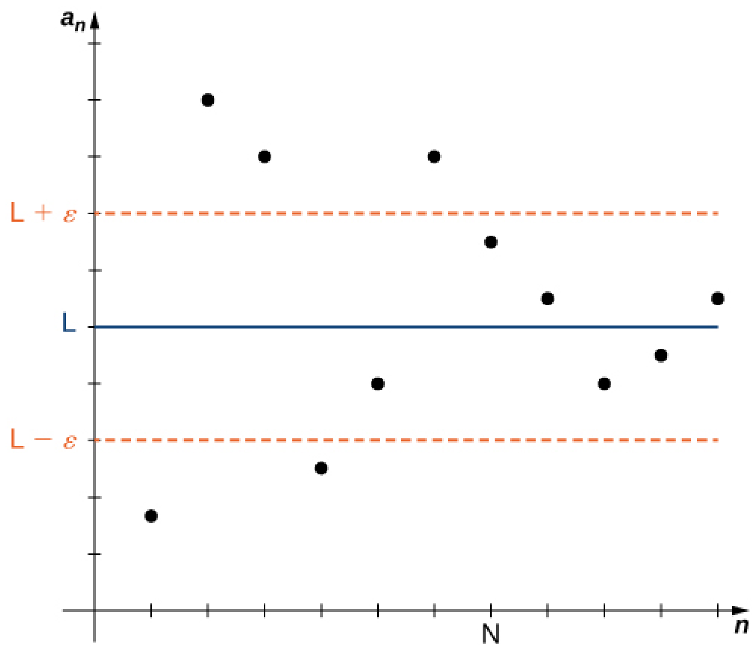
In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

Equation:

$$b_1, b_2, \ldots, b_N, a_1, a_2, \ldots,$$

this new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges. Further, if the sequence $\{a_n\}$ converges to L, this new sequence will also converge to L.



As n increases, the terms a_n become closer to L. For values of $n \geq N$, the distance between each point (n, a_n) and the line $y = L$ is less than ε.

As defined above, if a sequence does not converge, it is said to be a divergent sequence. For example, the sequences $\{1 + 3n\}$ and $\{(-1)^n\}$ shown in [link] diverge. However, different sequences can diverge in different ways. The sequence $\{(-1)^n\}$ diverges because the terms alternate between 1 and -1, but do not approach one value as $n \to \infty$. On the other hand, the sequence $\{1 + 3n\}$ diverges because the terms $1 + 3n \to \infty$ as $n \to \infty$. We say the sequence $\{1 + 3n\}$ diverges to infinity and write $\lim\limits_{n \to \infty} (1 + 3n) = \infty$. It is important to recognize that this notation does not imply the limit of the sequence $\{1 + 3n\}$ exists. The sequence is, in fact, divergent. Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $\{-5n + 2\}$ diverges to negative infinity because $-5n + 2 \to -\infty$ as $n \to -\infty$. We write this as $\lim\limits_{n \to \infty} (-5n + 2) = \to -\infty$.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence $\{a_n\}$ and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim\limits_{x \to \infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\left\{\frac{1}{n}\right\}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\left\{\frac{1}{n}\right\}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$.

Note:

Limit of a Sequence Defined by a Function
Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that
Equation:

$$\lim_{x\to\infty} f(x) = L,$$

then $\{a_n\}$ converges and
Equation:

$$\lim_{n\to\infty} a_n = L.$$

We can use this theorem to evaluate $\lim\limits_{n\to\infty} r^n$ for $0 \leq r \leq 1$. For example, consider the sequence $\{(1/2)^n\}$ and the related exponential function $f(x) = (1/2)^x$. Since $\lim\limits_{x\to\infty} (1/2)^x = 0$, we conclude that the sequence $\{(1/2)^n\}$ converges and its limit is 0. Similarly, for any real number r such that $0 \leq r < 1$, $\lim\limits_{x\to\infty} r^x = 0$, and therefore the sequence $\{r^n\}$ converges. On the other hand, if $r = 1$, then $\lim\limits_{x\to\infty} r^x = 1$, and therefore the limit of the sequence $\{1^n\}$ is 1. If $r > 1$, $\lim\limits_{x\to\infty} r^x = \infty$, and therefore we cannot apply this theorem. However, in this case, just as the function r^x grows without bound as $n \to \infty$, the terms r^n in the sequence become arbitrarily large as $n \to \infty$, and we conclude that the sequence $\{r^n\}$ diverges to infinity if $r > 1$.

We summarize these results regarding the geometric sequence $\{r^n\}$:
Equation:

$$\begin{aligned} r^n &\to 0 \, if \, 0 < r < 1 \\ r^n &\to 1 \, if \, r = 1 \\ r^n &\to \infty \, if \, r > 1. \end{aligned}$$

Later in this section we consider the case when $r < 0$.

We now consider slightly more complicated sequences. For example, consider the sequence $\{(2/3)^n + (1/4)^n\}$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $\{(2/3)^n\}$ and $\{(1/4)^n\}$. As we describe in the following algebraic limit laws, since $\{(2/3)^n\}$ and $\{1/4)^n\}$ both converge to 0, the sequence $\{(2/3)^n + (1/4)^n\}$ converges to $0 + 0 = 0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g (see Introduction to Limits), we are able to evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of $\{a_n\}$ and $\{b_n\}$.

Note:
Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

i. $\lim_{n\to\infty} c = c$
ii. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cA$
iii. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
iv. $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$
v. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Proof

We prove part iii.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, there exists a constant positive integer N_1 such that for all $n \geq N_1$. Since $\lim_{n\to\infty} b_n = B$, there exists a constant N_2 such that $|b_n - B| < \varepsilon/2$ for all $n \geq N_2$. Let N be the largest of N_1 and N_2. Therefore, for all $n \geq N$,

$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

□

The algebraic limit laws allow us to evaluate limits for many sequences. For example, consider the sequence $\left\{\frac{1}{n^2}\right\}$. As shown earlier, $\lim_{n\to\infty} 1/n = 0$. Similarly, for any positive integer k, we can conclude that
Equation:

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$

In the next example, we make use of this fact along with the limit laws to evaluate limits for other sequences.

Example:
Exercise:

Problem:
Determining Convergence and Finding Limits

For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

a. $\left\{5 - \frac{3}{n^2}\right\}$
b. $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$
c. $\left\{\frac{2^n}{n^2}\right\}$
d. $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$

Solution:

a. We know that $1/n \to 0$. Using this fact, we conclude that
Equation:

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{n}\right). \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0.$$

Therefore,
Equation:

$$\lim_{n\to\infty} \left(5 - \frac{3}{n^2}\right) = \lim_{n\to\infty} 5 - 3 \lim_{n\to\infty} \frac{1}{n^2} = 5 - 3.0 = 5.$$

The sequence converges and its limit is 5.

b. By factoring n^4 out of the numerator and denominator and using the limit laws above, we have
Equation:

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^4-7n^2+5}{6-4n^4} &= \lim_{n\to\infty} \frac{3-\frac{7}{n^2}+\frac{5}{n^4}}{\frac{6}{n^4}-4} \\
&= \frac{\lim_{n\to\infty}\left(3-\frac{7}{n^2}+\frac{5}{n^4}\right)}{\lim_{n\to\infty}\left(\frac{6}{n^4}-4\right)} \\
&= \frac{\left(\lim_{n\to\infty}(3)-\lim_{n\to\infty}\frac{7}{n^2}+\lim_{n\to\infty}\frac{5}{n^4}\right)}{\left(\lim_{n\to\infty}\frac{6}{n^4}-\lim_{n\to\infty}(4)\right)} \\
&= \frac{\left(\lim_{n\to\infty}(3)-7\cdot\lim_{n\to\infty}\frac{1}{n^2}+5\cdot\lim_{n\to\infty}\frac{1}{n^4}\right)}{\left(6\cdot\lim_{n\to\infty}\frac{1}{n^4}-\lim_{n\to\infty}(4)\right)} \\
&= \frac{3-7\cdot0+5\cdot0}{6\cdot0-4} = -\frac{3}{4}.
\end{aligned}$$

The sequence converges and its limit is $-3/4$.

c. Consider the related function $f(x) = 2^x/x^2$ defined on all real numbers $x > 0$. Since $2^x \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, apply L'Hôpital's rule and write
Equation:

$$\begin{aligned}
\lim_{x\to\infty} \frac{2^x}{x^2} &= \lim_{x\to\infty} \frac{2^x \ln 2}{2x} && \text{Take the derivatives of the numerator and denominator.} \\
&= \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} && \text{Take the derivatives again.} \\
&= \infty.
\end{aligned}$$

We conclude that the sequence diverges.

d. Consider the function $f(x) = \left(1 + \frac{4}{x}\right)^x$ defined on all real numbers $x > 0$. This function has the indeterminate form 1^∞ as $x \to \infty$. Let

Equation:

$$y = \lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x.$$

Now taking the natural logarithm of both sides of the equation, we obtain

Equation:

$$\ln(y) = \ln\left[\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x\right].$$

Since the function $f(x) = \ln x$ is continuous on its domain, we can interchange the limit and the natural logarithm. Therefore,

Equation:

$$\ln(y) = \lim_{x\to\infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right].$$

Using properties of logarithms, we write

Equation:

$$\lim_{x\to\infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right] = \lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right).$$

Since the right-hand side of this equation has the indeterminate form $\infty \cdot 0$, rewrite it as a fraction to apply L’Hôpital’s rule. Write

Equation:

$$\lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right) = \lim_{x\to\infty} \frac{\ln(1 + 4/x)}{1/x}.$$

Since the right-hand side is now in the indeterminate form $0/0$, we are able to apply L’Hôpital’s rule. We conclude that

Equation:

$$\lim_{x\to\infty}\frac{\ln(1+4/x)}{1/x} = \lim_{x\to\infty}\frac{4}{1+4/x} = 4.$$

Therefore, $\ln(y) = 4$ and $y = e^4$. Therefore, since $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x = e^4$, we can conclude that the sequence $\left\{\left(1+\frac{4}{n}\right)^n\right\}$ converges to e^4.

Note:
Exercise:

Problem:

Consider the sequence $\left\{\left(5n^2+1\right)/e^n\right\}$. Determine whether or not the sequence converges. If it converges, find its limit.

Solution:

The sequence converges, and its limit is 0.

Hint

Use L'Hôpital's rule.

Recall that if f is a continuous function at a value L, then $f(x) \to f(L)$ as $x \to L$. This idea applies to sequences as well. Suppose a sequence $a_n \to L$, and a function f is continuous at L. Then $f(a_n) \to f(L)$. This property often enables us to find limits for complicated sequences. For example, consider the sequence $\sqrt{5-\frac{3}{n^2}}$. From [link]a. we know the sequence $5-\frac{3}{n^2} \to 5$. Since $\sqrt{x}$ is a continuous function at $x = 5$,

Equation:

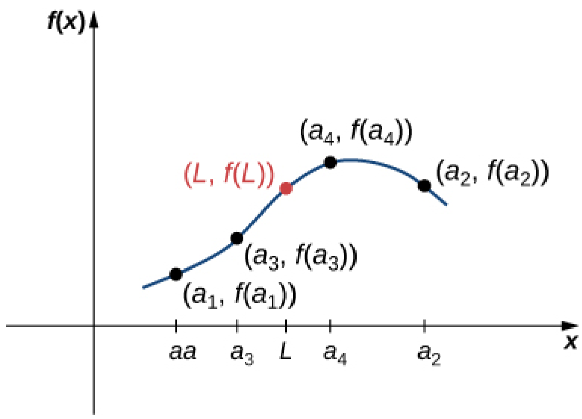
$$\lim_{n\to\infty}\sqrt{5-\frac{3}{n^2}} = \sqrt{\lim_{n\to\infty}\left(5-\frac{3}{n^2}\right)} = \sqrt{5}.$$

Note:
Continuous Functions Defined on Convergent Sequences
Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values a_n for $n \geq N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$ ([link]).

Proof

Let $\epsilon > 0$. Since f is continuous at L, there exists $\delta > 0$ such that $|f(x) - f(L)| < \varepsilon$ if $|x - L| < \delta$. Since the sequence $\{a_n\}$ converges to L, there exists N such that $|a_n - L| < \delta$ for all $n \geq N$. Therefore, for all $n \geq N$, $|a_n - L| < \delta$, which implies $|f(a_n) - f(L)| < \varepsilon$. We conclude that the sequence $\{f(a_n)\}$ converges to $f(L)$.

□



Because f is a continuous function as the inputs $a_1, a_2, a_3, \ldots$ approach L, the outputs $f(a_1), f(a_2), f(a_3), \ldots$ approach $f(L)$.

Example:
Exercise:

Problem:
Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $\left\{\cos\left(3/n^2\right)\right\}$ converges. If it converges, find its limit.

Solution:

Since the sequence $\left\{3/n^2\right\}$ converges to 0 and $\cos x$ is continuous at $x = 0$, we can conclude that the sequence $\left\{\cos\left(3/n^2\right)\right\}$ converges and

Equation:

$$\lim_{n\to\infty} \cos\left(\frac{3}{n^2}\right) = \cos(0) = 1.$$

Note:
Exercise:

Problem: Determine if the sequence $\left\{\sqrt{\frac{2n+1}{3n+5}}\right\}$ converges. If it converges, find its limit.

Solution:

The sequence converges, and its limit is $\sqrt{2/3}$.

Hint

Consider the sequence $\left\{\frac{2n+1}{3n+5}\right\}$.

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits discussed in Introduction to Limits.

Note:
Squeeze Theorem for Sequences
Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that
Equation:

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N.$$

If there exists a real number L such that
Equation:

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$ ([link]).

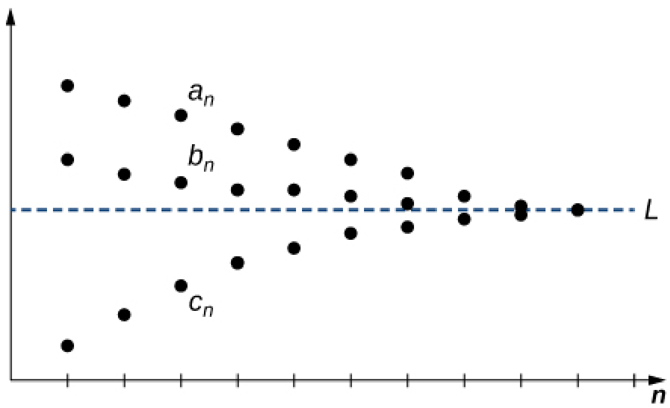
Proof

Let $\varepsilon > 0$. Since the sequence $\{a_n\}$ converges to L, there exists an integer N_1 such that $|a_n - L| < \varepsilon$ for all $n \geq N_1$. Similarly, since $\{c_n\}$ converges to L, there exists an integer N_2 such that $|c_n - L| < \varepsilon$ for all $n \geq N_2$. By assumption, there exists an integer N such that $a_n \leq b_n \leq c_n$ for all $n \geq N$. Let M be the largest of N_1, N_2, and N. We must show that $|b_n - L| < \varepsilon$ for all $n \geq M$. For all $n \geq M$,
Equation:

$$-\varepsilon < -|a_n - L| \leq a_n - L \leq b_n - L \leq c_n - L \leq |c_n - L| < \varepsilon.$$

Therefore, $-\varepsilon < b_n - L < \varepsilon$, and we conclude that $|b_n - L| < \varepsilon$ for all $n \geq M$, and we conclude that the sequence $\{b_n\}$ converges to L.

□



Each term b_n satisfies $a_n \leq b_n \leq c_n$ and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit, so the sequence $\{b_n\}$ must converge to the same limit as well.

Example:
Exercise:

Problem:
Using the Squeeze Theorem

Use the Squeeze Theorem to find the limit of each of the following sequences.

a. $\left\{\frac{\cos n}{n^2}\right\}$
b. $\left\{\left(-\frac{1}{2}\right)^n\right\}$

Solution:

a. Since $-1 \leq \cos n \leq 1$ for all integers n, we have
Equation:

$$-\frac{1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}.$$

Since $-1/n^2 \to 0$ and $1/n^2 \to 0$, we conclude that $\cos n/n^2 \to 0$ as well.

b. Since
Equation:

$$-\frac{1}{2^n} \leq \left(-\frac{1}{2}\right)^n \leq \frac{1}{2^n}$$

for all positive integers n, $-1/2^n \to 0$ and $1/2^n \to 0$, we can conclude that $(-1/2)^n \to 0$.

Note:
Exercise:

Problem: Find $\lim_{n\to\infty} \frac{2n - \sin n}{n}$.

Solution:

2

Hint

Use the fact that $-1 \leq \sin n \leq 1$.

Using the idea from [link]b. we conclude that $r^n \to 0$ for any real number r such that $-1 < r < 0$. If $r < -1$, the sequence $\{r^n\}$ diverges because the terms oscillate and become arbitrarily large in magnitude. If $r = -1$, the sequence $\{r^n\} = \{(-1)^n\}$ diverges, as discussed earlier. Here is a summary of the properties for geometric sequences.
Equation:

$$r^n \to 0 \text{ if } |r| < 1$$

Equation:

$$r^n \to 1 \text{ if } r = 1$$

Equation:

$$r^n \to \infty \text{ if } r > 1$$

Equation:

$$\{r^n\} \text{ diverges if } r \leq -1$$

Bounded Sequences

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Note:
Definition
A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that
Equation:

$$a_n \leq M$$

for all positive integers n.
A sequence $\{a_n\}$ is **bounded below** if there exists a real number M such that
Equation:

$$M \leq a_n$$

for all positive integers n.
A sequence $\{a_n\}$ is a **bounded sequence** if it is bounded above and bounded below.
If a sequence is not bounded, it is an **unbounded sequence**.

For example, the sequence $\{1/n\}$ is bounded above because $1/n \leq 1$ for all positive integers n. It is also bounded below because $1/n \geq 0$ for all positive integers n. Therefore, $\{1/n\}$ is a bounded sequence. On the other hand, consider the sequence $\{2^n\}$. Because $2^n \geq 2$ for all $n \geq 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2^n\}$ is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence $\{a_n\}$ is unbounded. Then it is not bounded above, or not bounded below, or both. In either case, there are terms a_n that are arbitrarily large in magnitude as n gets larger. As a result, the sequence $\{a_n\}$ cannot converge. Therefore, being bounded is a necessary condition for a sequence to converge.

Note:
Convergent Sequences Are Bounded
If a sequence $\{a_n\}$ converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $\{(-1)^n\}$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leq a_2 \leq a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are

two possibilities. The sequence could diverge to infinity, or it could converge. However, since the sequence is bounded, it is bounded above and the sequence cannot diverge to infinity. We conclude that $\{a_n\}$ converges. For example, consider the sequence
Equation:

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}.$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence
Equation:

$$\left\{2, 0, 3, 0, 4, 0, 1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}.$$

Even though the sequence is not increasing for all values of n, we see that $-1/2 < -1/3 < -1/4 < \cdots$. Therefore, starting with the eighth term, $a_8 = -1/2$, the sequence is increasing. In this case, we say the sequence is *eventually* increasing. Since the sequence is bounded above, it converges. It is also true that if a sequence is decreasing (or eventually decreasing) and bounded below, it also converges.

Note:
Definition
A sequence $\{a_n\}$ is increasing for all $n \geq n_0$ if
Equation:

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is decreasing for all $n \geq n_0$ if
Equation:

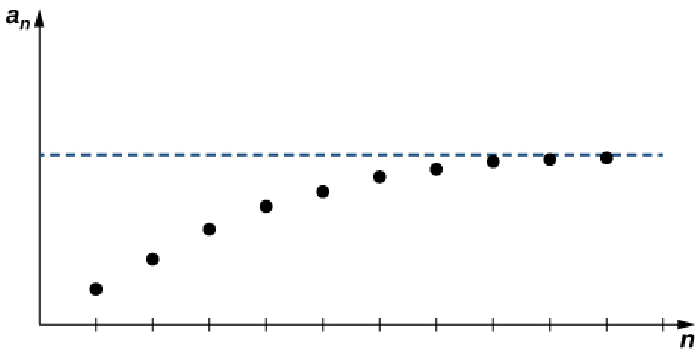
$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for convergence of a sequence.

Note:
Monotone Convergence Theorem
If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

The proof of this theorem is beyond the scope of this text. Instead, we provide a graph to show intuitively why this theorem makes sense ([link]).



Since the sequence $\{a_n\}$ is increasing and bounded above, it must converge.

In the following example, we show how the Monotone Convergence Theorem can be used to prove convergence of a sequence.

Example:
Exercise:

Problem:
Using the Monotone Convergence Theorem

For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit.

a. $\left\{\frac{4^n}{n!}\right\}$
b. $\{a_n\}$ defined recursively such that
Equation:

$$a_1 = 2 \text{ and } a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \text{ for all } n \geq 2.$$

Solution:

a. Writing out the first few terms, we see that
Equation:

$$\left\{\frac{4^n}{n!}\right\} = \left\{4, 8, \frac{32}{3}, \frac{32}{3}, \frac{128}{15}, \ldots\right\}.$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows.
Equation:

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \cdot \frac{4^n}{n!} = \frac{4}{n+1} \cdot a_n \leq a_n \, if \, n \geq 3.$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by 0 because $4^n/n! \geq 0$ for all positive integers n. Therefore, by the Monotone Convergence Theorem, the sequence converges.
To find the limit, we use the fact that the sequence converges and let $L = \lim\limits_{n\to\infty} a_n$. Now note this important observation. Consider $\lim\limits_{n\to\infty} a_{n+1}$. Since
Equation:

$$\{a_{n+1}\} = \{a_{2,} a_3, a_4, \ldots\},$$

the only difference between the sequences $\{a_{n+1}\}$ and $\{a_n\}$ is that $\{a_{n+1}\}$ omits the first term. Since a finite number of terms does not affect the convergence of a sequence,
Equation:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L.$$

Combining this fact with the equation
Equation:

$$a_{n+1} = \frac{4}{n+1} a_n$$

and taking the limit of both sides of the equation
Equation:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4}{n+1} a_n,$$

we can conclude that
Equation:

$$L = 0 \cdot L = 0.$$

b. Writing out the first several terms,
Equation:

$$\left\{2, \frac{5}{4}, \frac{41}{40}, \frac{3281}{3280}, \ldots\right\}.$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that
Equation:

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show this, first rewrite
Equation:

$$\frac{a_n}{2} + \frac{1}{2a_n} = \frac{a_n^2 + 1}{2a_n}.$$

Since $a_1 > 0$ and a_2 is defined as a sum of positive terms, $a_2 > 0$. Similarly, all terms $a_n > 0$. Therefore,
Equation:

$$\frac{a_n^2 + 1}{2a_n} \geq 1$$

if and only if
Equation:

$$a_n^2 + 1 \geq 2a_n.$$

Rewriting the inequality $a_n^2 + 1 \geq 2a_n$ as $a_n^2 - 2a_n + 1 \geq 0$, and using the fact that
Equation:

$$a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$$

because the square of any real number is nonnegative, we can conclude that
Equation:

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show that the sequence is decreasing, we must show that $a_{n+1} \leq a_n$ for all $n \geq 1$. Since $1 \leq a_n^2$, it follows that
Equation:

$$a_n^2 + 1 \le 2a_n^2.$$

Dividing both sides by $2a_n$, we obtain
Equation:

$$\frac{a_n}{2} + \frac{1}{2a_n} \le a_n.$$

Using the definition of a_{n+1}, we conclude that
Equation:

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \le a_n.$$

Since $\{a_n\}$ is bounded below and decreasing, by the Monotone Convergence Theorem, it converges.
To find the limit, let $L = \lim\limits_{n\to\infty} a_n$. Then using the recurrence relation and the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$, we have
Equation:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{a_n}{2} + \frac{1}{2a_n}\right),$$

and therefore
Equation:

$$L = \frac{L}{2} + \frac{1}{2L}.$$

Multiplying both sides of this equation by $2L$, we arrive at the equation
Equation:

$$2L^2 = L^2 + 1.$$

Solving this equation for L, we conclude that $L^2 = 1$, which implies $L = \pm 1$. Since all the terms are positive, the limit $L = 1$.

Note:
Exercise:

Problem:

Consider the sequence $\{a_n\}$ defined recursively such that $a_1 = 1$, $a_n = a_{n-1}/2$. Use the Monotone Convergence Theorem to show that this sequence converges and find its limit.

Solution:

0.

Hint

Show the sequence is decreasing and bounded below.

Note:
Fibonacci Numbers
The Fibonacci numbers are defined recursively by the sequence $\{F_n\}$ where $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$,
Equation:

$$F_n = F_{n-1} + F_{n-2}.$$

Here we look at properties of the Fibonacci numbers.

1. Write out the first twenty Fibonacci numbers.
2. Find a closed formula for the Fibonacci sequence by using the following steps.

    a. Consider the recursively defined sequence $\{x_n\}$ where $x_o = c$ and $x_{n+1} = ax_n$. Show that this sequence can be described by the closed formula $x_n = ca^n$ for all $n \geq 0$.
    b. Using the result from part a. as motivation, look for a solution of the equation
    Equation:

    $$F_n = F_{n-1} + F_{n-2}$$

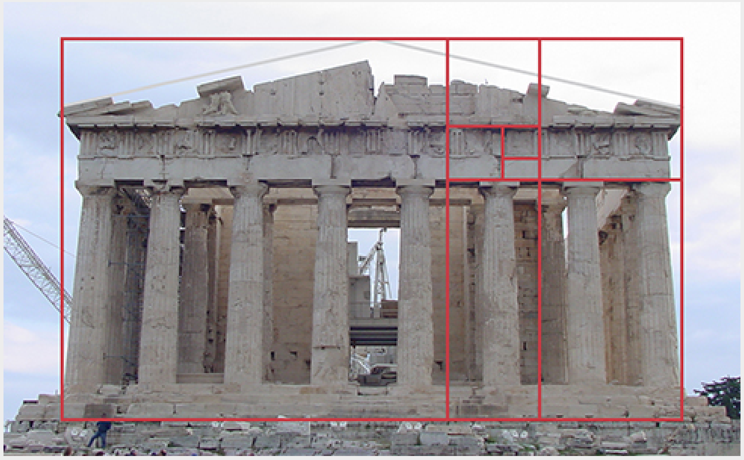
    of the form $F_n = c\lambda^n$. Determine what two values for λ will allow F_n to satisfy this equation.
    c. Consider the two solutions from part b.: λ_1 and λ_2. Let $F_n = c_1{\lambda_1}^n + c_2{\lambda_2}^n$. Use the initial conditions F_0 and F_1 to determine the values for the constants c_1 and c_2 and write the closed formula F_n.

3. Use the answer in 2 c. to show that
    Equation:

    $$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

    The number $\phi = \left(1+\sqrt{5}\right)/2$ is known as the golden ratio ([link] and [link]).

The seeds in a sunflower exhibit spiral patterns curving to the left and to the right. The number of spirals in each direction is always a Fibonacci number—always. (credit: modification of work by Esdras Calderan, Wikimedia Commons)



The proportion of the golden ratio appears in many famous examples of art and architecture. The ancient Greek temple known as the Parthenon was designed with these proportions, and the ratio appears again in many of the smaller details. (credit: modification of work by TravelingOtter, Flickr)

Key Concepts

- To determine the convergence of a sequence given by an explicit formula $a_n = f(n)$, we use the properties of limits for functions.
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences that converge to A and B, respectively, and c is any real number, then the sequence $\{ca_n\}$ converges to $c \cdot A$, the sequences $\{a_n \pm b_n\}$ converge to $A \pm B$, the sequence $\{a_n \cdot b_n\}$ converges to $A \cdot B$, and the sequence $\{a_n / b_n\}$ converges to A/B, provided $B \neq 0$.
- If a sequence is bounded and monotone, then it converges, but not all convergent sequences are monotone.
- If a sequence is unbounded, it diverges, but not all divergent sequences are unbounded.
- The geometric sequence $\{r^n\}$ converges if and only if $|r| < 1$ or $r = 1$.

Find the first six terms of each of the following sequences, starting with $n = 1$.

Exercise:

Problem: $a_n = 1 + (-1)^n$ for $n \geq 1$

Solution:

$a_n = 0$ if n is odd and $a_n = 2$ if n is even

Exercise:

Problem: $a_n = n^2 - 1$ for $n \geq 1$

Exercise:

Problem: $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

Solution:

$\{a_n\} = \{1, 3, 6, 10, 15, 21, \ldots\}$

Exercise:

Problem: $a_1 = 1$, $a_2 = 1$ and $a_{n+2} = a_n + a_{n+1}$ for $n \geq 1$

Exercise:

Problem: Find an explicit formula for a_n where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

Solution:

$a_n = \frac{n(n+1)}{2}$

Exercise:

Problem:

Find a formula a_n for the $n\text{th}$ term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n-1} - a_n = 17$ for $n \geq 1$.

Exercise:

Problem:

Find a formula a_n for the $n\text{th}$ term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n-1} - a_n = 4$ for $n \geq 1$.

Solution:

$a_n = 4n - 7$

Exercise:

Problem:

Find a formula a_n for the $n\text{th}$ term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.

Exercise:

Problem:

Find a formula a_n for the $n\text{th}$ term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

Solution:

$a_n = 3.10^{1-n} = 30.10^{-n}$

Exercise:

Problem:

Find an explicit formula for the $n\text{th}$ term of the sequence whose first several terms are $\{0, 3, 8, 15, 24, 35, 48, 63, 80, 99, \ldots\}$. (*Hint:* First add one to each term.)

Exercise:

Problem:

Find an explicit formula for the $n\text{th}$ term of the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Solution:

$a_n = 2^n - 1$

Find a formula for the general term a_n of each of the following sequences.

Exercise:

Problem: $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$ (*Hint:* Find where $\sin x$ takes these values)

Exercise:

Problem: $\{1, -1/3, 1/5, -1/7, \ldots\}$

Solution:

$a_n = \frac{(-1)^{n-1}}{2n-1}$

Find a function $f(n)$ that identifies the $n\text{th}$ term a_n of the following recursively defined sequences, as $a_n = f(n)$.

Exercise:

Problem: $a_1 = 1$ and $a_{n+1} = -a_n$ for $n \geq 1$

Exercise:

Problem: $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

Solution:

$f(n) = 2^n$

Exercise:

Problem: $a_1 = 1$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$

Exercise:

Problem: $a_1 = 2$ and $a_{n+1} = (n+1)a_n/2$ for $n \geq 1$

Solution:

$f(n) = n!/2^{n-2}$

Exercise:

Problem: $a_1 = 1$ and $a_{n+1} = a_n/2^n$ for $n \geq 1$

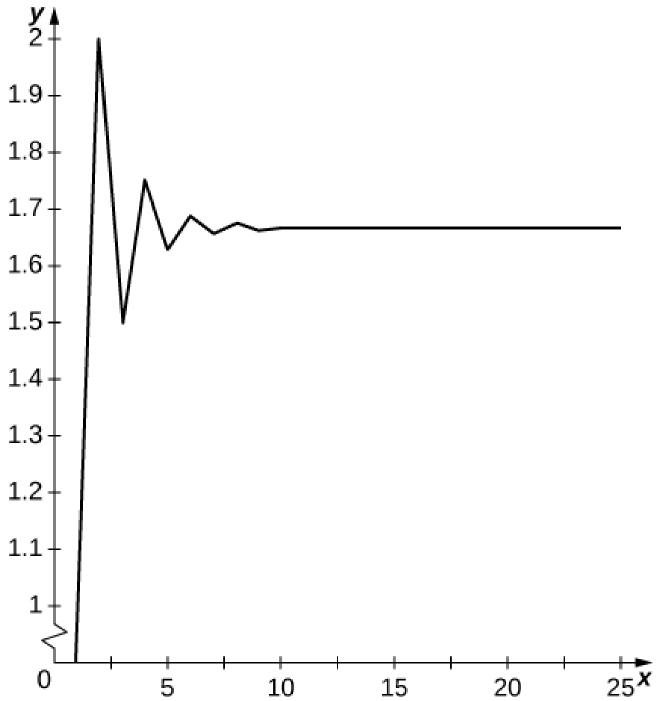
Plot the first N terms of each sequence. State whether the graphical evidence suggests that the sequence converges or diverges.

Exercise:

Problem: **[T]** $a_1 = 1$, $a_2 = 2$, and for $n \geq 2$, $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$; $N = 30$

Solution:

Terms oscillate above and below $5/3$ and appear to converge to $5/3$.



Exercise:

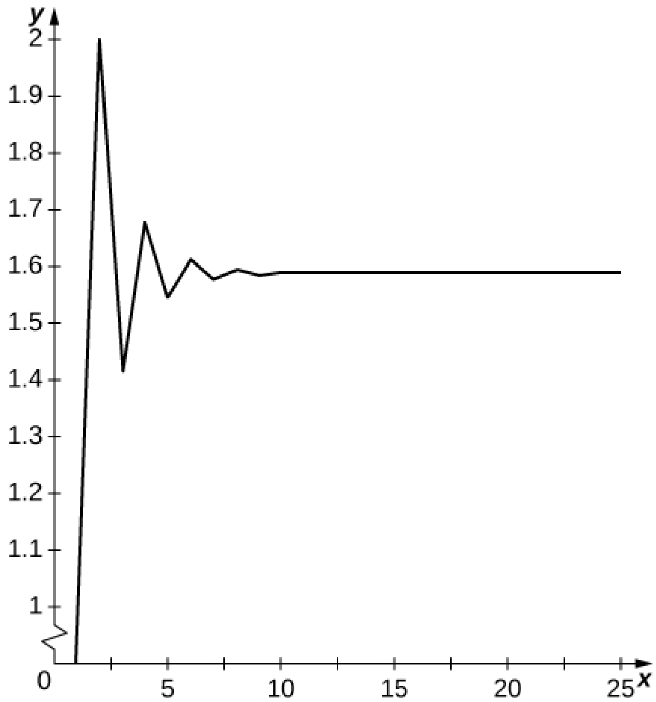
Problem: **[T]** $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and for $n \geq 4$, $a_n = \frac{1}{3}(a_{n-1} + a_{n-2} + a_{n-3})$, $N = 30$

Exercise:

Problem: **[T]** $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$, $a_n = \sqrt{a_{n-1}a_{n-2}}$; $N = 30$

Solution:

Terms oscillate above and below $y \approx 1.57\ldots$ and appear to converge to a limit.



Exercise:

Problem: **[T]** $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and for $n \geq 4$, $a_n = \sqrt{a_{n-1}a_{n-2}a_{n-3}}$; $N = 30$

Suppose that $\lim\limits_{n\to\infty} a_n = 1$, $\lim\limits_{n\to\infty} b_n = -1$, and $0 < -b_n < a_n$ for all n. Evaluate each of the following limits, or state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

Exercise:

Problem: $\lim\limits_{n\to\infty} (3a_n - 4b_n)$

Solution:

7

Exercise:

Problem: $\lim\limits_{n\to\infty} \left(\frac{1}{2}b_n - \frac{1}{2}a_n\right)$

Exercise:

Problem: $\lim\limits_{n\to\infty} \frac{a_n+b_n}{a_n-b_n}$

Solution:

0

Exercise:

Problem: $\lim\limits_{n\to\infty} \frac{a_n - b_n}{a_n + b_n}$

Find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

Exercise:

Problem: $\frac{n^2}{2^n}$

Solution:

0

Exercise:

Problem: $\frac{(n-1)^2}{(n+1)^2}$

Exercise:

Problem: $\frac{\sqrt{n}}{\sqrt{n+1}}$

Solution:

1

Exercise:

Problem: $n^{1/n}$ (*Hint:* $n^{1/n} = e^{\frac{1}{n}\ln n}$)

For each of the following sequences, whose $n\text{th}$ terms are indicated, state whether the sequence is bounded and whether it is eventually monotone, increasing, or decreasing.

Exercise:

Problem: $n/2^n$, $n \geq 2$

Solution:

bounded, decreasing for $n \geq 1$

Exercise:

Problem: $\ln\left(1 + \frac{1}{n}\right)$

Exercise:

Problem: $\sin n$

Solution:

bounded, not monotone

Exercise:

Problem: $\cos\left(n^2\right)$

Exercise:

Problem: $n^{1/n}$, $n \geq 3$

Solution:

bounded, decreasing

Exercise:

Problem: $n^{-1/n}$, $n \geq 3$

Exercise:

Problem: $\tan n$

Solution:

not monotone, not bounded

Exercise:

Problem:

Determine whether the sequence defined as follows has a limit. If it does, find the limit.

$a_1 = \sqrt{2}$, $a_2 = \sqrt{2\sqrt{2}}$, $a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$ etc.

Exercise:

Problem:

Determine whether the sequence defined as follows has a limit. If it does, find the limit.

$a_1 = 3$, $a_n = \sqrt{2a_{n-1}}$, $n = 2, 3, \ldots$.

Solution:

a_n is decreasing and bounded below by 2. The limit a must satisfy $a = \sqrt{2a}$ so $a = 2$, independent of the initial value.

Use the Squeeze Theorem to find the limit of each of the following sequences.

Exercise:

Problem: $n \sin(1/n)$

Exercise:

Problem: $\frac{\cos(1/n)-1}{1/n}$

Solution:

0

Exercise:

Problem: $a_n = \frac{n!}{n^n}$

Exercise:

Problem: $a_n = \sin n \sin(1/n)$

Solution:

$0 : |\sin x| \leq |x|$ and $|\sin x| \leq 1$ so $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$).

For the following sequences, plot the first 25 terms of the sequence and state whether the graphical evidence suggests that the sequence converges or diverges.

Exercise:

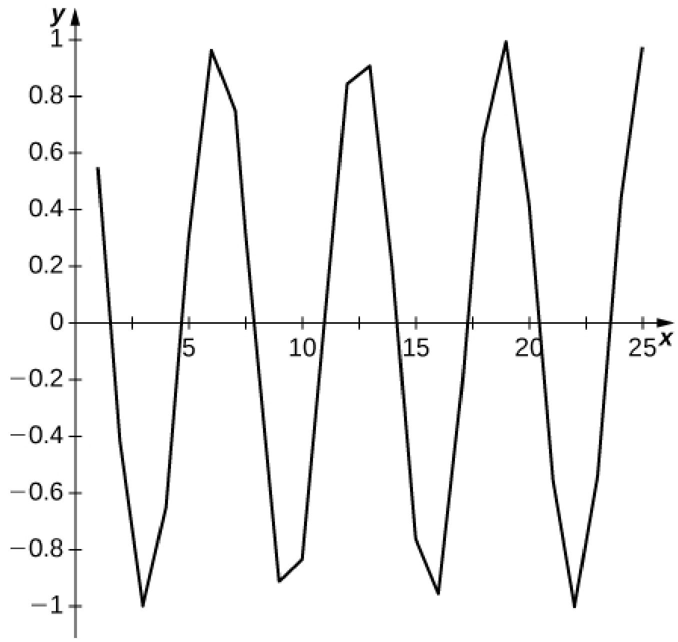
Problem: **[T]** $a_n = \sin n$

Exercise:

Problem: **[T]** $a_n = \cos n$

Solution:

Graph oscillates and suggests no limit.



Determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

Exercise:

Problem: $a_n = \tan^{-1}(n^2)$

Exercise:

Problem: $a_n = (2n)^{1/n} - n^{1/n}$

Solution:

$n^{1/n} \to 1$ and $2^{1/n} \to 1$, so $a_n \to 0$

Exercise:

Problem: $a_n = \frac{\ln(n^2)}{\ln(2n)}$

Exercise:

Problem: $a_n = \left(1 - \frac{2}{n}\right)^n$

Solution:

Since $(1 + 1/n)^n \to e$, one has $(1 - 2/n)^n \approx (1 + k)^{-2k} \to e^{-2}$ as $k \to \infty$.

Exercise:

Problem: $a_n = \ln\left(\frac{n+2}{n^2-3}\right)$

Exercise:

Problem: $a_n = \frac{2^n+3^n}{4^n}$

Solution:

$2^n + 3^n \le 2 \cdot 3^n$ and $3^n/4^n \to 0$ as $n \to \infty$, so $a_n \to 0$ as $n \to \infty$.

Exercise:

Problem: $a_n = \frac{(1000)^n}{n!}$

Exercise:

Problem: $a_n = \frac{(n!)^2}{(2n)!}$

Solution:

$\frac{a_{n+1}}{a_n} = n!/(n+1)(n+2)\cdots(2n) = \frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots(2n)} < 1/2^n$. In particular, $a_{n+1}/a_n \le 1/2$, so $a_n \to 0$ as $n \to \infty$.

Newton's method seeks to approximate a solution $f(x) = 0$ that starts with an initial approximation x_0 and successively defines a sequence $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. For the given choice of f and x_0, write out the formula for x_{n+1}. If the sequence appears to converge, give an exact formula for the solution x, then identify the limit x accurate to four decimal places and the smallest n such that x_n agrees with x up to four decimal places.

Exercise:

Problem: **[T]** $f(x) = x^2 - 2$, $x_0 = 1$

Exercise:

Problem: **[T]** $f(x) = (x-1)^2 - 2$, $x_0 = 2$

Solution:

$x_{n+1} = x_n - \left((x_n - 1)^2 - 2\right)/2(x_n - 1)$; $x = 1 + \sqrt{2}$, $x \approx 2.4142$, $n = 5$

Exercise:

Problem: **[T]** $f(x) = e^x - 2$, $x_0 = 1$

Exercise:

Problem: **[T]** $f(x) = \ln x - 1$, $x_0 = 2$

Solution:

$x_{n+1} = x_n - x_n(\ln(x_n) - 1)$; $x = e$, $x \approx 2.7183$, $n = 5$

Exercise:

Problem:

[T] Suppose you start with one liter of vinegar and repeatedly remove 0.1 L, replace with water, mix, and repeat.

a. Find a formula for the concentration after n steps.
b. After how many steps does the mixture contain less than 10% vinegar?

Exercise:

Problem:

[T] A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

a. Explain why the fish population after n months is modeled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000$.
b. How many fish will be in the pond after one year?

Solution:

a. Without losses, the population would obey $P_n = 1.06P_{n-1}$. The subtraction of 150 accounts for fish losses. b. After 12 months, we have $P_{12} \approx 1494$.

Exercise:

Problem:

[T] A bank account earns 5% interest compounded monthly. Suppose that \$1000 is initially deposited into the account, but that \$10 is withdrawn each month.

a. Show that the amount in the account after n months is $A_n = (1 + .05/12)A_{n-1} - 10$; $A_0 = 1000$.
b. How much money will be in the account after 1 year?
c. Is the amount increasing or decreasing?
d. Suppose that instead of \$10, a fixed amount d dollars is withdrawn each month. Find a value of d such that the amount in the account after each month remains \$1000.
e. What happens if d is greater than this amount?

Exercise:

Problem:

[T] A student takes out a college loan of \$10,000 at an annual percentage rate of 6%, compounded monthly.

a. If the student makes payments of $100 per month, how much does the student owe after 12 months?
b. After how many months will the loan be paid off?

Solution:

a. The student owes $9383 after 12 months. b. The loan will be paid in full after 139 months or eleven and a half years.

Exercise:

Problem:

[T] Consider a series combining geometric growth and arithmetic decrease. Let $a_1 = 1$. Fix $a > 1$ and $0 < b < a$. Set $a_{n+1} = a.\, a_n - b$. Find a formula for a_{n+1} in terms of a^n, a, and b and a relationship between a and b such that a_n converges.

Exercise:

Problem:

[T] The binary representation $x = 0.\, b_1 b_2 b_3 ...$ of a number x between 0 and 1 can be defined as follows. Let $b_1 = 0$ if $x < 1/2$ and $b_1 = 1$ if $1/2 \le x < 1$. Let $x_1 = 2x - b_1$. Let $b_2 = 0$ if $x_1 < 1/2$ and $b_2 = 1$ if $1/2 \le x < 1$. Let $x_2 = 2x_1 - b_2$ and in general, $x_n = 2x_{n-1} - b_n$ and $b_{n-1} = 0$ if $x_n < 1/2$ and $b_{n-1} = 1$ if $1/2 \le x_n < 1$. Find the binary expansion of $1/3$.

Solution:

$b_1 = 0$, $x_1 = 2/3$, $b_2 = 1$, $x_2 = 4/3 - 1 = 1/3$, so the pattern repeats, and $1/3 = 0.010101\ldots$.

Exercise:

Problem:

[T] To find an approximation for π, set $a_0 = \sqrt{2+1}$, $a_1 = \sqrt{2+a_0}$, and, in general, $a_{n+1} = \sqrt{2+a_n}$. Finally, set $p_n = 3.2^n \sqrt{2-a_n}$. Find the first ten terms of p_n and compare the values to π.

For the following two exercises, assume that you have access to a computer program or Internet source that can generate a list of zeros and ones of any desired length. Pseudorandom number generators (PRNGs) play an important role in simulating random noise in physical systems by creating sequences of zeros and ones that appear like the result of flipping a coin repeatedly. One of the simplest types of PRNGs recursively defines a random-looking sequence of N integers $a_1, a_2, \ldots, a_N$ by fixing two special integers K and M and letting a_{n+1} be the remainder after dividing $K.\, a_n$ into M, then creates a bit sequence of zeros and ones whose $n\text{th}$ term b_n is equal to one if a_n is odd and equal to zero if a_n is even. If the bits b_n are pseudorandom, then the behavior of their average $(b_1 + b_2 + \cdots + b_N)/N$ should be similar to behavior of averages of truly randomly generated bits.

Exercise:

Problem:

[T] Starting with $K = 16{,}807$ and $M = 2{,}147{,}483{,}647$, using ten different starting values of a_1, compute sequences of bits b_n up to $n = 1000$, and compare their averages to ten such sequences generated by a random bit generator.

Solution:

For the starting values $a_1 = 1$, $a_2 = 2$,. . ., $a_1 = 10$, the corresponding bit averages calculated by the method indicated are 0.5220, 0.5000, 0.4960, 0.4870, 0.4860, 0.4680, 0.5130, 0.5210, 0.5040, and 0.4840. Here is an example of ten corresponding averages of strings of 1000 bits generated by a random number generator: 0.4880, 0.4870, 0.5150, 0.5490, 0.5130, 0.5180, 0.4860, 0.5030, 0.5050, 0.4980. There is no real pattern in either type of average. The random-number-generated averages range between 0.4860 and 0.5490, a range of 0.0630, whereas the calculated PRNG bit averages range between 0.4680 and 0.5220, a range of 0.0540.

Exercise:

Problem:

[T] Find the first 1000 digits of π using either a computer program or Internet resource. Create a bit sequence b_n by letting $b_n = 1$ if the $n\text{th}$ digit of π is odd and $b_n = 0$ if the $n\text{th}$ digit of π is even. Compute the average value of b_n and the average value of $d_n = |b_{n+1} - b_n|$, $n = 1,\ldots, 999$. Does the sequence b_n appear random? Do the differences between successive elements of b_n appear random?

Glossary

arithmetic sequence
: a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

bounded above
: a sequence $\{a_n\}$ is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n

bounded below
: a sequence $\{a_n\}$ is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n

bounded sequence
: a sequence $\{a_n\}$ is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n

convergent sequence
: a convergent sequence is a sequence $\{a_n\}$ for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

divergent sequence
: a sequence that is not convergent is divergent

explicit formula
: a sequence may be defined by an explicit formula such that $a_n = f(n)$

geometric sequence
: a sequence $\{a_n\}$ in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence

index variable
: the subscript used to define the terms in a sequence is called the index

limit of a sequence
: the real number L to which a sequence converges is called the limit of the sequence

monotone sequence
: an increasing or decreasing sequence

recurrence relation
: a recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

sequence
: an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

term
: the number a_n in the sequence $\{a_n\}$ is called the $n\text{th}$ term of the sequence

unbounded sequence
: a sequence that is not bounded is called unbounded

Infinite Series

- Explain the meaning of the sum of an infinite series.
- Calculate the sum of a geometric series.
- Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. If you add these terms together, you get a series. In this section we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important types of series: the geometric series. We will use geometric series in the next chapter to write certain functions as polynomials with an infinite number of terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using polynomials that are easier to handle. We also discuss the harmonic series, arguably the most interesting divergent series because it just fails to converge.

Sums and Series

An infinite series is a sum of infinitely many terms and is written in the form
Equation:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms in the same way we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the *limit* of partial sums. A partial sum of an infinite series is a finite sum of the form
Equation:

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the following example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues such that each week half as much oil enters the lake as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that
Equation:

$$\begin{aligned}
&S_1 = 1\\
&S_2 = 1 + 0.5 = 1 + \tfrac{1}{2}\\
&S_3 = 1 + 0.5 + 0.25 = 1 + \tfrac{1}{2} + \tfrac{1}{4}\\
&S_4 = 1 + 0.5 + 0.25 + 0.125 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\\
&S_5 = 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}.
\end{aligned}$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is
Equation:

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

Equation:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

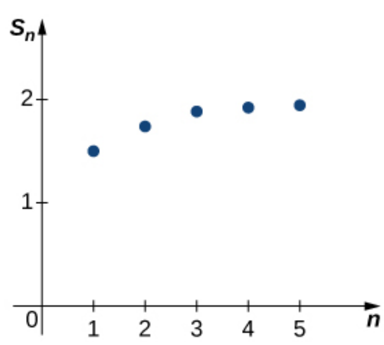
At the same time, as $k \to \infty$, the amount of oil in the lake can be calculated by evaluating $\lim_{k\to\infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums $\{S_k\}$. If the sequence of partial sums $\{S_k\}$ converges, we say that the infinite series converges, and its sum is given by $\lim_{k\to\infty} S_k$. If the sequence $\{S_k\}$ diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence $\{S_k\}$.

First, simplifying some of these partial sums, we see that

Equation:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \tfrac{1}{2} = \tfrac{3}{2} \\ S_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{7}{4} \\ S_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{15}{8} \\ S_5 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{31}{16}. \end{aligned}$$

Plotting some of these values in [link], it appears that the sequence $\{S_k\}$ could be approaching 2.



The graph shows the sequence of partial sums $\{S_k\}$. It appears that the sequence is approaching the value 2.

Let's look for more convincing evidence. In the following table, we list the values of S_k for several values of k.

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence $\{S_k\}$ converges to 2. Later we will provide an analytic argument that can be used to prove that $\lim_{k\to\infty} S_k = 2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does actually converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

Equation:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we summarize what it means for an infinite series to converge.

Note:
Definition
An **infinite series** is an expression of the form

Equation:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer k, the sum

Equation:

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the $k\text{th}$ **partial sum** of the infinite series. The partial sums form a sequence $\{S_k\}$. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

Equation:

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the **divergence of a series**.

Note:
This [website] shows a more whimsical approach to series.

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

Equation:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

can also be written as

Equation:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}.$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can reindex by making a change of variables. For example, consider the series

Equation:

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

By introducing the variable $m = n - 1$, so that $n = m + 1$, we can rewrite the series as

Equation:

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2}.$$

Example:

Exercise:

Problem:

Evaluating Limits of Sequences of Partial Sums

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution:

a. The sequence of partial sums $\{S_k\}$ satisfies

Equation:

$$\begin{aligned} S_1 &= \tfrac{1}{2} \\ S_2 &= \tfrac{1}{2} + \tfrac{2}{3} \\ S_3 &= \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} \\ S_4 &= \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5}. \end{aligned}$$

Notice that each term added is greater than $1/2$. As a result, we see that
Equation:

$$\begin{aligned} S_1 &= \tfrac{1}{2} \\ S_2 &= \tfrac{1}{2} + \tfrac{2}{3} > \tfrac{1}{2} + \tfrac{1}{2} = 2\left(\tfrac{1}{2}\right) \\ S_3 &= \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} > \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 3\left(\tfrac{1}{2}\right) \\ S_4 &= \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5} > \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 4\left(\tfrac{1}{2}\right). \end{aligned}$$

From this pattern we can see that $S_k > k\left(\frac{1}{2}\right)$ for every integer k. Therefore, $\{S_k\}$ is unbounded and consequently, diverges. Therefore, the infinite series $\sum_{n=1}^{\infty} n/(n+1)$ diverges.

b. The sequence of partial sums $\{S_k\}$ satisfies
Equation:

$$\begin{aligned} S_1 &= -1 \\ S_2 &= -1 + 1 = 0 \\ S_3 &= -1 + 1 - 1 = -1 \\ S_4 &= -1 + 1 - 1 + 1 = 0. \end{aligned}$$

From this pattern we can see the sequence of partial sums is
Equation:

$$\{S_k\} = \{-1, 0, -1, 0, \ldots\}.$$

Since this sequence diverges, the infinite series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

c. The sequence of partial sums $\{S_k\}$ satisfies
Equation:

$$\begin{aligned} S_1 &= \tfrac{1}{1\cdot 2} = \tfrac{1}{2} \\ S_2 &= \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} = \tfrac{1}{2} + \tfrac{1}{6} = \tfrac{2}{3} \\ S_3 &= \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \tfrac{1}{3\cdot 4} = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{12} = \tfrac{3}{4} \\ S_4 &= \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \tfrac{1}{3\cdot 4} + \tfrac{1}{4\cdot 5} = \tfrac{4}{5} \\ S_5 &= \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \tfrac{1}{3\cdot 4} + \tfrac{1}{4\cdot 5} + \tfrac{1}{5\cdot 6} = \tfrac{5}{6}. \end{aligned}$$

From this pattern, we can see that the $k\text{th}$ partial sum is given by the explicit formula
Equation:

$$S_k = \frac{k}{k+1}.$$

Since $k/(k+1) \to 1$, we conclude that the sequence of partial sums converges, and therefore the infinite series converges to 1. We have
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Note:
Exercise:

Problem: Determine whether the series $\sum_{n=1}^{\infty} (n+1)/n$ converges or diverges.

Solution:

The series diverges because the $k\text{th}$ partial sum $S_k > k$.

Hint

Look at the sequence of partial sums.

The Harmonic Series

A useful series to know about is the **harmonic series**. The harmonic series is defined as
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums $\{S_k\}$ approach infinity, but do so very slowly. We will show that the series diverges, but first we illustrate the slow growth of the terms in the sequence $\{S_k\}$ in the following table.

k	10	100	1000	10,000	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after 1,000,000 terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:
Equation:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \tfrac{1}{2} \\ S_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} \\ S_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}. \end{aligned}$$

Notice that for the last two terms in S_4,
Equation:

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}.$$

Therefore, we conclude that
Equation:

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea for S_8, we see that
Equation:

$$\begin{aligned} S_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + 3\left(\tfrac{1}{2}\right). \end{aligned}$$

From this pattern, we see that $S_1 = 1$, $S_2 = 1 + 1/2$, $S_4 > 1 + 2\,(1/2)$, and $S_8 > 1 + 3\,(1/2)$. More generally, it can be shown that $S_{2^j} > 1 + j(1/2)$ for all $j > 1$. Since $1 + j(1/2) \to \infty$, we conclude that the sequence $\{S_k\}$ is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since $\{S_k\}$ is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Note:
Algebraic Properties of Convergent Series
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold.

i. The series $\sum_{n=1}^{\infty}(a_n+b_n)$ converges and $\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n$. (Sum Rule)
ii. The series $\sum_{n=1}^{\infty}(a_n-b_n)$ converges and $\sum_{n=1}^{\infty}(a_n-b_n)=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n$. (Difference Rule)
iii. For any real number c, the series $\sum_{n=1}^{\infty}ca_n$ converges and $\sum_{n=1}^{\infty}ca_n=c\sum_{n=1}^{\infty}a_n$. (Constant Multiple Rule)

Example:
Exercise:

Problem:
Using Algebraic Properties of Convergent Series

Evaluate
Equation:

$$\sum_{n=1}^{\infty}\left[\frac{3}{n\,(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right].$$

Solution:

We showed earlier that
Equation:

$$\sum_{n=1}^{\infty}\frac{1}{n\,(n+1)}$$

and
Equation:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2.$$

Since both of those series converge, we can apply the properties of [link] to evaluate
Equation:

$$\sum_{n=1}^{\infty}\left[\frac{3}{n\,(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right].$$

Using the sum rule, write
Equation:

$$\sum_{n=1}^{\infty}\left[\frac{3}{n\left(n+1\right)}+\left(\frac{1}{2}\right)^{n-2}\right]=\sum_{n=1}^{\infty}\frac{3}{n\left(n+1\right)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}.$$

Then, using the constant multiple rule and the sums above, we can conclude that
Equation:

$$\begin{aligned}\sum_{n=1}^{\infty}\tfrac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^{n-2} &= 3\sum_{n=1}^{\infty}\tfrac{1}{n(n+1)}+\left(\tfrac{1}{2}\right)^{-1}\sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^{n-1}\\ &= 3\left(1\right)+\left(\tfrac{1}{2}\right)^{-1}\left(2\right)=3+2\left(2\right)=7.\end{aligned}$$

Note:
Exercise:

Problem: Evaluate $\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}$.

Solution:

10.

Hint

Rewrite as $\sum_{n=1}^{\infty}5\left(\frac{1}{2}\right)^{n-1}$.

Geometric Series

A **geometric series** is any series that we can write in the form
Equation:

$$a+ar+ar^2+ar^3+\cdots=\sum_{n=1}^{\infty}ar^{n-1}.$$

Because the ratio of each term in this series to the previous term is *r*, the number *r* is called the ratio. We refer to *a* as the initial term because it is the first term in the series. For example, the series
Equation:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

is a geometric series with initial term $a=1$ and ratio $r=1/2$.

In general, when does a geometric series converge? Consider the geometric series
Equation:

$$\sum_{n=1}^{\infty} ar^{n-1}$$

when $a > 0$. Its sequence of partial sums $\{S_k\}$ is given by
Equation:

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{k-1}.$$

Consider the case when $r = 1$. In that case,
Equation:

$$S_k = a + a\,(1) + a(1)^2 + \cdots + a(1)^{k-1} = ak.$$

Since $a > 0$, we know $ak \to \infty$ as $k \to \infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r = 1$. For $r \neq 1$, to find the limit of $\{S_k\}$, multiply [link] by $1 - r$. Doing so, we see that
Equation:

$$\begin{aligned}(1-r)S_k &= a\,(1-r)\left(1 + r + r^2 + r^3 + \cdots + r^{k-1}\right) \\ &= a[(1 + r + r^2 + r^3 + \cdots + r^{k-1}) - (r + r^2 + r^3 + \cdots + r^k)] \\ &= a\left(1 - r^k\right).\end{aligned}$$

All the other terms cancel out.

Therefore,
Equation:

$$S_k = \frac{a\left(1 - r^k\right)}{1 - r} \text{ for } r \neq 1.$$

From our discussion in the previous section, we know that the geometric sequence $r^k \to 0$ if $|r| < 1$ and that r^k diverges if $|r| > 1$ or $r = \pm 1$. Therefore, for $|r| < 1$, $S_k \to a/(1-r)$ and we have
Equation:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1.$$

If $|r| \geq 1$, S_k diverges, and therefore
Equation:

$$\sum_{n=1}^{\infty} ar^{n-1} \text{diverges if } |r| \geq 1.$$

Note:
Definition
A geometric series is a series of the form
Equation:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots.$$

If $|r| < 1$, the series converges, and
Equation:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$ or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form given by [link], it is a geometric series. For example, consider the series
Equation:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}.$$

To see that this is a geometric series, we write out the first several terms:
Equation:

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2} & = \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^3 + \left(\tfrac{2}{3}\right)^4 + \cdots \\ & = \tfrac{4}{9} + \tfrac{4}{9} \cdot \left(\tfrac{2}{3}\right) + \tfrac{4}{9} \cdot \left(\tfrac{2}{3}\right)^2 + \cdots. \end{aligned}$$

We see that the initial term is $a = 4/9$ and the ratio is $r = 2/3$. Therefore, the series can be written as
Equation:

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1}.$$

Since $r = 2/3 < 1$, this series converges, and its sum is given by
Equation:

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{4/9}{1 - 2/3} = \frac{4}{3}.$$

Example:
Exercise:

Problem:
Determining Convergence or Divergence of a Geometric Series

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

a. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$
b. $\sum_{n=1}^{\infty} e^{2n}$

Solution:

a. Writing out the first several terms in the series, we have
Equation:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} &= \frac{(-3)^2}{4^0} + \frac{(-3)^3}{4} + \frac{(-3)^4}{4^2} + \cdots \\ &= (-3)^2 + (-3)^2 \cdot \left(\frac{-3}{4}\right) + (-3)^2 \cdot \left(\frac{-3}{4}\right)^2 + \cdots \\ &= 9 + 9 \cdot \left(\frac{-3}{4}\right) + 9 \cdot \left(\frac{-3}{4}\right)^2 + \cdots.\end{aligned}$$

The initial term $a = -3$ and the ratio $r = -3/4$. Since $|r| = 3/4 < 1$, the series converges to
Equation:

$$\frac{9}{1-(-3/4)} = \frac{9}{7/4} = \frac{36}{7}.$$

b. Writing this series as
Equation:

$$e^2 \sum_{n=1}^{\infty} \left(e^2\right)^{n-1}$$

we can see that this is a geometric series where $r = e^2 > 1$. Therefore, the series diverges.

Note:
Exercise:

Problem:

Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Solution:

$5/7$

Hint

$r = -2/5$

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example:
Exercise:

Problem:
Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.\overline{26}$ as a fraction of integers.

Solution:

Since $3.\overline{26} = 3.262626\ldots$, first we write

Equation:

$$\begin{aligned} 3.262626\ldots &= 3 + \tfrac{26}{100} + \tfrac{26}{1000} + \tfrac{26}{100{,}000} + \cdots \\ &= 3 + \tfrac{26}{10^2} + \tfrac{26}{10^4} + \tfrac{26}{10^6} + \cdots. \end{aligned}$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a = 26/10^2$ and ratio $r = 1/10^2$. Therefore, the sum of this series is

Equation:

$$\frac{26/10^2}{1 - (1/10^2)} = \frac{26/10^2}{99/10^2} = \frac{26}{99}.$$

Thus,

Equation:

$$3.262626\ldots = 3 + \frac{26}{99} = \frac{323}{99}.$$

Note:
Exercise:

Problem: Write $5.2\overline{7}$ as a fraction of integers.

Solution:

$475/90$

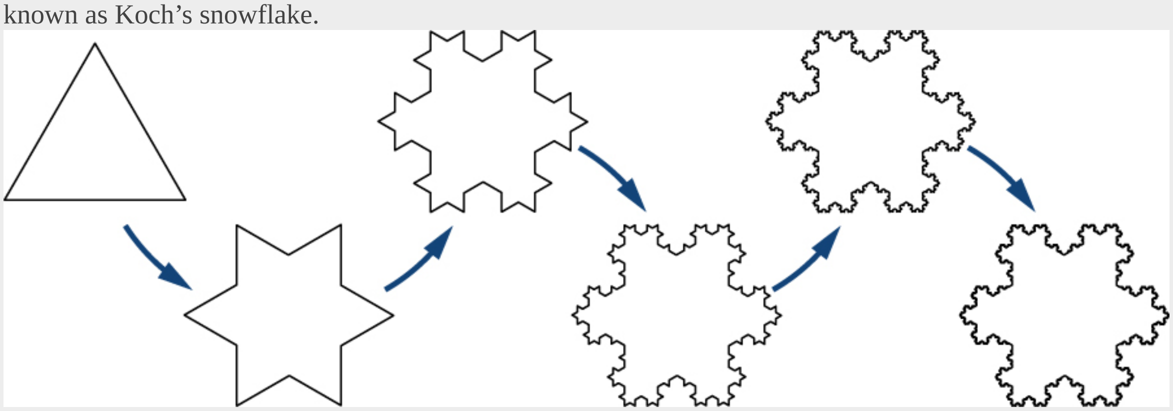
Hint

By expressing this number as a series, find a geometric series with initial term $a = 7/100$ and ratio $r = 1/10$.

Example:
Exercise:

Problem:
Chapter Opener: Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows ([link]). Let F_0 be an equilateral triangle with sides of length 1. For $n \geq 1$, let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as Koch's snowflake.



The first four figures, $F_0, F_1, F_2,$ and F_3, in the construction of the Koch snowflake.

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim\limits_{n\to\infty} L_n$ to find the length of the perimeter of Koch's snowflake.
b. Find the area A_n of figure F_n. Evaluate $\lim\limits_{n\to\infty} A_n$ to find the area of Koch's snowflake.

Solution:

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let l_n denote the length of each side of F_n. Since F_0 is an equilateral triangle with sides of length $l_0 = 1$, we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of F_0, we get four sides in F_1. Therefore, the number of sides for F_1 is
Equation:

$$N_1 = 4 \cdot 3.$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in F_0, the length of the line segments for F_1 is given by
Equation:

$$l_1 = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

Similarly, for F_2, since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by
Equation:

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3.$$

Since the length of each of these sides is $1/3$ the length of the sides of F_1, the length of each side of figure F_2 is given by
Equation:

$$l_2 = \frac{1}{3} \cdot l_1 = \frac{1}{3} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^2.$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3} l_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \frac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is
Equation:

$$N_n = 4^n \cdot 3$$

and the length of each side is
Equation:

$$l_n = \left(\frac{1}{3}\right)^n.$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by
Equation:

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n.$$

Therefore, the length of the perimeter of Koch's snowflake is
Equation:

$$L = \lim_{n \to \infty} L_n = \infty.$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0$, T_0 is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the sides of F_{n-1}, we have
Equation:

$$T_n = \left(\frac{1}{3}\right)^2 T_{n-1} = \frac{1}{9} \cdot T_{n-1}.$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},
Equation:

$$\begin{aligned} A_n &= A_{n-1} + N_{n-1} \cdot T_n \\ &= A_{n-1} + \left(3 \cdot 4^{n-1}\right) \cdot \left(\tfrac{1}{9}\right)^n \cdot \tfrac{\sqrt{3}}{4} \\ &= A_{n-1} + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)^n \cdot \tfrac{\sqrt{3}}{4}. \end{aligned}$$

Writing out the first few terms A_0, A_1, A_2, we see that
Equation:

$$\begin{array}{l} A_0 = \frac{\sqrt{3}}{4} \\ A_1 = A_0 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] \\ A_2 = A_1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2\right]. \end{array}$$

More generally,
Equation:

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4}\left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^n\right)\right].$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as
Equation:

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right)\right].$$

The expression $1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies
Equation:

$$1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1} = \frac{1 - (4/9)^n}{1 - (4/9)}.$$

Substituting this expression into the expression above and simplifying, we conclude that
Equation:

$$\begin{aligned} A_n &= \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{1}{3}\left(\tfrac{1-(4/9)^n}{1-(4/9)}\right)\right] \\ &= \tfrac{\sqrt{3}}{4}\left[\tfrac{8}{5} - \tfrac{3}{5}\left(\tfrac{4}{9}\right)^n\right]. \end{aligned}$$

Therefore, the area of Koch’s snowflake is

Equation:

$$A = \lim_{n\to\infty} A_n = \frac{2\sqrt{3}}{5}.$$

Analysis

The Koch snowflake is interesting because it has finite area, yet infinite perimeter. Although at first this may seem impossible, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y = 1/x^2$ and the x-axis on the interval $[1, \infty)$. Since the improper integral
Equation:

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Telescoping Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We discussed this series in [link], showing that the series converges by writing out the first several partial sums $S_1, S_2, \ldots, S_6$ and noticing that they are all of the form $S_k = \frac{k}{k+1}$. Here we use a different technique to show that this series converges. By using partial fractions, we can write
Equation:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, the series can be written as
Equation:

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots.$$

Writing out the first several terms in the sequence of partial sums $\{S_k\}$, we see that
Equation:

$$\begin{aligned} S_1 &= 1 - \tfrac{1}{2} \\ S_2 &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) = 1 - \tfrac{1}{3} \\ S_3 &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 1 - \tfrac{1}{4}. \end{aligned}$$

In general,
Equation:

$$S_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{k+1}.$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. In a sense, the series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series that has this property a telescoping series. For this series, since $S_k = 1 - 1/(k+1)$ and $1/(k+1) \to 0$ as $k \to \infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Note:
Definition
A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form
Equation:

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that
Equation:

$$\begin{aligned} S_1 &= b_1 - b_2 \\ S_2 &= (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3 \\ S_3 &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4. \end{aligned}$$

In general, the *k*th partial sum of this series is
Equation:

$$S_k = b_1 - b_{k+1}.$$

Since the *k*th partial sum can be simplified to the difference of these two terms, the sequence of partial sums $\{S_k\}$ will converge if and only if the sequence $\{b_{k+1}\}$ converges. Moreover, if the sequence b_{k+1} converges to some finite number B, then the sequence of partial sums converges to $b_1 - B$, and therefore
Equation:

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - B.$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example:
Exercise:

Problem:

Evaluating a Telescoping Series

Determine whether the telescoping series

Equation:

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution:

By writing out terms in the sequence of partial sums, we can see that

Equation:

$$\begin{aligned} S_1 &= \cos(1) - \cos\left(\tfrac{1}{2}\right) \\ S_2 &= \left(\cos(1) - \cos\left(\tfrac{1}{2}\right)\right) + \left(\cos\left(\tfrac{1}{2}\right) - \cos\left(\tfrac{1}{3}\right)\right) = \cos(1) - \cos\left(\tfrac{1}{3}\right) \\ S_3 &= \left(\cos(1) - \cos\left(\tfrac{1}{2}\right)\right) + \left(\cos\left(\tfrac{1}{2}\right) - \cos\left(\tfrac{1}{3}\right)\right) + \left(\cos\left(\tfrac{1}{3}\right) - \cos\left(\tfrac{1}{4}\right)\right) \\ &= \cos(1) - \cos\left(\tfrac{1}{4}\right). \end{aligned}$$

In general,

Equation:

$$S_k = \cos(1) - \cos\left(\frac{1}{k+1}\right).$$

Since $1/(k+1) \to 0$ as $k \to \infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1)) \to \cos(0) = 1$. Therefore, we conclude that $S_k \to \cos(1) - 1$. The telescoping series converges and the sum is given by

Equation:

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1.$$

Note:

Exercise:

Problem:

Determine whether $\sum_{n=1}^{\infty} \left[e^{1/n} - e^{1/(n+1)}\right]$ converges or diverges. If it converges, find its sum.

Solution:

$e - 1$

Hint

Write out the sequence of partial sums to see which terms cancel.

Note:
Euler’s Constant
We have shown that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Here we investigate the behavior of the partial sums S_k as $k \to \infty$. In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that
Equation:

$$\sum_{n=1}^{k} \frac{1}{n} - \ln k \to \gamma \text{ as } k \to \infty.$$

This constant γ is known as Euler’s constant.

1. Let $T_k = \sum_{n=1}^{k} \frac{1}{n} - \ln k$. Evaluate T_k for various values of k.
2. For T_k as defined in part 1. show that the sequence $\{T_k\}$ converges by using the following steps.
   a. Show that the sequence $\{T_k\}$ is monotone decreasing. (*Hint:* Show that $\ln(1 + 1/k > 1/(k+1))$
   b. Show that the sequence $\{T_k\}$ is bounded below by zero. (*Hint:* Express $\ln k$ as a definite integral.)
   c. Use the Monotone Convergence Theorem to conclude that the sequence $\{T_k\}$ converges. The limit γ is Euler’s constant.
3. Now estimate how far T_k is from γ for a given integer k. Prove that for $k \geq 1$, $0 < T_k - \gamma \leq 1/k$ by using the following steps.
   a. Show that $\ln(k+1) - \ln k < 1/k$.
   b. Use the result from part a. to show that for any integer k,
   Equation:

   $$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}.$$

   c. For any integers k and j such that $j > k$, express $T_k - T_j$ as a telescoping sum by writing
   Equation:

   $$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j).$$

   Use the result from part b. combined with this telescoping sum to conclude that
   Equation:

   $$T_k - T_j < \frac{1}{k} - \frac{1}{j}.$$

   d. Apply the limit to both sides of the inequality in part c. to conclude that
   Equation:

   $$T_k - \gamma \leq \frac{1}{k}.$$

e. Estimate γ to an accuracy of within 0.001.

Key Concepts

- Given the infinite series
Equation:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

and the corresponding sequence of partial sums $\{S_k\}$ where
Equation:

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k,$$

the series converges if and only if the sequence $\{S_k\}$ converges.
- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $|r| < 1$,
Equation:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- The harmonic series
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.
- A series of the form $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \cdots + [b_n - b_{n+1}] + \cdots$ is a telescoping series. The $k\text{th}$ partial sum of this series is given by $S_k = b_1 - b_{k+1}$. The series will converge if and only if $\lim_{k\to\infty} b_{k+1}$ exists. In that case,
Equation:

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - \lim_{k\to\infty}(b_{k+1}).$$

Key Equations

- **Harmonic series**
$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots$$
- **Sum of a geometric series**
$$\sum_{n=1}^{\infty} ar^{n-1}=\tfrac{a}{1-r} \text{ for } |r|<1$$

Using sigma notation, write the following expressions as infinite series.

Exercise:

Problem: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

Solution:

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

Exercise:

Problem: $1-1+1-1+\cdots$

Exercise:

Problem: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

Solution:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

Exercise:

Problem: $\sin 1+\sin 1/2+\sin 1/3+\sin 1/4+\cdots$

Compute the first four partial sums $S_1,\ldots,S_4$ for the series having nth term a_n starting with $n=1$ as follows.

Exercise:

Problem: $a_n=n$

Solution:

$1,3,6,10$

Exercise:

Problem: $a_n=1/n$

Exercise:

Problem: $a_n=\sin(n\pi/2)$

Solution:

$1, 1, 0, 0$

Exercise:

Problem: $a_n = (-1)^n$

In the following exercises, compute the general term a_n of the series with the given partial sum S_n. If the sequence of partial sums converges, find its limit S.

Exercise:

Problem: $S_n = 1 - \frac{1}{n}, n \geq 2$

Solution:

$a_n = S_n - S_{n-1} = \frac{1}{n-1} - \frac{1}{n}$. Series converges to $S = 1$.

Exercise:

Problem: $S_n = \frac{n(n+1)}{2}, n \geq 1$

Exercise:

Problem: $S_n = \sqrt{n}, n \geq 2$

Solution:

$a_n = S_n - S_{n-1} = \sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n-1} + \sqrt{n}}$. Series diverges because partial sums are unbounded.

Exercise:

Problem: $S_n = 2 - (n+2)/2^n, n \geq 1$

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$

Solution:

$S_1 = 1/3$, $S_2 = 1/3 + 2/4 > 1/3 + 1/3 = 2/3$, $S_3 = 1/3 + 2/4 + 3/5 > 3 \cdot (1/3) = 1$. In general $S_k > k/3$. Series diverges.

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} (1 - (-1)^n))$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ (*Hint:* Use a partial fraction decomposition like that for $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.)

Solution:

$S_1 = 1/(2.3) = 1/6 = 2/3 - 1/2,$
$S_2 = 1/(2.3) + 1/(3.4) = 2/12 + 1/12 = 1/4 = 3/4 - 1/2,$
$S_3 = 1/(2.3) + 1/(3.4) + 1/(4.5) = 10/60 + 5/60 + 3/60 = 3/10 = 4/5 - 1/2,$
$S_4 = 1/(2.3) + 1/(3.4) + 1/(4.5) + 1/(5.6) = 10/60 + 5/60 + 3/60 + 2/60 = 1/3 = 5/6 - 1/2.$

The pattern is $S_k = (k+1)/(k+2) - 1/2$ and the series converges to $1/2$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ (*Hint:* Follow the reasoning for $\sum_{n=1}^{\infty} \frac{1}{n}$.)

Suppose that $\sum_{n=1}^{\infty} a_n = 1$, that $\sum_{n=1}^{\infty} b_n = -1$, that $a_1 = 2$, and $b_1 = -3$. Find the sum of the indicated series.

Exercise:

Problem: $\sum_{n=1}^{\infty} (a_n + b_n)$

Solution:

0

Exercise:

Problem: $\sum_{n=1}^{\infty} (a_n - 2b_n)$

Exercise:

Problem: $\sum_{n=2}^{\infty} (a_n - b_n)$

Solution:

-3

Exercise:

Problem: $\sum_{n=1}^{\infty} (3a_{n+1} - 4b_{n+1})$

State whether the given series converges and explain why.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{n+1000}$ (*Hint:* Rewrite using a change of index.)

Solution:

diverges, $\sum_{n=1001}^{\infty} \frac{1}{n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{n+10^{80}}$ (*Hint:* Rewrite using a change of index.)

Exercise:

Problem: $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$

Solution:

convergent geometric series, $r = 1/10 < 1$

Exercise:

Problem: $1 + \frac{e}{\pi} + \frac{e^2}{\pi^2} + \frac{e^3}{\pi^3} + \cdots$

Exercise:

Problem: $1 + \frac{\pi}{e} + \frac{\pi^2}{e^4} + \frac{\pi^3}{e^6} + \frac{\pi^4}{e^8} + \cdots$

Solution:

convergent geometric series, $r = \pi/e^2 < 1$

Exercise:

Problem: $1 - \sqrt{\frac{\pi}{3}} + \sqrt{\frac{\pi^2}{9}} - \sqrt{\frac{\pi^3}{27}} + \cdots$

For a_n as follows, write the sum as a geometric series of the form $\sum_{n=1}^{\infty} ar^n$. State whether the series converges and if it does, find the value of $\sum a_n$.

Exercise:

Problem: $a_1 = -1$ and $a_n/a_{n+1} = -5$ for $n \geq 1$.

Solution:

$\sum_{n=1}^{\infty} 5 \cdot (-1/5)^n$, converges to $-5/6$

Exercise:

Problem: $a_1 = 2$ and $a_n/a_{n+1} = 1/2$ for $n \geq 1$.

Exercise:

Problem: $a_1 = 10$ and $a_n/a_{n+1} = 10$ for $n \geq 1$.

Solution:

$\sum_{n=1}^{\infty} 100 \cdot (1/10)^n$, converges to $100/9$

Exercise:

Problem: $a_1 = 1/10$ and $a_n/a_{n+1} = -10$ for $n \geq 1$.

Use the identity $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ to express the function as a geometric series in the indicated term.

Exercise:

Problem: $\frac{x}{1+x}$ in x

Solution:

$$x \sum_{n=0}^{\infty} (-x)^n = \sum_{n=1}^{\infty} (-1)^{n-1} x^n$$

Exercise:

Problem: $\frac{\sqrt{x}}{1-x^{3/2}}$ in $\sqrt{x}$

Exercise:

Problem: $\frac{1}{1+\sin^2 x}$ in $\sin x$

Solution:

$$\sum_{n=0}^{\infty} (-1)^n \sin^{2n}(x)$$

Exercise:

Problem: $\sec^2 x$ in $\sin x$

Evaluate the following telescoping series or state whether the series diverges.

Exercise:

Problem: $\sum_{n=1}^{\infty} 2^{1/n} - 2^{1/(n+1)}$

Solution:

$S_k = 2 - 2^{1/(k+1)} \to 1$ as $k \to \infty$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{n^{13}} - \frac{1}{(n+1)^{13}}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \left(\sqrt{n} - \sqrt{n+1}\right)$

Solution:

$S_k = 1 - \sqrt{k+1}$ diverges

Exercise:

Problem: $\sum_{n=1}^{\infty} (\sin n - \sin(n+1))$

Express the following series as a telescoping sum and evaluate its *n*th partial sum.

Exercise:

Problem: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Solution:

$\sum_{n=1}^{\infty} \ln n - \ln(n+1), S_k = -\ln(k+1)$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2}$ (*Hint:* Factor denominator and use partial fractions.)

Exercise:

Problem: $\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n \ln(n+1)}$

Solution:

$a_n = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$ and $S_k = \frac{1}{\ln(2)} - \frac{1}{\ln(k+1)} \to \frac{1}{\ln(2)}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(n+2)}{n(n+1)2^{n+1}}$ (*Hint:* Look at $1/(n2^n)$.)

A general telescoping series is one in which all but the first few terms cancel out after summing a given number of successive terms.

Exercise:

Problem: Let $a_n = f(n) - 2f(n+1) + f(n+2)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

Solution:

$\sum_{n=1}^{\infty} a_n = f(1) - f(2)$

Exercise:

Problem:

$a_n = f(n) - f(n+1) - f(n+2) + f(n+3)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

Exercise:

Problem:

Suppose that $a_n = c_0 f(n) + c_1 f(n+1) + c_2 f(n+2) + c_3 f(n+3) + c_4 f(n+4)$, where $f(n) \to 0$ as $n \to \infty$. Find a condition on the coefficients $c_0, \ldots, c_4$ that make this a general telescoping series.

Solution:

$c_0 + c_1 + c_2 + c_3 + c_4 = 0$

Exercise:

Problem: Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (*Hint:* $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$)

Exercise:

Problem: Evaluate $\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$.

Solution:

$\frac{2}{n^3-1} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$, $S_n = (1 - 1 + 1/3) + (1/2 - 2/3 + 1/4)$
$+ (1/3 - 2/4 + 1/5) + (1/4 - 2/5 + 1/6) + \cdots = 1/2$

Exercise:

Problem: Find a formula for $\sum_{n=1}^{\infty} \frac{1}{n\,(n+N)}$ where N is a positive integer.

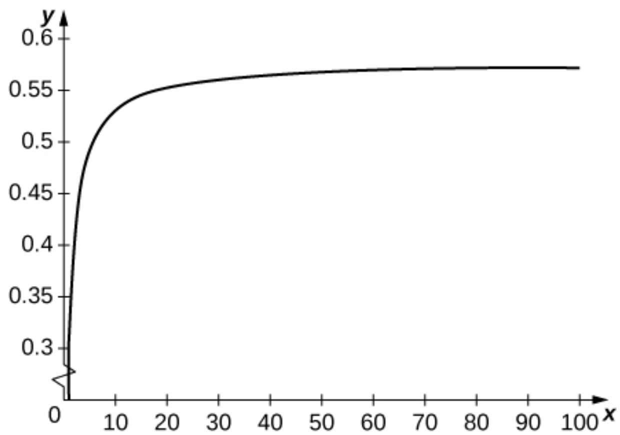
Exercise:

Problem:

[T] Define a sequence $t_k = \sum_{n=1}^{k-1}(1/k) - \ln k$. Use the graph of $1/x$ to verify that t_k is increasing. Plot t_k for $k = 1\ldots100$ and state whether it appears that the sequence converges.

Solution:

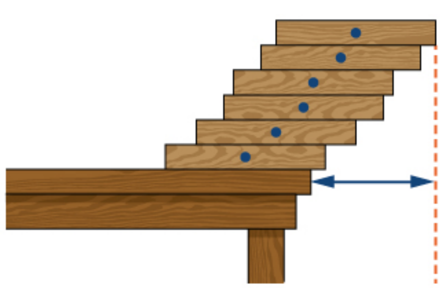
t_k converges to $0.57721\ldots t_k$ is a sum of rectangles of height $1/k$ over the interval $[k, k+1]$ which lie above the graph of $1/x$.



Exercise:

Problem:

[T] Suppose that N equal uniform rectangular blocks are stacked one on top of the other, allowing for some overhang. Archimedes' law of the lever implies that the stack of N blocks is stable as long as the center of mass of the top $(N-1)$ blocks lies at the edge of the bottom block. Let x denote the position of the edge of the bottom block, and think of its position as relative to the center of the next-to-bottom block. This implies that $(N-1)x = \left(\frac{1}{2} - x\right)$ or $x = 1/(2N)$. Use this expression to compute the maximum overhang (the position of the edge of the top block over the edge of the bottom block.) See the following figure.



Each of the following infinite series converges to the given multiple of π or $1/\pi$.

In each case, find the minimum value of N such that the $N\text{th}$ partial sum of the series accurately approximates the left-hand side to the given number of decimal places, and give the desired approximate value. Up to 15 decimals place, $\pi = 3.141592653589793....$

Exercise:

Problem: **[T]** $\pi = -3 + \sum_{n=1}^{\infty} \frac{n2^n n!^2}{(2n)!}$, error < 0.0001

Solution:

$N = 22$, $S_N = 6.1415$

Exercise:

Problem: **[T]** $\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}$, error $< 10^{-4}$

Exercise:

Problem: **[T]** $\frac{9801}{2\pi} = \frac{4}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}$, error $< 10^{-12}$

Solution:

$N = 3$, $S_N = 1.559877597243667...$

Exercise:

Problem: **[T]** $\frac{1}{12\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!\,(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$, error $< 10^{-15}$

Exercise:

Problem: **[T]** A fair coin is one that has probability $1/2$ of coming up heads when flipped.

a. What is the probability that a fair coin will come up tails n times in a row?
b. Find the probability that a coin comes up heads for the first time after an even number of coin flips.

Solution:

a. The probability of any given ordered sequence of outcomes for n coin flips is $1/2^n$. b. The probability of coming up heads for the first time on the n th flip is the probability of the sequence $TT...TH$ which is $1/2^n$. The probability of coming up heads for the first time on an even flip is $\sum_{n=1}^{\infty} 1/2^{2n}$ or $1/3$.

Exercise:

Problem:

[T] Find the probability that a fair coin is flipped a multiple of three times before coming up heads.

Exercise:

Problem:

[T] Find the probability that a fair coin will come up heads for the second time after an even number of flips.

Solution:

$5/9$

Exercise:

Problem:

[T] Find a series that expresses the probability that a fair coin will come up heads for the second time on a multiple of three flips.

Exercise:

Problem:

[T] The expected number of times that a fair coin will come up heads is defined as the sum over $n = 1, 2, \ldots$ of n times the probability that the coin will come up heads exactly n times in a row, or $n/2^{n+1}$. Compute the expected number of consecutive times that a fair coin will come up heads.

Solution:

$E = \sum_{n=1}^{\infty} n/2^{n+1} = 1$, as can be shown using summation by parts

Exercise:

Problem:

[T] A person deposits $\$10$ at the beginning of each quarter into a bank account that earns 4% annual interest compounded quarterly (four times a year).

a. Show that the interest accumulated after n quarters is $\$10\left(\frac{1.01^{n+1}-1}{0.01} - n\right)$.
b. Find the first eight terms of the sequence.
c. How much interest has accumulated after 2 years?

Exercise:

Problem:

[T] Suppose that the amount of a drug in a patient's system diminishes by a multiplicative factor $r < 1$ each hour. Suppose that a new dose is administered every N hours. Find an expression that gives the amount $A(n)$ in the patient's system after n hours for each n in terms of the dosage d and the ratio r. (*Hint:* Write $n = mN + k$, where $0 \le k < N$, and sum over values from the different doses administered.)

Solution:

The part of the first dose after n hours is dr^n, the part of the second dose is dr^{n-N}, and, in general, the part remaining of the $m\text{th}$ dose is dr^{n-mN}, so

$$A(n) = \sum_{l=0}^{m} dr^{n-lN} = \sum_{l=0}^{m} dr^{k+(m-l)N} = \sum_{q=0}^{m} dr^{k+qN} = dr^k \sum_{q=0}^{m} r^{Nq} = dr^k \frac{1-r^{(m+1)N}}{1-r^N}, n = k + mN.$$

Exercise:

Problem:

[T] A certain drug is effective for an average patient only if there is at least 1 mg per kg in the patient's system, while it is safe only if there is at most 2 mg per kg in an average patient's system. Suppose that the amount in a patient's system diminishes by a multiplicative factor of 0.9 each hour after a dose is administered. Find the maximum interval N of hours between doses, and corresponding dose range d (in mg/kg) for this N that will enable use of the drug to be both safe and effective in the long term.

Exercise:

Problem:

Suppose that $a_n \geq 0$ is a sequence of numbers. Explain why the sequence of partial sums of a_n is increasing.

Solution:

$S_{N+1} = a_{N+1} + S_N \geq S_N$

Exercise:

Problem:

[T] Suppose that a_n is a sequence of positive numbers and the sequence S_n of partial sums of a_n is bounded above. Explain why $\sum_{n=1}^{\infty} a_n$ converges. Does the conclusion remain true if we remove the hypothesis $a_n \geq 0$?

Exercise:

Problem:

[T] Suppose that $a_1 = S_1 = 1$ and that, for given numbers $S > 1$ and $0 < k < 1$, one defines $a_{n+1} = k(S - S_n)$ and $S_{n+1} = a_{n+1} + S_n$. Does S_n converge? If so, to what? (*Hint:* First argue that $S_n < S$ for all n and S_n is increasing.)

Solution:

Since $S > 1$, $a_2 > 0$, and since $k < 1$, $S_2 = 1 + a_2 < 1 + (S - 1) = S$. If $S_n > S$ for some *n*, then there is a smallest *n*. For this *n*, $S > S_{n-1}$, so $S_n = S_{n-1} + k(S - S_{n-1}) = kS + (1-k)S_{n-1} < S$, a contradiction. Thus $S_n < S$ and $a_{n+1} > 0$ for all *n*, so S_n is increasing and bounded by S. Let $S_* = \lim S_n$. If $S_* < S$, then $\delta = k(S - S_*) > 0$, but we can find *n* such that $S* - S_n < \delta/2$, which implies that $S_{n+1} = S_n + k(S - S_n) > S* + \delta/2$, contradicting that S_n is increasing to S_*. Thus $S_n \to S$.

Exercise:

Problem:

[T] A version of von Bertalanffy growth can be used to estimate the age of an individual in a homogeneous species from its length if the annual increase in year $n+1$ satisfies $a_{n+1} = k(S - S_n)$, with S_n as the length at year n, S as a limiting length, and k as a relative growth constant. If $S_1 = 3$, $S = 9$, and $k = 1/2$, numerically estimate the smallest value of n such that $S_n \geq 8$. Note that $S_{n+1} = S_n + a_{n+1}$. Find the corresponding n when $k = 1/4$.

Exercise:

Problem:

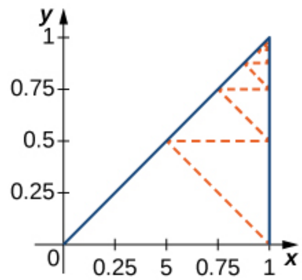
[T] Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Explain why $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} a_n = 0$.

Solution:

Let $S_k = \sum_{n=1}^{k} a_n$ and $S_k \to L$. Then S_k eventually becomes arbitrarily close to L, which means that $L - S_N = \sum_{n=N+1}^{\infty} a_n$ becomes arbitrarily small as $N \to \infty$.

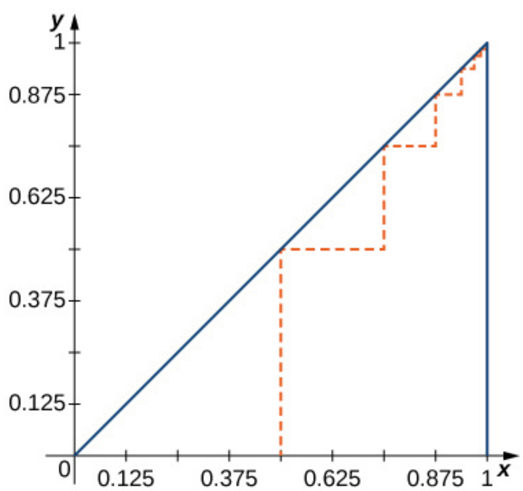
Exercise:

Problem: **[T]** Find the length of the dashed zig-zag path in the following figure.



Exercise:

Problem: **[T]** Find the total length of the dashed path in the following figure.



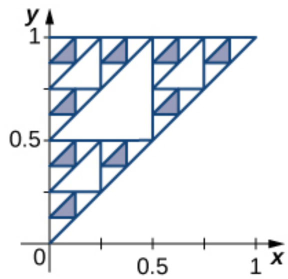
Solution:

$L = \left(1 + \frac{1}{2}\right) \sum_{n=1}^{\infty} 1/2^n = \frac{3}{2}$.

Exercise:

Problem:

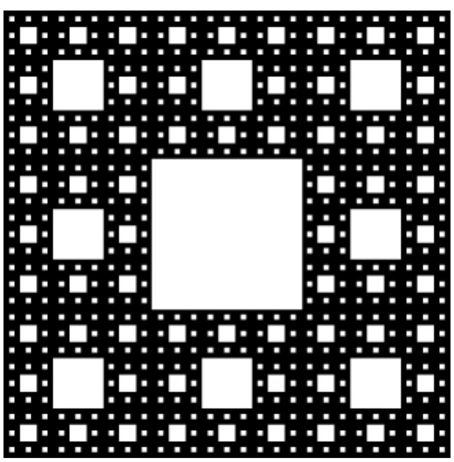
[T] The Sierpinski triangle is obtained from a triangle by deleting the middle fourth as indicated in the first step, by deleting the middle fourths of the remaining three congruent triangles in the second step, and in general deleting the middle fourths of the remaining triangles in each successive step. Assuming that the original triangle is shown in the figure, find the areas of the remaining parts of the original triangle after N steps and find the total length of all of the boundary triangles after N steps.



Exercise:

Problem:

[T] The Sierpinski gasket is obtained by dividing the unit square into nine equal sub-squares, removing the middle square, then doing the same at each stage to the remaining sub-squares. The figure shows the remaining set after four iterations. Compute the total area removed after N stages, and compute the length the total perimeter of the remaining set after N stages.



Solution:

At stage one a square of area $1/9$ is removed, at stage 2 one removes 8 squares of area $1/9^2$, at stage three one removes 8^2 squares of area $1/9^3$, and so on. The total removed area after N stages is

$$\sum_{n=0}^{N-1} 8^N/9^{N+1} = \frac{1}{8}\left(1-(8/9)^N\right)/\left(1-8/9\right) \to 1$$

as $N \to \infty$. The total perimeter is $4 + 4\sum_{n=0} 8^N/3^{N+1} \to \infty$.

Glossary

convergence of a series
: a series converges if the sequence of partial sums for that series converges

divergence of a series
: a series diverges if the sequence of partial sums for that series diverges

geometric series
: a geometric series is a series that can be written in the form

Equation:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

harmonic series
: the harmonic series takes the form

Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

infinite series
: an infinite series is an expression of the form

Equation:

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

partial sum
: the $k\text{th}$ partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum

Equation:

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

telescoping series
: a telescoping series is one in which most of the terms cancel in each of the partial sums

The Divergence and Integral Tests

- Use the divergence test to determine whether a series converges or diverges.
- Use the integral test to determine the convergence of a series.
- Estimate the value of a series by finding bounds on its remainder term.

In the previous section, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums $\{S_k\}$. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series. In this section, we discuss two of these tests: the divergence test and the integral test. We will examine several other tests in the rest of this chapter and then summarize how and when to use them.

Divergence Test

For a series $\sum_{n=1}^{\infty} a_n$ to converge, the $n\text{th}$ term a_n must satisfy $a_n \to 0$ as $n \to \infty$.

Therefore, from the algebraic limit properties of sequences,
Equation:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0.$$

Therefore, if $\sum\limits_{n=1}^{\infty} a_n$ converges, the $n\text{th}$ term $a_n \to 0$ as $n \to \infty$. An important consequence of this fact is the following statement:
Equation:

$$\text{If } a_n \nrightarrow 0 \text{ as } n \to \infty, \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

This test is known as the **divergence test** because it provides a way of proving that a series diverges.

Note:
Divergence Test
If $\lim\limits_{n\to\infty} a_n = c \neq 0$ or $\lim\limits_{n\to\infty} a_n$ does not exist, then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

It is important to note that the converse of this theorem is not true. That is, if $\lim\limits_{n\to\infty} a_n = 0$, we cannot make any conclusion about the convergence of $\sum\limits_{n=1}^{\infty} a_n$. For example, $\lim\limits_{n\to 0}(1/n) = 0$, but the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. In this section and the remaining sections of this chapter, we show many more examples of such series. Consequently, although we can use the divergence test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if $a_n \to 0$, the divergence test is inconclusive.

Example:
Exercise:

Problem:
Using the divergence test

For each of the following series, apply the divergence test. If the divergence test proves that the series diverges, state so. Otherwise, indicate that the divergence test is inconclusive.

a. $\sum_{n=1}^{\infty} \frac{n}{3n-1}$
b. $\sum_{n=1}^{\infty} \frac{1}{n^3}$
c. $\sum_{n=1}^{\infty} e^{1/n^2}$

Solution:

a. Since $n/(3n-1) \to 1/3 \neq 0$, by the divergence test, we can conclude that
Equation:

$$\sum_{n=1}^{\infty} \frac{n}{3n-1}$$

diverges.
b. Since $1/n^3 \to 0$, the divergence test is inconclusive.
c. Since $e^{1/n^2} \to 1 \neq 0$, by the divergence test, the series
Equation:

$$\sum_{n=1}^{\infty} e^{1/n^2}$$

diverges.

Note:
Exercise:

Problem: What does the divergence test tell us about the series $\sum_{n=1}^{\infty} \cos(1/n^2)$?

Solution:

The series diverges.

Hint

Look at $\lim_{n\to\infty} \cos(1/n^2)$.

Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums $\{S_k\}$ and showing that $S_{2^k} > 1 + k/2$ for all positive integers k. In this section we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the **integral test**, compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive.

To illustrate how the integral test works, use the harmonic series as an example. In [link], we depict the harmonic series by sketching a sequence of rectangles with areas $1, 1/2, 1/3, 1/4, \ldots$ along with the function $f(x) = 1/x$. From the graph, we see that

Equation:

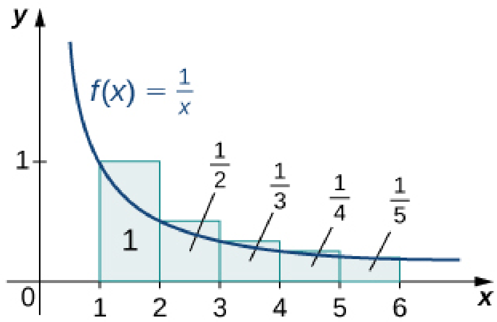
$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_{1}^{k+1} \frac{1}{x} dx.$$

Therefore, for each k, the $k\text{th}$ partial sum S_k satisfies

Equation:

$$S_k = \sum_{n=1}^{k} \frac{1}{n} > \int_1^{k+1} \frac{1}{x} dx = \ln x \Big|_1^{k+1} = \ln(k+1) - \ln(1) = \ln(k+1).$$

Since $\lim\limits_{k\to\infty} \ln(k+1) = \infty$, we see that the sequence of partial sums $\{S_k\}$ is unbounded. Therefore, $\{S_k\}$ diverges, and, consequently, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ also diverges.



The sum of the areas of the rectangles is greater than the area between the curve $f(x) = 1/x$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is infinite (as calculated by an improper integral), the sum of the areas of the rectangles is also infinite.

Now consider the series $\sum\limits_{n=1}^{\infty} 1/n^2$. We show how an integral can be used to prove that this series converges. In [link], we sketch a sequence of rectangles with areas $1, 1/2^2, 1/3^2, \ldots$ along with the function $f(x) = 1/x^2$. From the graph we see that

Equation:

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_1^k \frac{1}{x^2} dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

Equation:

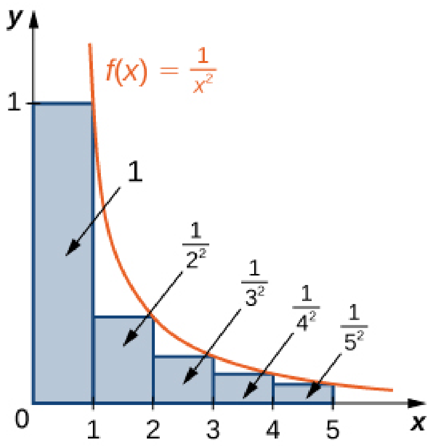
$$S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2}\,dx = 1 - \frac{1}{x}\Big|_1^k = 1 - \frac{1}{k} + 1 = 2 - \frac{1}{k} < 2.$$

We conclude that the sequence of partial sums $\{S_k\}$ is bounded. We also see that $\{S_k\}$ is an increasing sequence:

Equation:

$$S_k = S_{k-1} + \frac{1}{k^2} \text{ for } k \geq 2.$$

Since $\{S_k\}$ is increasing and bounded, by the Monotone Convergence Theorem, it converges. Therefore, the series $\sum_{n=1}^{\infty} 1/n^2$ converges.



The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area between the curve $f(x) = 1/x^2$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

We can extend this idea to prove convergence or divergence for many different series. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n such that there exists a continuous, positive, decreasing function f where $f(n) = a_n$ for all positive integers. Then, as in [link](a), for any integer k, the $k\text{th}$ partial sum S_k satisfies

Equation:

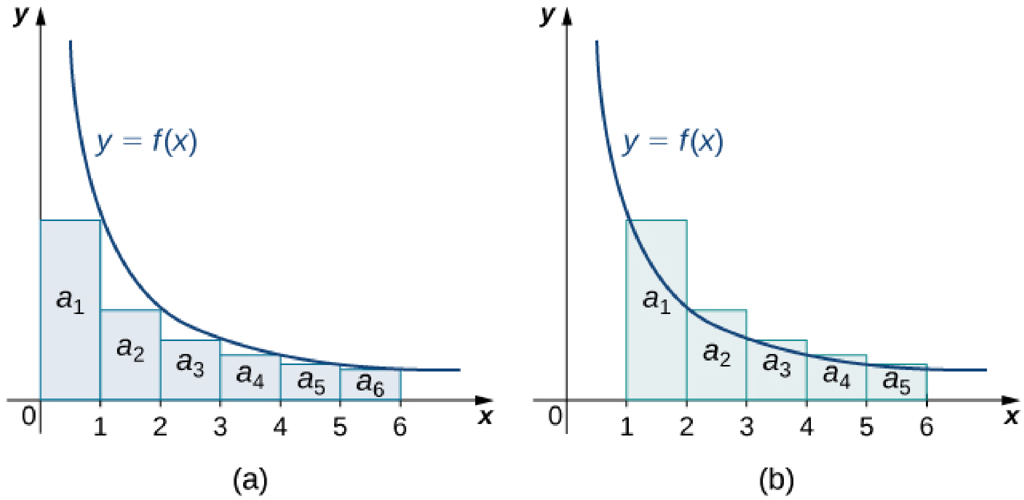
$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)dx < 1 + \int_1^{\infty} f(x)dx.$$

Therefore, if $\int_1^{\infty} f(x)dx$ converges, then the sequence of partial sums $\{S_k\}$ is bounded. Since $\{S_k\}$ is an increasing sequence, if it is also a bounded sequence, then by the Monotone Convergence Theorem, it converges. We conclude that if $\int_1^{\infty} f(x)dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. On the other hand, from [link](b), for any integer k, the $k\text{th}$ partial sum S_k satisfies

Equation:

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)dx.$$

If $\lim\limits_{k\to\infty} \int_1^{k+1} f(x)dx = \infty$, then $\{S_k\}$ is an unbounded sequence and therefore diverges. As a result, the series $\sum\limits_{n=1}^{\infty} a_n$ also diverges. Since f is a positive function, if $\int_1^{\infty} f(x)dx$ diverges, then $\lim\limits_{k\to\infty} \int_1^{k+1} f(x)dx = \infty$. We conclude that if $\int_1^{\infty} f(x)dx$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.



(a) If we can inscribe rectangles inside a region bounded by a curve $y = f(x)$ and the x-axis, and the area bounded by those curves for $x \geq 1$ is finite, then the sum of the areas of the rectangles is also finite. (b) If a set of rectangles circumscribes the region bounded by $y = f(x)$ and the x axis for $x \geq 1$ and the region has infinite area, then the sum of the areas of the rectangles is also infinite.

Note:
Integral Test
Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq N$.
Then
Equation:

$$\sum_{n=1}^{\infty} a_n \text{and} \int_N^{\infty} f(x)dx$$

both converge or both diverge (see [link]).

Although convergence of $\int_N^\infty f(x)dx$ implies convergence of the related series $\sum_{n=1}^\infty a_n$, it does not imply that the value of the integral and the series are the same. They may be different, and often are. For example,

Equation:

$$\sum_{n=1}^\infty \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots$$

is a geometric series with initial term $a = 1/e$ and ratio $r = 1/e$, which converges to

Equation:

$$\frac{1/e}{1-(1/e)} = \frac{1/e}{(e-1)/e} = \frac{1}{e-1}.$$

However, the related integral $\int_1^\infty (1/e)^x dx$ satisfies

Equation:

$$\int_1^\infty \left(\frac{1}{e}\right)^x dx = \int_1^\infty e^{-x} dx = \lim_{b\to\infty} \int_1^b e^{-x} dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty} \left[-e^{-b} + e^{-1}\right] = \frac{1}{e}.$$

Example:
Exercise:

Problem:
Using the Integral Test

For each of the following series, use the integral test to determine whether the series converges or diverges.

a. $\sum_{n=1}^\infty 1/n^3$
b. $\sum_{n=1}^\infty 1/\sqrt{2n-1}$

Solution:

a. Compare
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^3} dx.$$

We have
Equation:

$$\int_1^{\infty} \frac{1}{x^3} dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^3} dx = \lim_{b \to \infty} \left[-\frac{1}{2x^2} \Big|_1^b \right] = \lim_{b \to \infty} \left[-\frac{1}{2b^2} + \frac{1}{2} \right] = \frac{1}{2}.$$

Thus the integral $\int_1^{\infty} 1/x^3 dx$ converges, and therefore so does the series
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

b. Compare
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} \quad \text{and} \quad \int_1^{\infty} \frac{1}{\sqrt{2x-1}} dx.$$

Since
Equation:

$$\begin{aligned} \int_1^{\infty} \frac{1}{\sqrt{2x-1}} dx &= \lim_{b \to \infty} \int_1^{b} \frac{1}{\sqrt{2x-1}} dx = \lim_{b \to \infty} \sqrt{2x-1} \Big|_1^b \\ &= \lim_{b \to \infty} \left[\sqrt{2b-1} - 1 \right] = \infty, \end{aligned}$$

the integral $\int_1^{\infty} 1/\sqrt{2x-1}\, dx$ diverges, and therefore
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$$

diverges.

Note:
Exercise:

Problem:

Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$ converges or diverges.

Solution:

The series diverges.

Hint

Compare to the integral $\int_1^{\infty} \frac{x}{3x^2+1} dx.$

The *p*-Series

The harmonic series $\sum_{n=1}^{\infty} 1/n$ and the series $\sum_{n=1}^{\infty} 1/n^2$ are both examples of a type of series called a *p*-series.

Note:
Definition
For any real number p, the series
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a ***p*-series**.

We know the *p*-series converges if $p = 2$ and diverges if $p = 1$. What about other values of p? In general, it is difficult, if not impossible, to compute the exact value of most *p*-series. However, we can use the tests presented thus far to prove whether a *p*-series converges or diverges.

If $p < 0$, then $1/n^p \to \infty$, and if $p = 0$, then $1/n^p \to 1$. Therefore, by the divergence test,
Equation:

$$\sum_{n=1}^{\infty} 1/n^p \text{ diverges if } p \leq 0.$$

If $p > 0$, then $f(x) = 1/x^p$ is a positive, continuous, decreasing function. Therefore, for $p > 0$, we use the integral test, comparing
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ and } \int_1^{\infty} \frac{1}{x^p} dx.$$

We have already considered the case when $p = 1$. Here we consider the case when $p > 0, p \neq 1$. For this case,
Equation:

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p} dx = \lim_{b \to \infty} \frac{1}{1-p} x^{1-p} \Big|_1^b = \lim_{b \to \infty} \frac{1}{1-p} \left[b^{1-p} - 1 \right].$$

Because
Equation:

$$b^{1-p} \to 0 \text{ if } p > 1 \text{ and } b^{1-p} \to \infty \text{ if } p < 1,$$

we conclude that
Equation:

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1} \text{ if } p > 1 \\ \infty \text{ if } p < 1 \end{cases}.$$

Therefore, $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $0 < p < 1$.

In summary,
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}.$$

Example:
Exercise:

Problem:
Testing for Convergence of *p*-series

For each of the following series, determine whether it converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$
b. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Solution:

a. This is a p-series with $p = 4 > 1$, so the series converges.
b. Since $p = 2/3 < 1$, the series diverges.

Note:
Exercise:

Problem: Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converge or diverge?

Solution:

The series converges.

Hint

$p = 5/4$

Estimating the Value of a Series

Suppose we know that a series $\sum_{n=1}^{\infty} a_n$ converges and we want to estimate the sum of that series. Certainly we can approximate that sum using any finite sum $\sum_{n=1}^{N} a_n$ where N is any

positive integer. The question we address here is, for a convergent series $\sum_{n=1}^{\infty} a_n$, how good is the approximation $\sum_{n=1}^{N} a_n$? More specifically, if we let

Equation:

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$$

be the remainder when the sum of an infinite series is approximated by the $N\text{th}$ partial sum, how large is R_N? For some types of series, we are able to use the ideas from the integral test to estimate R_N.

Note:

Remainder Estimate from the Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq 1$.

Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

Equation:

$$S_N + \int_{N+1}^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)dx.$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

Equation:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_{N}^{\infty} f(x)dx.$$

This is known as the **remainder estimate**.

We illustrate [link] in [link]. In particular, by representing the remainder $R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by $\int_N^{\infty} f(x)dx$ and bounded below by $\int_{N+1}^{\infty} f(x)dx$. In other words,

Equation:

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)dx$$

and

Equation:

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^{\infty} f(x)dx.$$

We conclude that

Equation:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_N^{\infty} f(x)dx.$$

Since

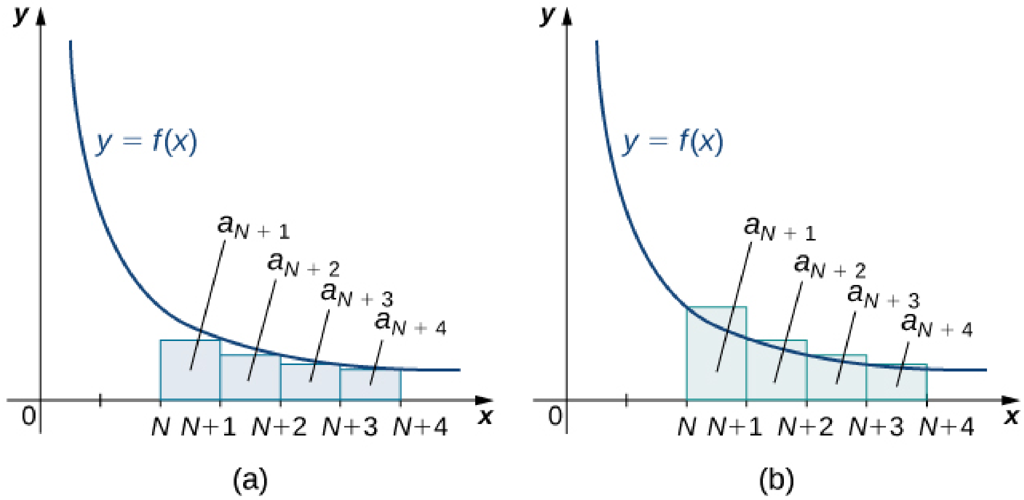
Equation:

$$\sum_{n=1}^{\infty} a_n = S_N + R_N,$$

where S_N is the $N\text{th}$ partial sum, we conclude that

Equation:

$$S_N + \int_{N+1}^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < S_N + \int_N^{\infty} f(x)dx.$$



Given a continuous, positive, decreasing function f and a sequence of positive terms a_n such that $a_n = f(n)$ for all positive integers n, (a) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^{\infty} f(x)dx$, or (b) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)dx$. Therefore, the integral is either an overestimate or an underestimate of the error.

Example:
Exercise:

Problem:
Estimating the Value of a Series

Consider the series $\sum_{n=1}^{\infty} 1/n^3$.

a. Calculate $S_{10} = \sum_{n=1}^{10} 1/n^3$ and estimate the error.
b. Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^{\infty} 1/n^3$ to within 0.001.

Solution:

a. Using a calculating utility, we have
Equation:

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.19753.$$

By the remainder estimate, we know
Equation:

$$R_N < \int_N^\infty \frac{1}{x^3} dx.$$

We have
Equation:

$$\int_{10}^\infty \frac{1}{x^3} dx = \lim_{b\to\infty} \int_{10}^b \frac{1}{x^3} dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_N^b = \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2N^2}\right] = \frac{1}{2N^2}.$$

Therefore, the error is $R_{10} < 1/2(10)^2 = 0.005$.

b. Find N such that $R_N < 0.001$. In part a. we showed that $R_N < 1/2N^2$. Therefore, the remainder $R_N < 0.001$ as long as $1/2N^2 < 0.001$. That is, we need $2N^2 > 1000$. Solving this inequality for N, we see that we need $N > 22.36$. To ensure that the remainder is within the desired amount, we need to round up to the nearest integer. Therefore, the minimum necessary value is $N = 23$.

Note:
Exercise:

Problem: For $\sum_{n=1}^\infty \frac{1}{n^4}$, calculate S_5 and estimate the error R_5.

Solution:

$S_5 \approx 1.09035$, $R_5 < 0.00267$

Hint

Use the remainder estimate $R_N < \int_N^\infty 1/x^4 dx$.

Key Concepts

- If $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\lim\limits_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.
- If $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n and f is a continuous, decreasing function such that $f(n) = a_n$ for all positive integers n, then
  Equation:

  $$\sum_{n=1}^{\infty} a_n \text{and} \int_1^{\infty} f(x)dx$$

  either both converge or both diverge. Furthermore, if $\sum_{n=1}^{\infty} a_n$ converges, then the $N\text{th}$ partial sum approximation S_N is accurate up to an error R_N where $\int_{N+1}^{\infty} f(x)dx < R_N < \int_N^{\infty} f(x)dx$.
- The p-series $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.

Key Equations

- **Divergence test**

  If $a_n \nrightarrow 0$ as $n \to \infty$, $\sum\limits_{n=1}^{\infty} a_n$ diverges.
- ***p*-series**
  $\sum\limits_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$
- **Remainder estimate from the integral test**
  $\int_{N+1}^{\infty} f(x)dx < R_N < \int_N^{\infty} f(x)dx$

For each of the following sequences, if the divergence test applies, either state that $\lim\limits_{n\to\infty} a_n$ does not exist or find $\lim\limits_{n\to\infty} a_n$. If the divergence test does not apply, state why.

Exercise:

Problem: $a_n = \frac{n}{n+2}$

Exercise:

Problem: $a_n = \frac{n}{5n^2-3}$

Solution:

$\lim_{n\to\infty} a_n = 0$. Divergence test does not apply.

Exercise:

Problem: $a_n = \frac{n}{\sqrt{3n^2+2n+1}}$

Exercise:

Problem: $a_n = \frac{(2n+1)(n-1)}{(n+1)^2}$

Solution:

$\lim_{n\to\infty} a_n = 2$. Series diverges.

Exercise:

Problem: $a_n = \frac{(2n+1)^{2n}}{(3n^2+1)^n}$

Exercise:

Problem: $a_n = \frac{2^n}{3^{n/2}}$

Solution:

$\lim_{n\to\infty} a_n = \infty$ (does not exist). Series diverges.

Exercise:

Problem: $a_n = \frac{2^n+3^n}{10^{n/2}}$

Exercise:

Problem: $a_n = e^{-2/n}$

Solution:

$\lim_{n\to\infty} a_n = 1$. Series diverges.

Exercise:

Problem: $a_n = \cos n$

Exercise:

Problem: $a_n = \tan n$

Solution:

$\lim\limits_{n\to\infty} a_n$ does not exist. Series diverges.

Exercise:

Problem: $a_n = \frac{1-\cos^2(1/n)}{\sin^2(2/n)}$

Exercise:

Problem: $a_n = \left(1 - \frac{1}{n}\right)^{2n}$

Solution:

$\lim\limits_{n\to\infty} a_n = 1/e^2$. Series diverges.

Exercise:

Problem: $a_n = \frac{\ln n}{n}$

Exercise:

Problem: $a_n = \frac{(\ln n)^2}{\sqrt{n}}$

Solution:

$\lim\limits_{n\to\infty} a_n = 0$. Divergence test does not apply.

State whether the given p-series converges.

Exercise:

Problem: $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Exercise:

Problem: $\sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Solution:

Series converges, $p > 1$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

Solution:

Series converges, $p = 4/3 > 1$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{n^\pi}{n^{2e}}$

Solution:

Series converges, $p = 2e - \pi > 1$.

Use the integral test to determine whether the following sums converge.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$

Solution:

Series diverges by comparison with $\int_1^\infty \frac{dx}{(x+5)^{1/3}}$.

Exercise:

Problem: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Solution:

Series diverges by comparison with $\int_1^\infty \frac{x}{1+x^2} dx$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$

Solution:

Series converges by comparison with $\int_1^\infty \frac{2x}{1+x^4} dx$.

Exercise:

Problem: $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Express the following sums as p-series and determine whether each converges.

Exercise:

Problem: $\sum_{n=1}^{\infty} 2^{-\ln n}$ (*Hint:* $2^{-\ln n} = 1/n^{\ln 2}$.)

Solution:

$2^{-\ln n} = 1/n^{\ln 2}$. Since $\ln 2 < 1$, diverges by p-series.

Exercise:

Problem: $\sum_{n=1}^{\infty} 3^{-\ln n}$ (*Hint:* $3^{-\ln n} = 1/n^{\ln 3}$.)

Exercise:

Problem: $\sum_{n=1}^{\infty} n 2^{-2\ln n}$

Solution:

$2^{-2\ln n} = 1/n^{2\ln 2}$. Since $2\ln 2 - 1 < 1$, diverges by p-series.

Exercise:

Problem: $\sum_{n=1}^{\infty} n 3^{-2\ln n}$

Use the estimate $R_N \leq \int_N^{\infty} f(t)dt$ to find a bound for the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ where $a_n = f(n)$.

Exercise:

Problem: $\sum_{n=1}^{1000} \frac{1}{n^2}$

Solution:

$$R_{1000} \leq \int_{1000}^{\infty} \frac{dt}{t^2} = -\frac{1}{t}\bigg|_{1000}^{\infty} = 0.001$$

Exercise:

Problem: $\sum_{n=1}^{1000} \frac{1}{n^3}$

Exercise:

Problem: $\sum_{n=1}^{1000} \frac{1}{1+n^2}$

Solution:

$$R_{1000} \leq \int_{1000}^{\infty} \frac{dt}{1+t^2} = \tan^{-1}\infty - \tan^{-1}(1000) = \pi/2 - \tan^{-1}(1000) \approx 0.000999$$

Exercise:

Problem: $\sum_{n=1}^{100} n/2^n$

[T] Find the minimum value of N such that the remainder estimate $\int_{N+1}^{\infty} f < R_N < \int_N^{\infty} f$ guarantees that $\sum_{n=1}^{N} a_n$ estimates $\sum_{n=1}^{\infty} a_n$, accurate to within the given error.

Exercise:

Problem: $a_n = \frac{1}{n^2}$, error $< 10^{-4}$

Solution:

$$R_N < \int_N^{\infty} \frac{dx}{x^2} = 1/N, N > 10^4$$

Exercise:

Problem: $a_n = \frac{1}{n^{1.1}}$, error $< 10^{-4}$

Exercise:

Problem: $a_n = \frac{1}{n^{1.01}}$, error $< 10^{-4}$

Solution:

$$R_N < \int_N^{\infty} \frac{dx}{x^{1.01}} = 100N^{-0.01}, N > 10^{600}$$

Exercise:

Problem: $a_n = \frac{1}{n \ln^2 n}$, error $< 10^{-3}$

Exercise:

Problem: $a_n = \frac{1}{1+n^2}$, error $< 10^{-3}$

Solution:

$$R_N < \int_N^\infty \frac{dx}{1+x^2} = \pi/2 - \tan^{-1}(N), N > \tan\left(\pi/2 - 10^{-3}\right) \approx 1000$$

In the following exercises, find a value of N such that R_N is smaller than the desired error. Compute the corresponding sum $\sum_{n=1}^{N} a_n$ and compare it to the given estimate of the infinite series.

Exercise:

Problem: $a_n = \frac{1}{n^{11}}$, error $< 10^{-4}$, $\sum_{n=1}^{\infty} \frac{1}{n^{11}} = 1.000494\ldots$

Exercise:

Problem: $a_n = \frac{1}{e^n}$, error $< 10^{-5}$, $\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} = 0.581976\ldots$

Solution:

$$R_N < \int_N^\infty \frac{dx}{e^x} = e^{-N}, N > 5\ln(10), \text{ okay if } N = 12; \sum_{n=1}^{12} e^{-n} = 0.581973....$$

Estimate agrees with $1/(e-1)$ to five decimal places.

Exercise:

Problem: $a_n = \frac{1}{e^{n^2}}$, error $< 10^{-5}$, $\sum_{n=1}^{\infty} n/e^{n2} = 0.40488139857\ldots$

Exercise:

Problem: $a_n = 1/n^4$, error $< 10^{-4}$, $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90 = 1.08232...$

Solution:

$R_N < \int_N^\infty dx/x^4 = 4/N^3, N > \left(4.10^4\right)^{1/3}$, okay if $N = 35$;

$\sum_{n=1}^{35} 1/n^4 = 1.08231\ldots$. Estimate agrees with the sum to four decimal places.

Exercise:

Problem: $a_n = 1/n^6$, error $< 10^{-6}$, $\sum_{n=1}^{\infty} 1/n^4 = \pi^6/945 = 1.01734306...$,

Exercise:

Problem:

Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$. (*Hint:* Compare to $\int_n^{2n} \frac{1}{t} dt$.)

Solution:

$\ln(2)$

Exercise:

Problem: Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}$

The next few exercises are intended to give a sense of applications in which partial sums of the harmonic series arise.

Exercise:

Problem:

In certain applications of probability, such as the so-called Watterson estimator for predicting mutation rates in population genetics, it is important to have an accurate estimate of the number $H_k = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right)$. Recall that $T_k = H_k - \ln k$ is decreasing. Compute $T = \lim_{k\to\infty} T_k$ to four decimal places. (*Hint:* $\frac{1}{k+1} < \int_k^{k+1} \frac{1}{x} dx$.)

Solution:

$T = 0.5772...$

Exercise:

Problem:

[T] Complete sampling with replacement, sometimes called the *coupon collector's problem*, is phrased as follows: Suppose you have N unique items in a bin. At each step, an item is chosen at random, identified, and put back in the bin. The problem asks what is the expected number of steps $E(N)$ that it takes to draw each unique item at least once. It turns out that $E(N) = N. H_N = N\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right)$. Find $E(N)$ for $N = 10, 20, \text{and } 50$.

Exercise:

Problem:

[T] The simplest way to shuffle cards is to take the top card and insert it at a random place in the deck, called top random insertion, and then repeat. We will consider a deck to be randomly shuffled once enough top random insertions have been made that the card originally at the bottom has reached the top and then been randomly inserted. If the deck has n cards, then the probability that the insertion will be below the card initially at the bottom (call this card B) is $1/n$. Thus the expected number of top random insertions before B is no longer at the bottom is n. Once one card is below B, there are two places below B and the probability that a randomly inserted card will fall below B is $2/n$. The expected number of top random insertions before this happens is $n/2$. The two cards below B are now in random order. Continuing this way, find a formula for the expected number of top random insertions needed to consider the deck to be randomly shuffled.

Solution:

The expected number of random insertions to get B to the top is $n + n/2 + n/3 + \cdots + n/(n-1)$. Then one more insertion puts B back in at random. Thus, the expected number of shuffles to randomize the deck is $n(1 + 1/2 + \cdots + 1/n)$.

Exercise:

Problem:

Suppose a scooter can travel 100 km on a full tank of fuel. Assuming that fuel can be transferred from one scooter to another but can only be carried in the tank, present a procedure that will enable one of the scooters to travel $100H_N$ km, where $H_N = 1 + 1/2 + \cdots + 1/N$.

Exercise:

Problem:

Show that for the remainder estimate to apply on $[N, \infty)$ it is sufficient that $f(x)$ be decreasing on $[N, \infty)$, but f need not be decreasing on $[1, \infty)$.

Solution:

Set $b_n = a_{n+N}$ and $g(t) = f(t+N)$ such that f is decreasing on $[t, \infty)$.

Exercise:

Problem:

[T] Use the remainder estimate and integration by parts to approximate $\sum_{n=1}^{\infty} n/e^n$ within an error smaller than 0.0001.

Exercise:

Problem: Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge if p is large enough? If so, for which p?

Solution:

The series converges for $p > 1$ by integral test using change of variable.

Exercise:

Problem:

[T] Suppose a computer can sum one million terms per second of the divergent series $\sum_{n=1}^{N} \frac{1}{n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

Exercise:

Problem:

[T] A fast computer can sum one million terms per second of the divergent series $\sum_{n=2}^{N} \frac{1}{n \ln n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

Solution:

$N = e^{e^{100}} \approx e^{10^{43}}$ terms are needed.

Glossary

divergence test

if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges

integral test

for a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n, if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, then

Equation:

$$\sum_{n=1}^{\infty} a_n \text{and} \int_1^{\infty} f(x)dx$$

either both converge or both diverge

p-series

a series of the form $\sum_{n=1}^{\infty} 1/n^p$

remainder estimate

for a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n and a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate:

Equation:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_N^{\infty} f(x)dx$$

Comparison Tests

- Use the comparison test to test a series for convergence.
- Use the limit comparison test to determine convergence of a series.

We have seen that the integral test allows us to determine the convergence or divergence of a series by comparing it to a related improper integral. In this section, we show how to use comparison tests to determine the convergence or divergence of a series by comparing it to a series whose convergence or divergence is known. Typically these tests are used to determine convergence of series that are similar to geometric series or *p*-series.

Comparison Test

In the preceding two sections, we discussed two large classes of series: geometric series and *p*-series. We know exactly when these series converge and when they diverge. Here we show how to use the convergence or divergence of these series to prove convergence or divergence for other series, using a method called the **comparison test**.

For example, consider the series
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

This series looks similar to the convergent series
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the terms in each of the series are positive, the sequence of partial sums for each series is monotone increasing. Furthermore, since
Equation:

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for all positive integers n, the $k\text{th}$ partial sum S_k of $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ satisfies
Equation:

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(See [link](a) and [link].) Since the series on the right converges, the sequence $\{S_k\}$ is bounded above. We conclude that $\{S_k\}$ is a monotone increasing sequence that is bounded above. Therefore, by the Monotone Convergence Theorem, $\{S_k\}$ converges, and thus
Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges.

Similarly, consider the series

Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}.$$

This series looks similar to the divergent series

Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums for each series is monotone increasing and

Equation:

$$\frac{1}{n-1/2} > \frac{1}{n} > 0$$

for every positive integer n. Therefore, the $k\text{th}$ partial sum S_k of $\sum_{n=1}^{\infty} \frac{1}{n-1/2}$ satisfies

Equation:

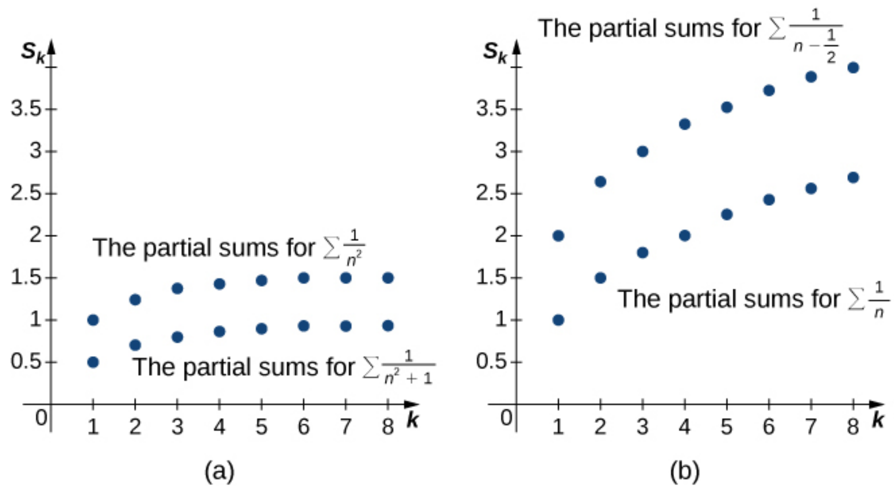
$$S_k = \sum_{n=1}^{k} \frac{1}{n-1/2} > \sum_{n=1}^{k} \frac{1}{n}.$$

(See [link](b) and [link].) Since the series $\sum_{n=1}^{\infty} 1/n$ diverges to infinity, the sequence of partial sums $\sum_{n=1}^{k} 1/n$ is unbounded. Consequently, $\{S_k\}$ is an unbounded sequence, and therefore diverges. We conclude that

Equation:

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}$$

diverges.



(a) Each of the partial sums for the given series is less than the corresponding partial sum for the converging $p-$ series. (b) Each of the partial sums for the given series is greater than the corresponding partial sum for the diverging harmonic series.

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n^2+1}$	0.5	0.7	0.8	0.8588	0.8973	0.9243	0.9443	0.9597
$\sum_{n=1}^{k} \frac{1}{n^2}$	1	1.25	1.3611	1.4236	1.4636	1.4914	1.5118	1.5274

Comparing a series with a p-series (p = 2)

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n-1/2}$	2	2.6667	3.0667	3.3524	3.5746	3.7564	3.9103	4.0436
$\sum_{n=1}^{k} \frac{1}{n}$	1	1.5	1.8333	2.0933	2.2833	2.45	2.5929	2.7179

Comparing a series with the harmonic series

Note:
Comparison Test

i. Suppose there exists an integer N such that $0 \leq a_n \leq b_n$ for all $n \geq N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
ii. Suppose there exists an integer N such that $a_n \geq b_n \geq 0$ for all $n \geq N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

We prove part i. The proof of part ii. is the contrapositive of part i. Let $\{S_k\}$ be the sequence of partial sums associated with $\sum_{n=1}^{\infty} a_n$, and let $L = \sum_{n=1}^{\infty} b_n$. Since the terms $a_n \geq 0$,
Equation:

$$S_k = a_1 + a_2 + \cdots + a_k \leq a_1 + a_2 + \cdots + a_k + a_{k+1} = S_{k+1}.$$

Therefore, the sequence of partial sums is increasing. Further, since $a_n \leq b_n$ for all $n \geq N$, then
Equation:

$$\sum_{n=N}^{k} a_n \leq \sum_{n=N}^{k} b_n \leq \sum_{n=1}^{\infty} b_n = L.$$

Therefore, for all $k \geq 1$,
Equation:

$$S_k = (a_1 + a_2 + \cdots + a_{N-1}) + \sum_{n=N}^{k} a_n \leq (a_1 + a_2 + \cdots + a_{N-1}) + L.$$

Since $a_1 + a_2 + \cdots + a_{N-1}$ is a finite number, we conclude that the sequence $\{S_k\}$ is bounded above. Therefore, $\{S_k\}$ is an increasing sequence that is bounded above. By the Monotone Convergence Theorem, we conclude that $\{S_k\}$ converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges.

□

To use the comparison test to determine the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$, it is necessary to find a suitable series with which to compare it. Since we know the convergence properties of geometric series and *p*-series, these series are often used. If there exists an integer N such that for all $n \geq N$, each term a_n is less than each corresponding term of a known convergent series, then $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if there exists an integer N such that for all $n \geq N$, each term a_n is greater than each corresponding term of a known divergent series, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example:
Exercise:

Problem:
Using the Comparison Test

For each of the following series, use the comparison test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$
b. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$
c. $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

Solution:

a. Compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$ Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3$, it converges. Further,
Equation:

$$\frac{1}{n^3 + 3n + 1} < \frac{1}{n^3}$$

for every positive integer n. Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges.

b. Compare to $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $r = 1/2$ and $|1/2| < 1$, it converges. Also,
Equation:

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

for every positive integer n. Therefore, we see that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

c. Compare to $\sum_{n=2}^{\infty} \frac{1}{n}$. Since
Equation:

$$\frac{1}{\ln(n)} > \frac{1}{n}$$

for every integer $n \geq 2$ and $\sum_{n=2}^{\infty} 1/n$ diverges, we have that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges.

Note:
Exercise:

Problem: Use the comparison test to determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^3+n+1}$ converges or diverges.

Solution:

The series converges.

Hint

Find a value p such that $\frac{n}{n^3+n+1} \leq \frac{1}{n^p}$.

Limit Comparison Test

The comparison test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult. Consider the series

Equation:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

It is natural to compare this series with the convergent series

Equation:

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

However, this series does not satisfy the hypothesis necessary to use the comparison test because

Equation:

$$\frac{1}{n^2-1} > \frac{1}{n^2}$$

for all integers $n \geq 2$. Although we could look for a different series with which to compare $\sum_{n=2}^{\infty} 1/(n^2-1)$, instead we show how we can use the **limit comparison test** to compare

Equation:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \text{ and } \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Let us examine the idea behind the limit comparison test. Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. with positive terms a_n and b_n and evaluate

Equation:

$$\lim_{n \to \infty} \frac{a_n}{b_n}.$$

If

Equation:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0,$$

then, for n sufficiently large, $a_n \approx Lb_n$. Therefore, either both series converge or both series diverge. For the series $\sum_{n=2}^{\infty} 1/(n^2-1)$ and $\sum_{n=2}^{\infty} 1/n^2$, we see that
Equation:

$$\lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1.$$

Since $\sum_{n=2}^{\infty} 1/n^2$ converges, we conclude that
Equation:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

converges.

The limit comparison test can be used in two other cases. Suppose
Equation:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

In this case, $\{a_n/b_n\}$ is a bounded sequence. As a result, there exists a constant M such that $a_n \le Mb_n$. Therefore, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, suppose
Equation:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty.$$

In this case, $\{a_n/b_n\}$ is an unbounded sequence. Therefore, for every constant M there exists an integer N such that $a_n \ge Mb_n$ for all $n \ge N$. Therefore, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.

Note:
Limit Comparison Test
Let $a_n,\ b_n \ge 0$ for all $n \ge 1$.

i. If $\lim\limits_{n\to\infty} a_n/b_n = L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
ii. If $\lim\limits_{n\to\infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
iii. If $\lim\limits_{n\to\infty} a_n/b_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note that if $a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $a_n/b_n \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information. For example, consider the two series $\sum\limits_{n=1}^{\infty} 1/\sqrt{n}$ and $\sum\limits_{n=1}^{\infty} 1/n^2$. These series are both *p*-series with $p = 1/2$ and $p = 2$, respectively. Since

$p = 1/2 > 1$, the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges. On the other hand, since $p = 2 < 1$, the series $\sum_{n=1}^{\infty} 1/n^2$ converges. However, suppose we attempted to apply the limit comparison test, using the convergent $p-$series $\sum_{n=1}^{\infty} 1/n^3$ as our comparison series. First, we see that

Equation:

$$\frac{1/\sqrt{n}}{1/n^3} = \frac{n^3}{\sqrt{n}} = n^{5/2} \to \infty \text{ as } n \to \infty.$$

Similarly, we see that

Equation:

$$\frac{1/n^2}{1/n^3} = n \to \infty \text{ as } n \to \infty.$$

Therefore, if $a_n/b_n \to \infty$ when $\sum_{n=1}^{\infty} b_n$ converges, we do not gain any information on the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Example:

Exercise:

Problem:

Using the Limit Comparison Test

For each of the following series, use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$
b. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$
c. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Solution:

a. Compare this series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Calculate

Equation:

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

b. Compare this series to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$. We see that

Equation:

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} \left[1+\left(\frac{1}{2}\right)^n\right] = 1.$$

Therefore,
Equation:

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = 1.$$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges.

c. Since $\ln n < n$, compare with $\sum_{n=1}^{\infty} \frac{1}{n}$. We see that
Equation:

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\ln n}{n}.$$

In order to evaluate $\lim\limits_{n\to\infty} \ln n/n$, evaluate the limit as $x \to \infty$ of the real-valued function $\ln(x)/x$. These two limits are equal, and making this change allows us to use L'Hôpital's rule. We obtain
Equation:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore, $\lim\limits_{n\to\infty} \ln n/n = 0$, and, consequently,
Equation:

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n} = 0.$$

Since the limit is 0 but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit comparison test does not provide any information. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ instead. In this case,
Equation:

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n^2} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \ln n = \infty.$$

Since the limit is ∞ but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the test still does not provide any information.
So now we try a series between the two we already tried. Choosing the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, we see that
Equation:

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}.$$

As above, in order to evaluate $\lim\limits_{n\to\infty} \ln n/\sqrt{n}$, evaluate the limit as $x \to \infty$ of the real-valued function

$\ln x/\sqrt{x}$. Using L'Hôpital's rule,
Equation:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Since the limit is 0 and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we can conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

Note:
Exercise:

Problem:

Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{5^n}{3^n+2}$ converges or diverges.

Solution:

The series diverges.

Hint

Compare with a geometric series.

Key Concepts

- The comparison tests are used to determine convergence or divergence of series with positive terms.
- When using the comparison tests, a series $\sum_{n=1}^{\infty} a_n$ is often compared to a geometric or *p*-series.

Use the comparison test to determine whether the following series converge.
Exercise:

Problem: $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{2}{n(n+1)}$

Exercise:

Problem: $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n(n+1/2)}$

Solution:

Converges by comparison with $1/n^2$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{2(n+1)}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Solution:

Diverges by comparison with harmonic series, since $2n-1 \geq n$.

Exercise:

Problem: $\sum_{n=2}^{\infty} \frac{1}{(n \ln n)^2}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$

Solution:

$a_n = 1/\left(n+1\right)\left(n+2\right) < 1/n^2$. Converges by comparison with p-series, $p=2$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{n!}$

Exercise:

Problem: $$\sum_{n=1}^{\infty} \frac{\sin\left(1/n\right)}{n}$$

Solution:

$\sin\left(1/n\right) \leq 1/n$, so converges by comparison with p-series, $p=2$.

Exercise:

Problem: $$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$$

Exercise:

Problem: $$\sum_{n=1}^{\infty} \frac{\sin\left(1/n\right)}{\sqrt{n}}$$

Solution:

$\sin\left(1/n\right) \leq 1$, so converges by comparison with p-series, $p=3/2$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{n^{1.2}-1}{n^{2.3}+1}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

Solution:

Since $\sqrt{n+1}-\sqrt{n}=1/\left(\sqrt{n+1}+\sqrt{n}\right)\leq 2/\sqrt{n}$, series converges by comparison with p-series for $p=1.5$.

Exercise:

Problem: $\sum_{n=1}^{\infty}\frac{\sqrt[4]{n}}{\sqrt[3]{n^4+n^2}}$

Use the limit comparison test to determine whether each of the following series converges or diverges.

Exercise:

Problem: $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^2$

Solution:

Converges by limit comparison with p-series for $p>1$.

Exercise:

Problem: $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^{0.6}}\right)^2$

Exercise:

Problem: $\sum_{n=1}^{\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{n}$

Solution:

Converges by limit comparison with p-series, $p=2$.

Exercise:

Problem: $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2}\right)$

Exercise:

Problem: $\sum_{n=1}^{\infty}\frac{1}{4^n-3^n}$

Solution:

Converges by limit comparison with 4^{-n}.

Exercise:

Problem: $\sum_{n=1}^{\infty}\frac{1}{n^2-n\sin n}$

Exercise:

Problem: $\sum_{n=1}^{\infty}\frac{1}{e^{(1.1)n}-3^n}$

Solution:

Converges by limit comparison with $1/e^{1.1n}$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{e^{(1.01)n} - 3^n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Solution:

Diverges by limit comparison with harmonic series.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{2^{1+1/n} n^{1+1/n}}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)$

Solution:

Converges by limit comparison with p-series, $p = 3$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{n}\left(\tan^{-1} n - \frac{\pi}{2}\right)$

Solution:

Converges by limit comparison with p-series, $p = 3$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n.n}$ $(Hint: \left(1 - \frac{1}{n}\right)^n \to 1/e.)$

Exercise:

Problem: $\sum_{n=1}^{\infty} \left(1 - e^{-1/n}\right)$ $(Hint: 1/e \approx (1 - 1/n)^n$, so $1 - e^{-1/n} \approx 1/n.)$

Solution:

Diverges by limit comparison with $1/n$.

Exercise:

Problem: Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ converge if p is large enough? If so, for which p?

Exercise:

Problem: Does $\sum_{n=1}^{\infty}\left(\frac{(\ln n)}{n}\right)^p$ converge if p is large enough? If so, for which p?

Solution:

Converges for $p > 1$ by comparison with a p series for slightly smaller p.

Exercise:

Problem: For which p does the series $\sum_{n=1}^{\infty} 2^{pn}/3^n$ converge?

Exercise:

Problem: For which $p > 0$ does the series $\sum_{n=1}^{\infty} \frac{n^p}{2^n}$ converge?

Solution:

Converges for all $p > 0$.

Exercise:

Problem: For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{r^{n^2}}{2^n}$ converge?

Exercise:

Problem: For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{2^n}{r^{n^2}}$ converge?

Solution:

Converges for all $r > 1$. If $r > 1$ then $r^n > 4$, say, once $n > \ln(2)/\ln(r)$ and then the series converges by limit comparison with a geometric series with ratio $1/2$.

Exercise:

Problem: Find all values of p and q such that $\sum_{n=1}^{\infty} \frac{n^p}{(n!)^q}$ converges.

Exercise:

Problem: Does $\sum_{n=1}^{\infty} \frac{\sin^2(nr/2)}{n}$ converge or diverge? Explain.

Solution:

The numerator is equal to 1 when n is odd and 0 when n is even, so the series can be rewritten $\sum_{n=1}^{\infty} \frac{1}{2n+1}$, which diverges by limit comparison with the harmonic series.

Exercise:

Problem:

Explain why, for each n, at least one of $\{|\sin n|, |\sin(n+1)|, \ldots, |\sin n + 6|\}$ is larger than $1/2$. Use this relation to test convergence of $\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n}}$.

Exercise:

Problem:

Suppose that $a_n \geq 0$ and $b_n \geq 0$ and that $\sum_{n=1}^{\infty} a^2{}_n$ and $\sum_{n=1}^{\infty} b^2{}_n$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges and $\sum_{n=1}^{\infty} a_n\, b_n \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$.

Solution:

$(a-b)^2 = a^2 - 2ab + b^2$ or $a^2 + b^2 \geq 2ab$, so convergence follows from comparison of $2a_n b_n$ with $a^2{}_n + b^2{}_n$. Since the partial sums on the left are bounded by those on the right, the inequality holds for the infinite series.

Exercise:

Problem: Does $\sum_{n=1}^{\infty} 2^{-\ln \ln n}$ converge? (*Hint:* Write $2^{\ln \ln n}$ as a power of $\ln n$.)

Exercise:

Problem: Does $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ converge? (*Hint:* Use $t = e^{\ln(t)}$ to compare to a $p -$ series.)

Solution:

$(\ln n)^{-\ln n} = e^{-\ln(n)\ln\ln(n)}$. If n is sufficiently large, then $\ln \ln n > 2$, so $(\ln n)^{-\ln n} < 1/n^2$, and the series converges by comparison to a $p -$ series.

Exercise:

Problem: Does $\sum_{n=2}^{\infty} (\ln n)^{-\ln \ln n}$ converge? (*Hint:* Compare a_n to $1/n$.)

Exercise:

Problem:

Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a^2{}_n$ converges. If $\sum_{n=1}^{\infty} a^2{}_n$ converges, does $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Solution:

$a_n \to 0$, so $a^2{}_n \leq |a_n|$ for large n. Convergence follows from limit comparison. $\sum 1/n^2$ converges, but $\sum 1/n$ does not, so the fact that $\sum_{n=1}^{\infty} a^2{}_n$ converges does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

Exercise:

Problem:

Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of zeros and ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

Exercise:

Problem:

Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that b_n is an arbitrary sequence of zeros and ones with infinitely many terms equal to one. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily diverge?

Solution:

No. $\sum_{n=1}^{\infty} 1/n$ diverges. Let $b_k = 0$ unless $k = n^2$ for some n. Then $\sum_k b_k/k = \sum 1/k^2$ converges.

Exercise:

Problem:

Complete the details of the following argument: If $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to a finite sum s, then $\frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ and $s - \frac{1}{2}s = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$. Why does this lead to a contradiction?

Exercise:

Problem: Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a^2{}_n$ converges, then $\sum_{n=1}^{\infty} \sin^2(a_n)$ converges.

Solution:

$|\sin t| \leq |t|$, so the result follows from the comparison test.

Exercise:

Problem:

Suppose that $a_n/b_n \to 0$ in the comparison test, where $a_n \geq 0$ and $b_n \geq 0$. Prove that if $\sum b_n$ converges, then $\sum a_n$ converges.

Exercise:

Problem:

Let b_n be an infinite sequence of zeros and ones. What is the largest possible value of $x = \sum_{n=1}^{\infty} b_n/2^n$?

Solution:

By the comparison test, $x = \sum_{n=1}^{\infty} b_n/2^n \leq \sum_{n=1}^{\infty} 1/2^n = 1.$

Exercise:

Problem:

Let d_n be an infinite sequence of digits, meaning d_n takes values in $\{0,\ 1, \ldots, 9\}$. What is the largest possible value of $x = \sum_{n=1}^{\infty} d_n/10^n$ that converges?

Exercise:

Problem: Explain why, if $x > 1/2$, then x cannot be written $x = \sum_{n=2}^{\infty} \frac{b_n}{2^n} (b_n = 0 \text{ or } 1,\ b_1 = 0)$.

Solution:

If $b_1 = 0$, then, by comparison, $x \leq \sum_{n=2}^{\infty} 1/2^n = 1/2$.

Exercise:

Problem:

[T] Evelyn has a perfect balancing scale, an unlimited number of 1 -kg weights, and one each of $1/2$ -kg, $1/4$ -kg, $1/8$ -kg, and so on weights. She wishes to weigh a meteorite of unspecified origin to arbitrary precision. Assuming the scale is big enough, can she do it? What does this have to do with infinite series?

Exercise:

Problem:

[T] Robert wants to know his body mass to arbitrary precision. He has a big balancing scale that works perfectly, an unlimited collection of 1 -kg weights, and nine each of 0.1 -kg, 0.01 -kg, 0.001 -kg, and so on weights. Assuming the scale is big enough, can he do this? What does this have to do with infinite series?

Solution:

Yes. Keep adding 1 -kg weights until the balance tips to the side with the weights. If it balances perfectly, with Robert standing on the other side, stop. Otherwise, remove one of the 1 -kg weights, and add 0.1 -kg weights one at a time. If it balances after adding some of these, stop. Otherwise if it tips to the weights, remove the last 0.1 -kg weight. Start adding 0.01 -kg weights. If it balances, stop. If it tips to the side with the weights, remove the last 0.01 -kg weight that was added. Continue in this way for the 0.001 -kg weights, and so on. After a finite number of steps, one has a finite series of the form $A + \sum_{n=1}^{N} s_n/10^n$ where A is the number of full kg weights and d_n is the number of $1/10^n$ -kg weights that were added. If at some state this series is Robert's exact weight, the process will stop. Otherwise it represents the $N\text{th}$ partial sum of an infinite series that gives Robert's exact weight, and the error of this sum is at most $1/10^N$.

Exercise:

Problem:

The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is half the harmonic series and hence diverges. It is obtained from the harmonic series by deleting all terms in which n is odd. Let $m > 1$ be fixed. Show, more generally, that deleting all terms $1/n$ where $n = mk$ for some integer k also results in a divergent series.

Exercise:

Problem:

In view of the previous exercise, it may be surprising that a subseries of the harmonic series in which about one in every five terms is deleted might converge. A *depleted harmonic series* is a series obtained from $\sum_{n=1}^{\infty} \frac{1}{n}$ by removing any term $1/n$ if a given digit, say 9, appears in the decimal expansion of n. Argue that this depleted harmonic series converges by answering the following questions.

a. How many whole numbers n have d digits?
b. How many d-digit whole numbers $h(d)$. do not contain 9 as one or more of their digits?
c. What is the smallest d-digit number $m(d)$?
d. Explain why the deleted harmonic series is bounded by $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$.
e. Show that $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$ converges.

Solution:

a. $10^d - 10^{d-1} < 10^d$ b. $h(d) < 9^d$ c. $m(d) = 10^{d-1} + 1$ d. Group the terms in the deleted harmonic series together by number of digits. $h(d)$ bounds the number of terms, and each term is at most $1/m(d)$. $\sum_{d=1}^{\infty} h(d)/m(d) \le \sum_{d=1}^{\infty} 9^d/(10)^{d-1} \le 90$. One can actually use comparison to estimate the value to smaller than 80. The actual value is smaller than 23.

Exercise:

Problem:

Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{n+1} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (*Hint:* S_n is monotone.)

Exercise:

Problem:

Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{(n+1)^2} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (*Hint:*
$S_2 = a_2 + a_1 = a_2 + S_1 = a_2 + 1 = 1 + 1/4 = (1 + 1/4)S_1$,
$S_3 = \frac{1}{3^2} S_2 + S_2 = (1 + 1/9)S_2 = (1 + 1/9)(1 + 1/4)S_1$, etc. Look at $\ln(S_n)$, and use $\ln(1 + t) \le t$, $t > 0$.)

Solution:

Continuing the hint gives $S_N = \left(1 + 1/N^2\right) \left(1 + 1/(N-1)^2 \ldots (1 + 1/4)\right)$. Then $\ln(S_N) = \ln\left(1 + 1/N^2\right) + \ln\left(1 + 1/(N-1)^2\right) + \cdots + \ln(1 + 1/4)$. Since $\ln(1 + t)$ is bounded by a constant times t, when $0 < t < 1$ one has $\ln(S_N) \leq C \sum_{n=1}^{N} \frac{1}{n^2}$, which converges by comparison to the p-series for $p = 2$.

Glossary

comparison test
: if $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \geq b_n \geq 0$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

limit comparison test
: suppose $a_n,\ b_n \geq 0$ for all $n \geq 1$. If $\lim_{n\to\infty} a_n/b_n \to L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge; if $\lim_{n\to\infty} a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n\to\infty} a_n/b_n \to \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Alternating Series

- Use the alternating series test to test an alternating series for convergence.
- Estimate the sum of an alternating series.
- Explain the meaning of absolute convergence and conditional convergence.

So far in this chapter, we have primarily discussed series with positive terms. In this section we introduce alternating series—those series whose terms alternate in sign. We will show in a later chapter that these series often arise when studying power series. After defining alternating series, we introduce the alternating series test to determine whether such a series converges.

The Alternating Series Test

A series whose terms alternate between positive and negative values is an **alternating series**. For example, the series
Equation:

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n=-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$$

and
Equation:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

are both alternating series.

Note:
Definition
Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form
Equation:

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n=b_1-b_2+b_3-b_4+\cdots$$

or
Equation:

$$\sum_{n-1}^{\infty}(-1)^{n}b_n=-b_1+b_2-b_3+b_4-\cdots$$

Where $b_n \geq 0$ for all positive integers *n*.

Series (1), shown in [link], is a geometric series. Since $|r| = |-1/2| < 1$, the series converges. Series (2), shown in [link], is called the alternating harmonic series. We will show that whereas the harmonic series diverges, the alternating harmonic series converges.

To prove this, we look at the sequence of partial sums $\{S_k\}$ ([link]).

Proof

Consider the odd terms S_{2k+1} for $k \geq 0$. Since $1/(2k+1) < 1/2k$,

Equation:

$$S_{2k+1} = S_{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} < S_{2k-1}.$$

Therefore, $\{S_{2k+1}\}$ is a decreasing sequence. Also,

Equation:

$$S_{2k+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \frac{1}{2k+1} > 0.$$

Therefore, $\{S_{2k+1}\}$ is bounded below. Since $\{S_{2k+1}\}$ is a decreasing sequence that is bounded below, by the Monotone Convergence Theorem, $\{S_{2k+1}\}$ converges. Similarly, the even terms $\{S_{2k}\}$ form an increasing sequence that is bounded above because

Equation:

$$S_{2k} = S_{2k-2} + \frac{1}{2k-1} - \frac{1}{2k} > S_{2k-2}$$

and

Equation:

$$S_{2k} = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) - \frac{1}{2k} < 1.$$

Therefore, by the Monotone Convergence Theorem, the sequence $\{S_{2k}\}$ also converges. Since

Equation:

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1},$$

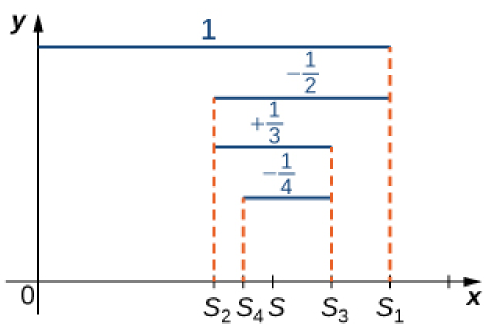
we know that
Equation:

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k+1}.$$

Letting $S = \lim_{k\to\infty} S_{2k+1}$ and using the fact that $1/\,(2k+1) \to 0$, we conclude that $\lim_{k\to\infty} S_{2k} = S$. Since the odd terms and the even terms in the sequence of partial sums converge to the same limit S, it can be shown that the sequence of partial sums converges to S, and therefore the alternating harmonic series converges to S.

It can also be shown that $S = \ln 2$, and we can write
Equation:

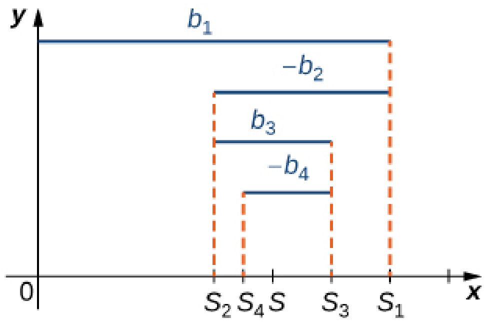
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2).$$



For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

□

More generally, any alternating series of form (3) ([link]) or (4) ([link]) converges as long as $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $b_n \to 0$ ([link]). The proof is similar to the proof for the alternating harmonic series.



For an alternating series $b_1 - b_2 + b_3 - \cdots$ in which $b_1 > b_2 > b_3 > \cdots$, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

Note:
Alternating Series Test
An alternating series of the form
Equation:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ or } \sum_{n=1}^{\infty} (-1)^n b_n$$

converges if

i. $0 \le b_{n+1} \le b_n$ for all $n \ge 1$ and
ii. $\lim\limits_{n \to \infty} b_n = 0$.

This is known as the **alternating series test**.

We remark that this theorem is true more generally as long as there exists some integer N such that $0 \le b_{n+1} \le b_n$ for all $n \ge N$.

Example:
Exercise:

Problem:

Convergence of Alternating Series

For each of the following alternating series, determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$
b. $\sum_{n=1}^{\infty} (-1)^{n+1}n/(n+1)$

Solution:

a. Since
$$\frac{1}{(n+1)^2} < \frac{1}{n^2} \quad \text{and} \quad \frac{1}{n^2} \to 0,$$
the series converges.
b. Since $n/(n+1) \nrightarrow 0$ as $n \to \infty$, we cannot apply the alternating series test. Instead, we use the *n*th term test for divergence. Since
$$\lim_{n\to\infty} \frac{(-1)^{n+1}n}{n+1} \neq 0,$$
the series diverges.

Note:
Exercise:

Problem: Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1}n/2^n$ converges or diverges.

Solution:

The series converges.

Hint

Is $\{n/2^n\}$ decreasing? What is $\lim\limits_{n\to\infty} n/2^n$?

Remainder of an Alternating Series

It is difficult to explicitly calculate the sum of most alternating series, so typically the sum is approximated by using a partial sum. When doing so, we are interested in the amount of error in our approximation. Consider an alternating series

Equation:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

satisfying the hypotheses of the alternating series test. Let S denote the sum of this series and $\{S_k\}$ be the corresponding sequence of partial sums. From [link], we see that for any integer $N \geq 1$, the remainder R_N satisfies

Equation:

$$|R_N| = |S - S_N| \leq |S_{N+1} - S_N| = b_{n+1}.$$

Note:

Remainders in Alternating Series

Consider an alternating series of the form

Equation:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{or} \sum_{n=1}^{\infty} (-1)^{n} b_n$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the $N\text{th}$ partial sum. For any integer $N \geq 1$, the remainder $R_N = S - S_N$ satisfies

Equation:

$$|R_N| \leq b_{N+1}.$$

In other words, if the conditions of the alternating series test apply, then the error in approximating the infinite series by the $N\text{th}$ partial sum S_N is in magnitude at most the size of the next term b_{N+1}.

Example:

Exercise:

Problem:

Estimating the Remainder of an Alternating Series

Consider the alternating series

Equation:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Use the remainder estimate to determine a bound on the error R_{10} if we approximate the sum of the series by the partial sum S_{10}.

Solution:

From the theorem stated above,

$|R_{10}| \leq b_{11} = \frac{1}{11^2} \approx 0.008265.$

Note:
Exercise:

Problem: Find a bound for R_{20} when approximating $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ by S_{20}.

Solution:

0.04762

Hint

$|R_{20}| \leq b_{21}$

Absolute and Conditional Convergence

Consider a series $\sum_{n=1}^{\infty} a_n$ and the related series $\sum_{n=1}^{\infty} |a_n|$. Here we discuss possibilities for the relationship between the convergence of these two series. For example, consider the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$. The series whose terms are the absolute value of these terms is the harmonic series, since $\sum_{n=1}^{\infty} |(-1)^{n+1}/n| = \sum_{n=1}^{\infty} 1/n$. Since the alternating harmonic series converges, but the harmonic series diverges, we say the alternating harmonic series exhibits conditional convergence.

By comparison, consider the series $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$. The series whose terms are the absolute values of the terms of this series is the series $\sum_{n=1}^{\infty} 1/n^2$. Since both of these series converge, we say the series $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$ exhibits absolute convergence.

Note:
Definition
A series $\sum_{n=1}^{\infty} a_n$ exhibits **absolute convergence** if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ exhibits **conditional convergence** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

As shown by the alternating harmonic series, a series $\sum_{n=1}^{\infty} a_n$ may converge, but $\sum_{n=1}^{\infty} |a_n|$ may diverge. In the following theorem, however, we show that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Note:
Absolute Convergence Implies Convergence
If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof

Suppose that $\sum\limits_{n=1}^{\infty} |a_n|$ converges. We show this by using the fact that $a_n = |a_n|$ or $a_n = -|a_n|$ and therefore $|a_n| + a_n = 2|a_n|$ or $|a_n| + a_n = 0$. Therefore, $0 \le |a_n| + a_n \le 2|a_n|$. Consequently, by the comparison test, since $2\sum_{n=1}^{\infty} |a_n|$ converges, the series
Equation:

$$\sum_{n=1}^{\infty} (|a_n| + a_n)$$

converges. By using the algebraic properties for convergent series, we conclude that
Equation:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (|a_n| + a_n) - \sum_{n=1}^{\infty} |a_n|$$

converges.

□

Example:
Exercise:

Problem:
Absolute versus Conditional Convergence

For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} (-1)^{n+1}/(3n+1)$
b. $\sum_{n=1}^{\infty} \cos(n)/n^2$

Solution:

a. We can see that
Equation:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{3n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+1}$$

diverges by using the limit comparison test with the harmonic series. In fact,
Equation:

$$\lim_{n \to \infty} \frac{1/(3n+1)}{1/n} = \frac{1}{3}.$$

Therefore, the series does not converge absolutely. However, since
Equation:

$$\frac{1}{3(n+1)+1} < \frac{1}{3n+1} \text{ and } \frac{1}{3n+1} \to 0,$$

the series converges. We can conclude that $\sum_{n=1}^{\infty} (-1)^{n+1}/(3n+1)$ converges conditionally.

b. Noting that $|\cos n| \leq 1$, to determine whether the series converges absolutely, compare
Equation:

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$$

with the series $\sum_{n=1}^{\infty} 1/n^2$. Since $\sum_{n=1}^{\infty} 1/n^2$ converges, by the comparison test,

$\sum_{n=1}^{\infty} |\cos n/n^2|$ converges, and therefore $\sum_{n=1}^{\infty} \cos n/n^2$ converges absolutely.

Note:
Exercise:

Problem:

Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} n/(2n^3+1)$ converges absolutely, converges conditionally, or diverges.

Solution:

The series converges absolutely.

Hint

Check for absolute convergence first.

To see the difference between absolute and conditional convergence, look at what happens when we *rearrange* the terms of the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$. We show that we can rearrange the terms so that the new series diverges. Certainly if we rearrange the terms of a finite sum, the sum does not change. When we work with an infinite sum, however, interesting things can happen.

Begin by adding enough of the positive terms to produce a sum that is larger than some real number $M > 0$. For example, let $M = 10$, and find an integer k such that

Equation:

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} > 10.$$

(We can do this because the series $\sum_{n=1}^{\infty} 1/(2n-1)$ diverges to infinity.) Then subtract $1/2$. Then add more positive terms until the sum reaches 100. That is, find another integer $j > k$ such that

Equation:

$$1 + \frac{1}{3} + \cdots + \frac{1}{2k-1} - \frac{1}{2} + \frac{1}{2k+1} + \cdots + \frac{1}{2j+1} > 100.$$

Then subtract $1/4$. Continuing in this way, we have found a way of rearranging the terms in the alternating harmonic series so that the sequence of partial sums for the rearranged series is

unbounded and therefore diverges.

The terms in the alternating harmonic series can also be rearranged so that the new series converges to a different value. In [link], we show how to rearrange the terms to create a new series that converges to $3\ln(2)/2$. We point out that the alternating harmonic series can be rearranged to create a series that converges to any real number r; however, the proof of that fact is beyond the scope of this text.

In general, any series $\sum_{n=1}^{\infty} a_n$ that converges conditionally can be rearranged so that the new series diverges or converges to a different real number. A series that converges absolutely does not have this property. For any series $\sum_{n=1}^{\infty} a_n$ that converges absolutely, the value of $\sum_{n=1}^{\infty} a_n$ is the same for any rearrangement of the terms. This result is known as the Riemann Rearrangement Theorem, which is beyond the scope of this book.

Example:
Exercise:

Problem:
Rearranging Series

Use the fact that
Equation:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2$$

to rearrange the terms in the alternating harmonic series so the sum of the rearranged series is $3\ln(2)/2$.

Solution:

Let
Equation:

$$\sum_{n=1}^{\infty} a_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots.$$

Since $\sum\limits_{n=1}^{\infty} a_n=\ln(2)$, by the algebraic properties of convergent series,
Equation:

$$\sum_{n=1}^{\infty}\frac{1}{2}a_n=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots=\frac{1}{2}\sum_{n=1}^{\infty}a_n=\frac{\ln 2}{2}.$$

Now introduce the series $\sum_{n=1}^{\infty} b_n$ such that for all $n \geq 1$, $b_{2n-1} = 0$ and $b_{2n} = a_n/2$. Then

Equation:

$$\sum_{n=1}^{\infty} b_n = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{\ln 2}{2}.$$

Then using the algebraic limit properties of convergent series, since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, the series $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and

Equation:

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \ln 2 + \frac{\ln 2}{2} = \frac{3 \ln 2}{2}.$$

Now adding the corresponding terms, a_n and b_n, we see that

Equation:

$$\begin{aligned} \sum_{n=1}^{\infty} (a_n + b_n) \quad &= (1 + 0) + \left(-\tfrac{1}{2} + \tfrac{1}{2}\right) + \left(\tfrac{1}{3} + 0\right) + \left(-\tfrac{1}{4} - \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + 0\right) + \left(-\tfrac{1}{6} + \tfrac{1}{6}\right) \\ &\quad + \left(\tfrac{1}{7} + 0\right) + \left(\tfrac{1}{8} - \tfrac{1}{8}\right) + \cdots \\ &= 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots. \end{aligned}$$

We notice that the series on the right side of the equal sign is a rearrangement of the alternating harmonic series. Since $\sum_{n=1}^{\infty} (a_n + b_n) = 3 \ln(2)/2$, we conclude that

Equation:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3 \ln(2)}{2}.$$

Therefore, we have found a rearrangement of the alternating harmonic series having the desired property.

Key Concepts

- For an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, if $b_{k+1} \leq b_k$ for all k and $b_k \to 0$ as $k \to \infty$, the alternating series converges.
- If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Key Equations

- **Alternating series**
  $$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots \text{ or}$$
  $$\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

State whether each of the following series converges absolutely, conditionally, or not at all.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+3}$

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{\sqrt{n}+3}$

Solution:

Does not converge by divergence test. Terms do not tend to zero.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+3}}$

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n+3}}{n}$

Solution:

Converges conditionally by alternating series test, since $\sqrt{n+3}/n$ is decreasing. Does not converge absolutely by comparison with p-series, $p = 1/2$.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^n}{n!}$

Solution:

Converges absolutely by limit comparison to $3^n/4^n$, for example.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n-1}{n}\right)^n$

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n+1}{n}\right)^n$

Solution:

Diverges by divergence test since $\lim\limits_{n\to\infty}|a_n| = e$.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2 n$

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2 n$

Solution:

Does not converge. Terms do not tend to zero.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2(1/n)$

Solution:

$\lim\limits_{n\to\infty}\cos^2(1/n) = 1$. Diverges by divergence test.

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} \ln (1/n)$

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} \ln \left(1+\frac{1}{n}\right)$

Solution:

Converges by alternating series test.

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{1+n^4}$

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^e}{1+n^\pi}$

Solution:

Converges conditionally by alternating series test. Does not converge absolutely by limit comparison with p-series, $p=\pi-e$

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} 2^{1/n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} n^{1/n}$

Solution:

Diverges; terms do not tend to zero.

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n} \left(1-n^{1/n}\right)$ (*Hint:* $n^{1/n} \approx 1+\ln(n)/n$ for large n.)

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} n \left(1-\cos\left(\frac{1}{n}\right)\right)$ (*Hint:* $\cos(1/n) \approx 1-1/n^2$ for large n.)

Solution:

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$ (*Hint:* Rationalize the numerator.)

Exercise:

Problem:

$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (*Hint:* Find common denominator then rationalize numerator.)

Solution:

Converges absolutely by limit comparison with p-series, $p=3/2$, after applying the hint.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\ln(n+1)-\ln n\right)$

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}n\left(\tan^{-1}(n+1)-\tan^{-1}n\right)$ (*Hint:* Use Mean Value Theorem.)

Solution:

Converges by alternating series test since $n(\tan^{-1}(n+1)-\tan^{-1}n)$ is decreasing to zero for large n. Does not converge absolutely by limit comparison with harmonic series after applying hint.

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\left((n+1)^2-n^2\right)$

Exercise:

Problem: $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Solution:

Converges absolutely, since $a_n=\frac{1}{n}-\frac{1}{n+1}$ are terms of a telescoping series.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{1/n}}$

Solution:

Terms do not tend to zero. Series diverges by divergence test.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$

Exercise:

Problem: $\sum_{n=1}^{\infty} \sin(n\pi/2) \sin(1/n)$

Solution:

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

In each of the following problems, use the estimate $|R_N| \leq b_{N+1}$ to find a value of N that guarantees that the sum of the first N terms of the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_N for this N.

Exercise:

Problem: **[T]** $b_n = 1/n$, error $< 10^{-5}$

Exercise:

Problem: **[T]** $b_n = 1/\ln(n)$, $n \geq 2$, error $< 10^{-1}$

Solution:

$\ln(N+1) > 10$, $N+1 > e^{10}$, $N \geq 22026$; $S_{22026} = 0.0257\ldots$

Exercise:

Problem: **[T]** $b_n = 1/\sqrt{n}$, error $< 10^{-3}$

Exercise:

Problem: **[T]** $b_n = 1/2^n$, error $< 10^{-6}$

Solution:

$2^{N+1} > 10^6$ or $N+1 > 6\ln(10)/\ln(2) = 19.93.$ or $N \geq 19$; $S_{19} = 0.333333969\ldots$

Exercise:

Problem: **[T]** $b_n = \ln\left(1+\frac{1}{n}\right)$, error $< 10^{-3}$

Exercise:

Problem: **[T]** $b_n = 1/n^2$, error $< 10^{-6}$

Solution:

$(N+1)^2 > 10^6$ or $N > 999$; $S_{1000} \approx 0.822466$.

For the following exercises, indicate whether each of the following statements is true or false. If the statement is false, provide an example in which it is false.

Exercise:

Problem:

If $b_n \geq 0$ is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, then $\sum\limits_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

Exercise:

Problem: If $b_n \geq 0$ is decreasing, then $\sum\limits_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

Solution:

True. b_n need not tend to zero since if $c_n = b_n - \lim b_n$, then $c_{2n-1} - c_{2n} = b_{2n-1} - b_{2n}$.

Exercise:

Problem: If $b_n \geq 0$ and $\lim\limits_{n\to\infty} b_n = 0$ then $\sum\limits_{n=1}^{\infty}(\frac{1}{2}(b_{3n-2} + b_{3n-1}) - b_{3n})$ converges.

Exercise:

Problem:

If $b_n \geq 0$ is decreasing and $\sum\limits_{n=1}^{\infty}(b_{3n-2} + b_{3n-1} - b_{3n})$ converges then $\sum\limits_{n=1}^{\infty} b_{3n-2}$ converges.

Solution:

True. $b_{3n-1} - b_{3n} \geq 0$, so convergence of $\sum b_{3n-2}$ follows from the comparison test.

Exercise:

Problem:

If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges conditionally but not absolutely, then b_n does not tend to zero.

Exercise:

Problem:

Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^- = -a_n$ if $a_n < 0$. (Also, $a_n^+ = 0$ if $a_n < 0$ and $a_n^- = 0$ if $a_n \geq 0$.) If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, then neither $\sum_{n=1}^{\infty} a_n^+$ nor $\sum_{n=1}^{\infty} a_n^-$ converge.

Solution:

True. If one converges, then so must the other, implying absolute convergence.

Exercise:

Problem: Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges.

Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

Exercise:

Problem:

Suppose that a_n is a sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every possible sequence b_n of zeros and ones. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely?

Solution:

Yes. Take $b_n = 1$ if $a_n \geq 0$ and $b_n = 0$ if $a_n < 0$. Then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n:a_n \geq 0} a_n$ converges. Similarly, one can show $\sum_{n:a_n<0} a_n$ converges. Since both series converge, the series must converge absolutely.

The following series do not satisfy the hypotheses of the alternating series test as stated.

In each case, state which hypothesis is not satisfied. State whether the series converges absolutely.

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin^2 n}{n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos^2 n}{n}$

Solution:

Not decreasing. Does not converge absolutely.

Exercise:

Problem: $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$

Exercise:

Problem: $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

Solution:

Not alternating. Can be expressed as $\sum_{n=1}^{\infty}\left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}\right)$, which diverges by comparison with $\sum \frac{1}{3n-2}$.

Exercise:

Problem: Show that the alternating series $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

Exercise:

Problem:

Suppose that $\sum a_n$ converges absolutely. Show that the series consisting of the positive terms a_n also converges.

Solution:

Let $a^+{}_n = a_n$ if $a_n \geq 0$ and $a^+{}_n = 0$ if $a_n < 0$. Then $a^+{}_n \leq |a_n|$ for all n so the sequence of partial sums of $a^+{}_n$ is increasing and bounded above by the sequence of partial sums of $|a_n|$, which converges; hence, $\sum_{n=1}^{\infty} a^+{}_n$ converges.

Exercise:

Problem:

Show that the alternating series $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

Exercise:

Problem:

The formula $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\cos\theta$ by the fifth partial sum $1 - \theta^2/2! + \theta^4/4! – \theta^6/6! + \theta^8/8!$ for $\theta = 1$, $\theta = \pi/6$, and $\theta = \pi$.

Solution:

For $N = 5$ one has $R_N\ b_6 = \theta^{10}/10!$. When $\theta = 1$, $R_5 \leq 1/10! \approx 2.75 \times 10^{-7}$. When $\theta = \pi/6$, $R_5 \leq (\pi/6)^{10}/10! \approx 4.26 \times 10^{-10}$. When $\theta = \pi$, $R_5 \leq \pi^{10}/10! = 0.0258$.

Exercise:

Problem:

The formula $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\sin\theta$ by the fifth partial sum $\theta – \theta^3/3! + \theta^5/5! – \theta^7/7! + \theta^9/9!$ for $\theta = 1$, $\theta = \pi/6$, and $\theta = \pi$.

Exercise:

Problem:

How many terms in $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ are needed to approximate $\cos 1$ accurate to an error of at most 0.00001?

Solution:

Let $b_n = 1/(2n-2)!$. Then $R_N \le 1/(2N)! < 0.00001$ when $(2N)! > 10^5$ or $N = 5$ and $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = 0.540325\ldots$, whereas $\cos 1 = 0.5403023\ldots$

Exercise:

Problem:

How many terms in $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ are needed to approximate $\sin 1$ accurate to an error of at most 0.00001?

Exercise:

Problem:

Sometimes the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges to a certain fraction of an absolutely convergent series $\sum_{n=1}^{\infty} b_n$ at a faster rate. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $S = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$. Which of the series $6\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $S\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ gives a better estimation of π^2 using 1000 terms?

Solution:

Let $T = \sum \frac{1}{n^2}$. Then $T - S = \frac{1}{2}T$, so $S = T/2$. $\sqrt{6 \times \sum_{n=1}^{1000} 1/n^2} = 3.140638\ldots$; $\sqrt{12 \times \sum_{n=1}^{1000} (-1)^{n-1}/n^2} = 3.141591\ldots$;

$\pi = 3.141592\ldots$. The alternating series is more accurate for 1000 terms.

The following alternating series converge to given multiples of π. Find the value of N predicted by the remainder estimate such that the Nth partial sum of the series accurately approximates the left-hand side to within the given error. Find the minimum N for which the error bound holds, and give the desired approximate value in each case. Up to 15 decimals places, $\pi = 3.141592653589793\ldots$.

Exercise:

Problem: [T] $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, error < 0.0001

Exercise:

Problem: **[T]** $\frac{\pi}{\sqrt{12}} = \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$, error < 0.0001

Solution:

$N = 6$, $S_N = 0.9068$

Exercise:

Problem:

[T] The series $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ plays an important role in signal processing. Show that $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ converges whenever $0 < x < \pi$. (*Hint:* Use the formula for the sine of a sum of angles.)

Exercise:

Problem:

[T] If $\sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n} \to \ln 2$, what is $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \cdots$?

Solution:

$\ln(2)$. The $3n\text{th}$ partial sum is the same as that for the alternating harmonic series.

Exercise:

Problem:

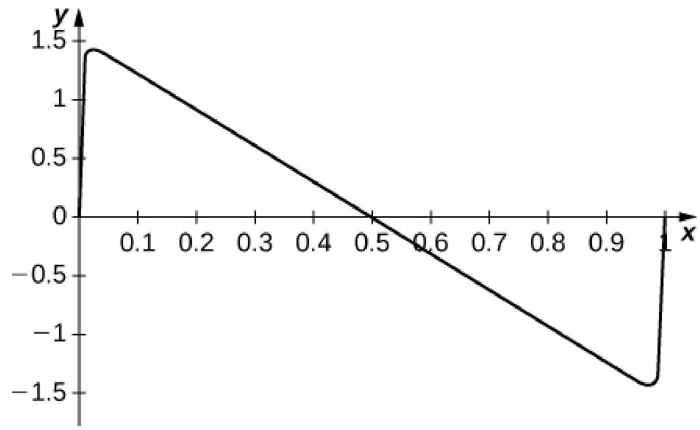
[T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ for $0 \le x < 1$. Explain why $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ diverges when $x = 0, 1$. How does the series behave for other x?

Exercise:

Problem: **[T]** Plot the series $\sum_{n=1}^{100} \frac{\sin(2\pi nx)}{n}$ for $0 \le x < 1$ and comment on its behavior

Solution:

The series jumps rapidly near the endpoints. For x away from the endpoints, the graph looks like $\pi(1/2 - x)$.



Exercise:

Problem: **[T]** Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n^2}$ for $0 \leq x < 1$ and describe its graph.

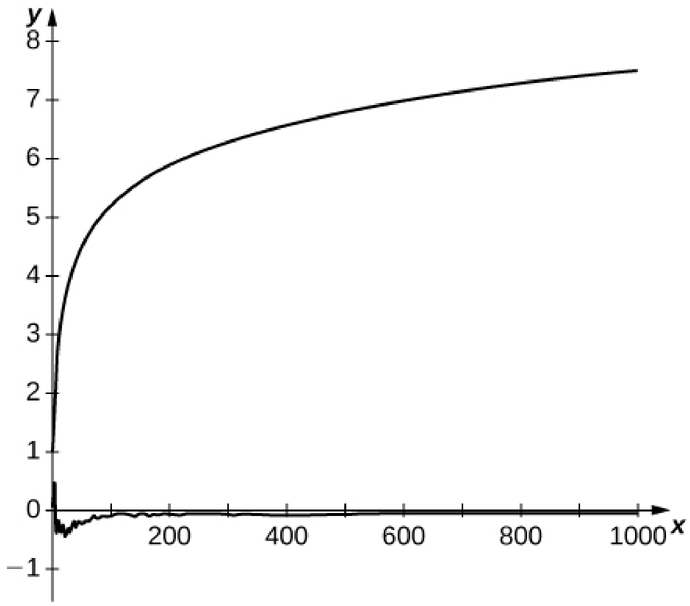
Exercise:

Problem:

[T] The alternating harmonic series converges because of cancellation among its terms. Its sum is known because the cancellation can be described explicitly. A random harmonic series is one of the form $\sum_{n=1}^{\infty} \frac{S_n}{n}$, where s_n is a randomly generated sequence of ± 1's in which the values ± 1 are equally likely to occur. Use a random number generator to produce 1000 random ± 1s and plot the partial sums $S_N = \sum_{n=1}^{N} \frac{s_n}{n}$ of your random harmonic sequence for $N = 1$ to 1000. Compare to a plot of the first 1000 partial sums of the harmonic series.

Solution:

Here is a typical result. The top curve consists of partial sums of the harmonic series. The bottom curve plots partial sums of a random harmonic series.



Exercise:

Problem:

[T] Estimates of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be *accelerated* by writing its partial sums as $\sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n(n+1)} + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$ and recalling that $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$ converges to one as $N \to \infty$. Compare the estimate of $\pi^2/6$ using the sums $\sum_{n=1}^{1000} \frac{1}{n^2}$ with the estimate using $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$.

Exercise:

Problem:

[T] The *Euler transform* rewrites $S = \sum_{n=0}^{\infty} (-1)^n b_n$ as $S = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} \sum_{m=0}^{n} \binom{n}{m} b_{n-m}$. For the alternating harmonic series, it takes the form $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Compute partial sums of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ until they approximate $\ln(2)$ accurate to within 0.0001. How many terms are needed? Compare this answer to the number of terms of the alternating harmonic series are needed to estimate $\ln(2)$.

Solution:

By the alternating series test, $|S_n - S| \le b_{n+1}$, so one needs 10^4 terms of the alternating harmonic series to estimate $\ln(2)$ to within 0.0001. The first 10 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ are (up to four decimals)
$0.5000, 0.6250, 0.6667, 0.6823, 0.6885, 0.6911, 0.6923, 0.6928, 0.6930, 0.6931$ and the tenth partial sum is within 0.0001 of $\ln(2) = 0.6931\ldots$.

Exercise:

Problem:

[T] In the text it was stated that a conditionally convergent series can be rearranged to converge to any number. Here is a slightly simpler, but similar, fact. If $a_n \ge 0$ is such that $a_n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} a_n$ diverges, then, given any number A there is a sequence s_n of ± 1's such that $\sum_{n=1}^{\infty} a_n s_n \to A$. Show this for $A > 0$ as follows.

a. Recursively define s_n by $s_n = 1$ if $S_{n-1} = \sum_{k=1}^{n-1} a_k s_k < A$ and $s_n = -1$ otherwise.
b. Explain why eventually $S_n \ge A$, and for any m larger than this n, $A - a_m \le S_m \le A + a_m$.
c. Explain why this implies that $S_n \to A$ as $n \to \infty$.

Glossary

absolute convergence
: if the series $\sum_{n=1}^{\infty} |a_n|$ converges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely

alternating series
: a series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n \ge 0$, is called an alternating series

alternating series test
: for an alternating series of either form, if $b_{n+1} \le b_n$ for all integers $n \ge 1$ and $b_n \to 0$, then an alternating series converges

conditional convergence
: if the series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally

Ratio and Root Tests

- Use the ratio test to determine absolute convergence of a series.
- Use the root test to determine absolute convergence of a series.
- Describe a strategy for testing the convergence of a given series.

In this section, we prove the last two series convergence tests: the ratio test and the root test. These tests are particularly nice because they do not require us to find a comparable series. The ratio test will be especially useful in the discussion of power series in the next chapter.

Throughout this chapter, we have seen that no single convergence test works for all series. Therefore, at the end of this section we discuss a strategy for choosing which convergence test to use for a given series.

Ratio Test

Consider a series $\sum_{n=1}^{\infty} a_n$. From our earlier discussion and examples, we know that $\lim_{n\to\infty} a_n = 0$ is not a sufficient condition for the series to converge. Not only do we need $a_n \to 0$, but we need $a_n \to 0$ quickly enough. For example, consider the series $\sum_{n=1}^{\infty} 1/n$ and the series $\sum_{n=1}^{\infty} 1/n^2$. We know that $1/n \to 0$ and $1/n^2 \to 0$. However, only the series $\sum_{n=1}^{\infty} 1/n^2$ converges. The series $\sum_{n=1}^{\infty} 1/n$ diverges because the terms in the sequence $\{1/n\}$ do not approach zero fast enough as $n \to \infty$. Here we introduce the **ratio test**, which provides a way of measuring how fast the terms of a series approach zero.

Note:
Ratio Test
Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let
Equation:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
iii. If $\rho = 1$, the test does not provide any information.

Proof

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.

We begin with the proof of part i. In this case, $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$. Since $0 \leq \rho < 1$, there exists R such that $0 \leq \rho < R < 1$. Let $\varepsilon = R - \rho > 0$. By the definition of limit of a sequence, there exists some integer N such that

Equation:

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < \varepsilon \text{ for all } n \geq N.$$

Therefore,

Equation:

$$\left| \frac{a_{n+1}}{a_n} \right| < \rho + \varepsilon = R \text{ for all } n \geq N$$

and, thus,

Equation:

$$\begin{aligned}
&|a_{N+1}| < R|a_N| \\
&|a_{N+2}| < R|a_{N+1}| < R^2|a_N| \\
&|a_{N+3}| < R|a_{N+2}| < R^2|a_{N+1}| < R^3|a_N| \\
&|a_{N+4}| < R|a_{N+3}| < R^2|a_{N+2}| < R^3|a_{N+1}| < R^4|a_N| \\
&\vdots
\end{aligned}$$

Since $R < 1$, the geometric series

Equation:

$$R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

converges. Given the inequalities above, we can apply the comparison test and conclude that the series

Equation:

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + |a_{N+4}| + \cdots$$

converges. Therefore, since

Equation:

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n|$$

where $\sum_{n=1}^{N}|a_n|$ is a finite sum and $\sum_{n=N+1}^{\infty}|a_n|$ converges, we conclude that $\sum_{n=1}^{\infty}|a_n|$ converges.

For part ii.
Equation:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1.$$

Since $\rho > 1$, there exists R such that $\rho > R > 1$. Let $\varepsilon = \rho - R > 0$. By the definition of the limit of a sequence, there exists an integer N such that
Equation:

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon \text{ for all } n \geq N.$$

Therefore,
Equation:

$$R = \rho - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| \text{ for all } n \geq N,$$

and, thus,
Equation:

$$\begin{aligned}
&|a_{N+1}| > R|a_N| \\
&|a_{N+2}| > R|a_{N+1}| > R^2|a_N| \\
&|a_{N+3}| > R|a_{N+2}| > R^2|a_{N+1}| > R^3|a_N| \\
&|a_{N+4}| > R|a_{N+3}| > R^2|a_{N+2}| > R^3|a_{N+1}| > R^4|a_N|.
\end{aligned}$$

Since $R > 1$, the geometric series
Equation:

$$R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

diverges. Applying the comparison test, we conclude that the series
Equation:

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

diverges, and therefore the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

For part iii. we show that the test does not provide any information if $\rho = 1$ by considering the $p-$ series $\sum_{n=1}^{\infty} 1/n^p$. For any real number p,

Equation:

$$\rho = \lim_{n\to\infty} \frac{1/(n+1)^p}{1/n^p} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = 1.$$

However, we know that if $p \leq 1$, the $p-$ series $\sum_{n=1}^{\infty} 1/n^p$ diverges, whereas $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$.

□

The ratio test is particularly useful for series whose terms contain factorials or exponentials, where the ratio of terms simplifies the expression. The ratio test is convenient because it does not require us to find a comparative series. The drawback is that the test sometimes does not provide any information regarding convergence.

Example:
Exercise:

Problem:
Using the Ratio Test

For each of the following series, use the ratio test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
b. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$
c. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

Solution:

a. From the ratio test, we can see that
Equation:

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}.$$

Since $(n+1)! = (n+1) \cdot n!$,
Equation:

$$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

Since $\rho < 1$, the series converges.

b. We can see that
Equation:

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} \\ &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e. \end{aligned}$$

Since $\rho > 1$, the series diverges.

c. Since
Equation:

$$\begin{aligned} \left| \frac{(-1)^{n+1}((n+1)!)^2/(2(n+1))!}{(-1)^n(n!)^2/(2n)!} \right| &= \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!} \\ &= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \end{aligned}$$

we see that
Equation:

$$\rho = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}.$$

Since $\rho < 1$, the series converges.

Note:
Exercise:

Problem:

Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges or diverges.

Solution:

The series converges.

Hint

Evaluate $\lim\limits_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$.

Root Test

The approach of the **root test** is similar to that of the ratio test. Consider a series $\sum_{n=1}^{\infty} a_n$ such that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \rho$ for some real number ρ. Then for N sufficiently large, $\left|a_N\right| \approx \rho^N$. Therefore, we can approximate $\sum\limits_{n=N}^{\infty} |a_n|$ by writing

Equation:

$$\left|a_N\right| + \left|a_{N+1}\right| + \left|a_{N+2}\right| + \cdots \approx \rho^N + \rho^{N+1} + \rho^{N+2} + \cdots.$$

The expression on the right-hand side is a geometric series. As in the ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $0 \le \rho < 1$ and the series diverges if $\rho \ge 1$. If $\rho = 1$, the test does not provide any information. For example, for any p-series, $\sum\limits_{n=1}^{\infty} 1/n^p$, we see that

Equation:

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^p}\right|} = \lim_{n\to\infty} \frac{1}{n^{p/n}}.$$

To evaluate this limit, we use the natural logarithm function. Doing so, we see that

Equation:

$$\ln\rho = \ln\left(\lim_{n\to\infty} \frac{1}{n^{p/n}}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right)^{p/n} = \lim_{n\to\infty} \frac{p}{n} \cdot \ln\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{p\ln(1/n)}{n}.$$

Using L'Hôpital's rule, it follows that $\ln\rho = 0$, and therefore $\rho = 1$ for all p. However, we know that the p-series only converges if $p > 1$ and diverges if $p < 1$.

Note:
Root Test
Consider the series $\sum_{n=1}^{\infty} a_n$. Let
Equation:

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

i. If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
iii. If $\rho = 1$, the test does not provide any information.

The root test is useful for series whose terms involve exponentials. In particular, for a series whose terms a_n satisfy $|a_n| = b_n^n$, then $\sqrt[n]{|a_n|} = b_n$ and we need only evaluate $\lim\limits_{n\to\infty} b_n$.

Example:
Exercise:

Problem:
Using the Root Test

For each of the following series, use the root test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$
b. $\sum_{n=1}^{\infty} \frac{n^n}{(\ln(n))^n}$

Solution:

a. To apply the root test, we compute
Equation:

$$\rho = \lim_{n\to\infty} \sqrt[n]{(n^2+3n)^n/(4n^2+5)^n} = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} = \frac{1}{4}.$$

Since $\rho < 1$, the series converges absolutely.
b. We have
Equation:

$$\rho = \lim_{n\to\infty} \sqrt[n]{n^n/(\ln n)^n} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty \text{ by L'Hôpital's rule.}$$

Since $\rho = \infty$, the series diverges.

Note:
Exercise:

Problem:

Use the root test to determine whether the series $\sum_{n=1}^{\infty} 1/n^n$ converges or diverges.

Solution:

The series converges.

Hint

Evaluate $\lim\limits_{n\to\infty} \sqrt[n]{1/n^n}$ using L'Hôpital's rule.

Choosing a Convergence Test

At this point, we have a long list of convergence tests. However, not all tests can be used for all series. When given a series, we must determine which test is the best to use. Here is a strategy for finding the best test to apply.

Note:
Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series $\sum\limits_{n=1}^{\infty} a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\sum\limits_{n=1}^{\infty} a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a $p-$ series or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If

we are interested in absolute convergence, proceed to step 3, considering the series of absolute values $\sum_{n=1}^{\infty}|a_n|$.

3. Is the series similar to a $p-$ series or geometric series? If so, try the comparison test or limit comparison test.
4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = b_n^n$, try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.

Note:
Visit this website for more information on testing series for convergence, plus general information on sequences and series.

Example:
Exercise:

Problem:
Using Convergence Tests

For each of the following series, determine which convergence test is the best to use and explain why. Then determine if the series converges or diverges. If the series is an alternating series, determine whether it converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$
b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$
c. $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$
d. $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

Solution:

a. Step 1. The series is not a $p-$ series or geometric series.
Step 2. The series is not alternating.
Step 3. For large values of n, we approximate the series by the expression
Equation:

$$\frac{n^2+2n}{n^3+3n^2+1} \approx \frac{n^2}{n^3} = \frac{1}{n}.$$

Therefore, it seems reasonable to apply the comparison test or limit comparison test using the series $\sum_{n=1}^{\infty} 1/n$. Using the limit comparison test, we see that

Equation:

$$\lim_{n\to\infty} \frac{(n^2+2n)/(n^3+3n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+3n^2+1} = 1.$$

Since the series $\sum_{n=1}^{\infty} 1/n$ diverges, this series diverges as well.

b. Step 1.The series is not a familiar series.
Step 2. The series is alternating. Since we are interested in absolute convergence, consider the series
Equation:

$$\sum_{n=1}^{\infty} \frac{3n}{(n+1)!}.$$

Step 3. The series is not similar to a *p*-series or geometric series.
Step 4. Since each term contains a factorial, apply the ratio test. We see that
Equation:

$$\lim_{n\to\infty} \frac{(3\,(n+1))/\,(n+1)!}{(3n+1)/n!} = \lim_{n\to\infty} \frac{3n+3}{(n+1)!} \cdot \frac{n!}{3n+1} = \lim_{n\to\infty} \frac{3n+3}{(n+1)\,(3n+1)} = 0.$$

Therefore, this series converges, and we conclude that the original series converges absolutely, and thus converges.

c. Step 1. The series is not a familiar series.
Step 2. It is not an alternating series.
Step 3. There is no obvious series with which to compare this series.
Step 4. There is no factorial. There is a power, but it is not an ideal situation for the root test.
Step 5. To apply the divergence test, we calculate that
Equation:

$$\lim_{n\to\infty} \frac{e^n}{n^3} = \infty.$$

Therefore, by the divergence test, the series diverges.

d. Step 1. This series is not a familiar series.
Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.
Step 4. Since each term is a power of n, we can apply the root test. Since
Equation:

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{3}{n+1}\right)^n}=\lim_{n\to\infty}\frac{3}{n+1}=0,$$

by the root test, we conclude that the series converges.

Note:
Exercise:

Problem:

For the series $\sum_{n=1}^{\infty}\frac{2^n}{3^n+n}$, determine which convergence test is the best to use and explain why.

Solution:

The comparison test because $2^n/\left(3^n+n\right)<2^n/3^n$ for all positive integers n. The limit comparison test could also be used.

Hint

The series is similar to the geometric series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$.

In [link], we summarize the convergence tests and when each can be applied. Note that while the comparison test, limit comparison test, and integral test require the series $\sum\limits_{n=1}^{\infty}a_n$ to have nonnegative terms, if $\sum\limits_{n=1}^{\infty}a_n$ has negative terms, these tests can be applied to $\sum\limits_{n=1}^{\infty}|a_n|$ to test for absolute convergence.

Series or Test	Conclusions	Comments

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim\limits_{n\to\infty} a_n$.	If $\lim\limits_{n\to\infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim\limits_{n\to\infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\|r\| < 1$, the series converges to $a/\left(1-r\right)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\|r\| \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	

Series or Test	Conclusions	Comments
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim\limits_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx$.	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim\limits_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	

Series or Test	Conclusions	Comments
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n \to \infty} \sqrt[n]{\lvert a_n \rvert}$.	If $0 \le \rho < 1$, the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n$.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	

Summary of Convergence Tests

Note:

Series Converging to π and $1/\pi$

Dozens of series exist that converge to π or an algebraic expression containing π. Here we look at several examples and compare their rates of convergence. By rate of convergence, we mean the number of terms necessary for a partial sum to be within a certain amount of the actual value. The series representations of π in the first two examples can be explained using Maclaurin series, which are discussed in the next chapter. The third example relies on material beyond the scope of this text.

1. The series
   Equation:

$$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

   was discovered by Gregory and Leibniz in the late 1600s. This result follows from the Maclaurin series for $f(x) = \tan^{-1} x$. We will discuss this series in the next chapter.

   a. Prove that this series converges.
   b. Evaluate the partial sums S_n for $n = 10, 20, 50, 100$.
   c. Use the remainder estimate for alternating series to get a bound on the error R_n.
   d. What is the smallest value of N that guarantees $\lvert R_N \rvert < 0.01$? Evaluate S_N.

2. The series
   Equation:

$$\begin{aligned}\pi &= 6\sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)} \\ &= 6\left(\frac{1}{2} + \frac{1}{2\cdot 3}\left(\frac{1}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left(\frac{1}{2}\right)^7 + \cdots\right)\end{aligned}$$

has been attributed to Newton in the late 1600s. The proof of this result uses the Maclaurin series for $f(x) = \sin^{-1} x$.

a. Prove that the series converges.
b. Evaluate the partial sums S_n for $n = 5, 10, 20$.
c. Compare S_n to π for $n = 5, 10, 20$ and discuss the number of correct decimal places.

3. The series
Equation:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

was discovered by Ramanujan in the early 1900s. William Gosper, Jr., used this series to calculate π to an accuracy of more than 17 million digits in the mid-1980s. At the time, that was a world record. Since that time, this series and others by Ramanujan have led mathematicians to find many other series representations for π and $1/\pi$.

a. Prove that this series converges.
b. Evaluate the first term in this series. Compare this number with the value of π from a calculating utility. To how many decimal places do these two numbers agree? What if we add the first two terms in the series?
c. Investigate the life of Srinivasa Ramanujan $(1887–1920)$ and write a brief summary. Ramanujan is one of the most fascinating stories in the history of mathematics. He was basically self-taught, with no formal training in mathematics, yet he contributed in highly original ways to many advanced areas of mathematics.

Key Concepts

- For the ratio test, we consider
Equation:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, the series diverges. If $\rho = 1$, the test does not provide any information. This test is useful for series whose terms involve factorials.

- For the root test, we consider
Equation:

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, the series diverges. If $\rho = 1$, the test does not provide any information. The root test is useful for series whose terms involve powers.

- For a series that is similar to a geometric series or $p-$ series, consider one of the comparison tests.

Use the ratio test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is given in the following problems. State if the ratio test is inconclusive.

Exercise:

Problem: $a_n = 1/n!$

Solution:

$a_{n+1}/a_n \to 0$. Converges.

Exercise:

Problem: $a_n = 10^n/n!$

Exercise:

Problem: $a_n = n^2/2^n$

Solution:

$\frac{a_{n+1}}{a_n} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \to 1/2 < 1$. Converges.

Exercise:

Problem: $a_n = n^{10}/2^n$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n!)}$

Solution:

$\frac{a_{n+1}}{a_n} \to 1/27 < 1$. Converges.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n!)}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

Solution:

$\frac{a_{n+1}}{a_n} \to 4/e^2 < 1$. Converges.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}$

Solution:

$\frac{a_{n+1}}{a_n} \to 1$. Ratio test is inconclusive.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(2n)!}{(n/e)^{2n}}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(2^n n!)^2}{(2n)^{2n}}$

Solution:

$\frac{a_n}{a_{n+1}} \to 1/e^2$. Converges.

Use the root test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is as follows.

Exercise:

Problem: $a_k = \left(\frac{k-1}{2k+3}\right)^k$

Exercise:

Problem: $a_k = \left(\frac{2k^2-1}{k^2+3}\right)^k$

Solution:

$(a_k)^{1/k} \to 2 > 1$. Diverges.

Exercise:

Problem: $a_n = \frac{(\ln n)^{2n}}{n^n}$

Exercise:

Problem: $a_n = n/2^n$

Solution:

$(a_n)^{1/n} \to 1/2 < 1$. Converges.

Exercise:

Problem: $a_n = n/e^n$

Exercise:

Problem: $a_k = \frac{k^e}{e^k}$

Solution:

$(a_k)^{1/k} \to 1/e < 1$. Converges.

Exercise:

Problem: $a_k = \frac{\pi^k}{k^\pi}$

Exercise:

Problem: $a_n = \left(\frac{1}{e} + \frac{1}{n}\right)^n$

Solution:

$a_n^{1/n} = \frac{1}{e} + \frac{1}{n} \to \frac{1}{e} < 1$. Converges.

Exercise:

Problem: $a_k = \frac{1}{(1+\ln k)^k}$

Exercise:

Problem: $a_n = \frac{(\ln(1+\ln n))^n}{(\ln n)^n}$

Solution:

$a_n^{1/n} = \frac{(\ln(1+\ln n))}{(\ln n)} \to 0$ by L'Hôpital's rule. Converges.

In the following exercises, use either the ratio test or the root test as appropriate to determine whether the series $\sum_{k=1}^{\infty} a_k$ with given terms a_k converges, or state if the test is inconclusive.

Exercise:

Problem: $a_k = \frac{k!}{1\cdot3\cdot5\cdots(2k-1)}$

Exercise:

Problem: $a_k = \frac{2\cdot4\cdot6\cdots2k}{(2k)!}$

Solution:

$\frac{a_{k+1}}{a_k} = \frac{1}{2k+1} \to 0$. Converges by ratio test.

Exercise:

Problem: $a_k = \frac{1\cdot4\cdot7\cdots(3k-2)}{3^k k!}$

Exercise:

Problem: $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$

Solution:

$(a_n)^{1/n} \to 1/e$. Converges by root test.

Exercise:

Problem: $a_k = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right)^k$ (*Hint:* Compare $a_k^{1/k}$ to $\displaystyle\int_k^{2k} \frac{dt}{t}$.)

Exercise:

Problem: $a_k = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k}\right)^k$

Solution:

$a_k^{1/k} \to \ln(3) > 1$. Diverges by root test.

Exercise:

Problem: $a_n = \left(n^{1/n} - 1\right)^n$

Use the ratio test to determine whether $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, or state if the ratio test is inconclusive.

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n^2}}{2^{n^3}}$

Solution:

$\frac{a_{n+1}}{a_n} = \frac{3^{2n+1}}{2^{3n^2+3n+1}} \to 0$. Converge.

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n n!}$

Use the root and limit comparison tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges.

Exercise:

Problem: $a_n = 1/x_n^n$ where $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$, $x_1 = 1$ (*Hint:* Find limit of $\{x_n\}$.)

Solution:

Converges by root test and limit comparison test since $x_n \to \sqrt{2}$.

In the following exercises, use an appropriate test to determine whether the series converges.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(n+1)}{n^3 + n^2 + n + 1}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^3 + 3n^2 + 3n + 1}$

Solution:

Converges absolutely by limit comparison with $p-$ series, $p = 2$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3 + (1.1)^n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(n-1)^n}{(n+1)^n}$

Solution:

$\lim\limits_{n\to\infty} a_n = 1/e^2 \neq 0$. Series diverges.

Exercise:

Problem: $a_n = \left(1 + \frac{1}{n^2}\right)^n$ (*Hint:* $\left(1 + \frac{1}{n^2}\right)^{n^2} \approx e$.)

Exercise:

Problem: $a_k = 1/2^{\sin^2 k}$

Solution:

Terms do not tend to zero: $a_k \geq 1/2$, since $\sin^2 x \leq 1$.

Exercise:

Problem: $a_k = 2^{-\sin(1/k)}$

Exercise:

Problem: $a_n = 1/\binom{n+2}{n}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Solution:

$a_n = \frac{2}{(n+1)(n+2)}$, which converges by comparison with $p-$series for $p = 2$.

Exercise:

Problem: $a_k = 1/\binom{2k}{k}$

Exercise:

Problem: $a_k = 2^k/\binom{3k}{k}$

Solution:

$a_k = \frac{2^k 1\cdot 2\cdots k}{(2k+1)(2k+2)\cdots 3k} \leq (2/3)^k$ converges by comparison with geometric series.

Exercise:

Problem: $a_k = \left(\frac{k}{k+\ln k}\right)^k$ (*Hint:* $a_k = \left(1+\frac{\ln k}{k}\right)^{-(k/\ln k)\ln k} \approx e^{-\ln k}$.)

Exercise:

Problem: $a_k = \left(\frac{k}{k+\ln k}\right)^{2k}$ (*Hint:* $a_k = \left(1+\frac{\ln k}{k}\right)^{-(k/\ln k)\ln k^2}$.)

Solution:

$a_k \approx e^{-\ln k^2} = 1/k^2$. Series converges by limit comparison with $p-$series, $p = 2$.

The following series converge by the ratio test. Use summation by parts, $\sum_{k=1}^{n} a_k \left(b_{k+1} - b_k\right) = \left[a_{n+1}b_{n+1} - a_1b_1\right] - \sum_{k=1}^{n} b_{k+1}\left(a_{k+1} - a_k\right)$, to find the sum of the given series.

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{k}{2^k}$ (*Hint:* Take $a_k = k$ and $b_k = 2^{1-k}$.)

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{k}{c^k}$, where $c > 1$ (*Hint:* Take $a_k = k$ and $b_k = c^{1-k}/(c-1)$.)

Solution:

If $b_k = c^{1-k}/(c-1)$ and $a_k = k$, then $b_{k+1} - b_k = -c^{-k}$ and $\sum_{n=1}^{\infty} \frac{k}{c^k} = a_1b_1 + \frac{1}{c-1}\sum_{k=1}^{\infty} c^{-k} = \frac{c}{(c-1)^2}$.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(n+1)^2}{2^n}$

Solution:

$6 + 4 + 1 = 11$

The kth term of each of the following series has a factor x^k. Find the range of x for which the ratio test implies that the series converges.

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$

Solution:

$|x| \leq 1$

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{x^{2k}}{3^k}$

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{x^{k}}{k!}$

Solution:

$|x| < \infty$

Exercise:

Problem: Does there exist a number p such that $\sum_{n=1}^{\infty} \frac{2^n}{n^p}$ converges?

Exercise:

Problem: Let $0 < r < 1$. For which real numbers p does $\sum_{n=1}^{\infty} n^p r^n$ converge?

Solution:

All real numbers p by the ratio test.

Exercise:

Problem: Suppose that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p$. For which values of p must $\sum_{n=1}^{\infty} 2^n a_n$ converge?

Exercise:

Problem:

Suppose that $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of $r > 0$ is $\sum\limits_{n=1}^{\infty} r^n a_n$ guaranteed to converge?

Solution:

$r < 1/p$

Exercise:

Problem:

Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \leq (n+1)^p$ for all $n = 1, 2, \ldots$ where p is a fixed real number. For which values of p is $\sum\limits_{n=1}^{\infty} n!\, a_n$ guaranteed to converge?

Exercise:

Problem:

For which values of $r > 0$, if any, does $\sum\limits_{n=1}^{\infty} r^{\sqrt{n}}$ converge? (*Hint:* $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{k=1}^{\infty} \sum\limits_{n=k^2}^{(k+1)^2-1} a_n.$)

Solution:

$0 < r < 1$. Note that the ratio and root tests are inconclusive. Using the hint, there are $2k$ terms $r^{\sqrt{n}}$ for $k^2 \leq n < (k+1)^2$, and for $r < 1$ each term is at least r^k. Thus, $\sum\limits_{n=1}^{\infty} r^{\sqrt{n}} = \sum\limits_{k=1}^{\infty} \sum\limits_{n=k^2}^{(k+1)^2-1} r^{\sqrt{n}} \geq \sum\limits_{k=1}^{\infty} 2kr^k$, which converges by the ratio test for $r < 1$. For $r \geq 1$ the series diverges by the divergence test.

Exercise:

Problem: Suppose that $\left|\frac{a_{n+2}}{a_n}\right| \leq r < 1$ for all n. Can you conclude that $\sum\limits_{n=1}^{\infty} a_n$ converges?

Exercise:

Problem:

Let $a_n = 2^{-[n/2]}$ where $[x]$ is the greatest integer less than or equal to x. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and justify your answer.

Solution:

One has $a_1 = 1$, $a_2 = a_3 = 1/2$,$\ldots a_{2n} = a_{2n+1} = 1/2^n$. The ratio test does not apply because $a_{n+1}/a_n = 1$ if n is even. However, $a_{n+2}/a_n = 1/2$, so the series converges according to the previous exercise. Of course, the series is just a duplicated geometric series.

The following *advanced* exercises use a generalized ratio test to determine convergence of some series that arise in particular applications when tests in this chapter, including the ratio and root test, are not powerful enough to determine their convergence. The test states that if $\lim\limits_{n\to\infty} \frac{a_{2n}}{a_n} < 1/2$, then $\sum a_n$ converges, while if $\lim\limits_{n\to\infty} \frac{a_{2n+1}}{a_n} > 1/2$, then $\sum a_n$ diverges.

Exercise:

Problem:

Let $a_n = \frac{1}{4}\,\frac{3}{6}\,\frac{5}{8}\cdots\frac{2n-1}{2n+2} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^n(n+1)!}$. Explain why the ratio test cannot determine convergence of $\sum_{n=1}^{\infty} a_n$. Use the fact that $1 - 1/(4k)$ is increasing k to estimate $\lim\limits_{n\to\infty} \frac{a_{2n}}{a_n}$.

Exercise:

Problem:

Let $a_n = \frac{1}{1+x}\,\frac{2}{2+x}\cdots\frac{n}{n+x}\,\frac{1}{n} = \frac{(n-1)!}{(1+x)(2+x)\cdots(n+x)}$. Show that $a_{2n}/a_n \le e^{-x/2}/2$. For which $x > 0$ does the generalized ratio test imply convergence of $\sum_{n=1}^{\infty} a_n$? (*Hint:* Write $2a_{2n}/a_n$ as a product of n factors each smaller than $1/\left(1 + x/\left(2n\right)\right).$)

Solution:

$a_{2n}/a_n = \frac{1}{2}\cdot\frac{n+1}{n+1+x}\,\frac{n+2}{n+2+x}\cdots\frac{2n}{2n+x}$. The inverse of the $k\text{th}$ factor is $(n+k+x)/(n+k) > 1 + x/(2n)$ so the product is less than $\left(1 + x/\left(2n\right)\right)^{-n} \approx e^{-x/2}$. Thus for $x > 0$, $\frac{a_{2n}}{a_n} \le \frac{1}{2}e^{-x/2}$. The series converges for $x > 0$.

Exercise:

Problem: Let $a_n = \frac{n^{\ln n}}{(\ln n)^n}$. Show that $\frac{a_{2n}}{a_n} \to 0$ as $n \to \infty$.

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem: If $\lim_{n \to \infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Solution:

false

Exercise:

Problem: If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Exercise:

Problem: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Solution:

true

Exercise:

Problem: If $\sum_{n=1}^{\infty} 2^n a_n$ converges, then $\sum_{n=1}^{\infty} (-2)^n a_n$ converges.

Is the sequence bounded, monotone, and convergent or divergent? If it is convergent, find the limit.

Exercise:

Problem: $a_n = \frac{3+n^2}{1-n}$

Solution:

unbounded, not monotone, divergent

Exercise:

Problem: $a_n = \ln\left(\frac{1}{n}\right)$

Exercise:

Problem: $a_n = \frac{\ln(n+1)}{\sqrt{n+1}}$

Solution:

bounded, monotone, convergent, 0

Exercise:

Problem: $a_n = \frac{2^{n+1}}{5^n}$

Exercise:

Problem: $a_n = \frac{\ln(\cos n)}{n}$

Solution:

unbounded, not monotone, divergent

Is the series convergent or divergent?

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 4}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Solution:

diverges

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

Solution:

converges

Exercise:

Problem: $\sum_{n=1}^{\infty} n^{-(n+1/n)}$

Is the series convergent or divergent? If convergent, is it absolutely convergent?

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution:

converges, but not absolutely

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Solution:

converges absolutely

Exercise:

Problem: $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

Exercise:

Problem: $\sum_{n=1}^{\infty} \cos(\pi n) e^{-n}$

Solution:

converges absolutely

Evaluate

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{2^{n+4}}{7^n}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

Solution:

$\frac{1}{2}$

Exercise:

Problem:

A legend from India tells that a mathematician invented chess for a king. The king enjoyed the game so much he allowed the mathematician to demand any payment. The mathematician asked for one grain of rice for the first square on the chessboard, two grains of rice for the second square on the chessboard, and so on. Find an exact expression for the total payment (in grains of rice) requested by the mathematician. Assuming there are 30,000 grains of rice in 1 pound, and 2000 pounds in 1 ton, how many tons of rice did the mathematician attempt to receive?

The following problems consider a simple population model of the housefly, which can be exhibited by the recursive formula $x_{n+1} = bx_n$, where x_n is the population of houseflies at generation n, and b is the average number of offspring per housefly who survive to the next generation. Assume a starting population x_0.

Exercise:

Problem: Find $\lim_{n\to\infty} x_n$ if $b > 1$, $b < 1$, and $b = 1$.

Solution:

∞, 0, x_0

Exercise:

Problem:

Find an expression for $S_n = \sum_{i=0}^{n} x_i$ in terms of b and x_0. What does it physically represent?

Exercise:

Problem: If $b = \frac{3}{4}$ and $x_0 = 100$, find S_{10} and $\lim_{n \to \infty} S_n$

Solution:

$S_{10} \approx 383, \lim_{n \to \infty} S_n = 400$

Exercise:

Problem:

For what values of b will the series converge and diverge? What does the series converge to?

Glossary

ratio test
: for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} |a_{n+1}/a_n|$; if $0 \leq \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

root test
: for a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|}$; if $0 \leq \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

Introduction
class="introduction"

If you win a lottery, do you get more money by taking a lump-sum payment or by accepting fixed payments over time? (credit: modification of work by Robert Huffstutter, Flickr)

When winning a lottery, sometimes an individual has an option of receiving winnings in one lump-sum payment or receiving smaller payments over fixed time intervals. For example, you might have the option of receiving 20 million dollars today or receiving 1.5 million dollars each year for the next 20 years. Which is the better deal? Certainly 1.5 million dollars over 20 years is equivalent to 30 million dollars. However, receiving the 20 million dollars today would allow you to invest the money.

Alternatively, what if you were guaranteed to receive 1 million dollars every year indefinitely (extending to your heirs) or receive 20 million dollars today. Which would be the better deal? To answer these questions, you need to know how to use infinite series to calculate the value of periodic payments over time in terms of today's dollars (see [link]).

An infinite series of the form is known as a power series. Since

the terms contain the variable x, power series can be used to define functions. They can be used to represent given functions, but they are also

important because they allow us to write functions that cannot be expressed any other way than as “infinite polynomials.” In addition, power series can be easily differentiated and integrated, thus being useful in solving differential equations and integrating complicated functions. An infinite series can also be truncated, resulting in a finite polynomial that we can use to approximate functional values. Power series have applications in a variety of fields, including physics, chemistry, biology, and economics. As we will see in this chapter, representing functions using power series allows us to solve mathematical problems that cannot be solved with other techniques.

Power Series and Functions

- Identify a power series and provide examples of them.
- Determine the radius of convergence and interval of convergence of a power series.
- Use a power series to represent a function.

A power series is a type of series with terms involving a variable. More specifically, if the variable is *x*, then all the terms of the series involve powers of *x*. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

Form of a Power Series

A series of the form
Equation:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,$$

where x is a variable and the coefficients c_n are constants, is known as a **power series**. The series
Equation:

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

is an example of a power series. Since this series is a geometric series with ratio $r = |x|$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

Note:
Definition
A series of the form
Equation:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

is a power series centered at $x = 0$. A series of the form
Equation:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

is a power series centered at $x = a$.

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a$, respectively.

The series
Equation:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and
Equation:

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots$$

are both power series centered at $x = 0$. The series

Equation:

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \cdots$$

is a power series centered at $x = 2$.

Convergence of a Power Series

Since the terms in a power series involve a variable *x*, the series may converge for certain values of *x* and diverge for other values of *x*. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of *x*. Most power series, however, converge for more than one value of *x*. In that case, the power series either converges for all real numbers *x* or converges for all *x* in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all *x* in the interval $(-1, 1)$, but diverges for all *x* outside that interval. We now summarize these three possibilities for a general power series.

Note:

Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.
ii. The series converges for all real numbers *x*.
iii. There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. At the values *x* where $|x-a| = R$, the series may converge or diverge.

Proof

Suppose that the power series is centered at $a = 0$. (For a series centered at a value of a other than zero, the result follows by letting $y = x - a$ and considering the series $\sum_{n=1}^{\infty} c_n y^n$.) We must first prove the following fact:

If there exists a real number $d \neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all *x* such that $|x| < |d|$.

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the *n*th term $c_n d^n \to 0$ as $n \to \infty$. Therefore, there exists an integer *N* such that $|c_n d^n| \leq 1$ for all $n \geq N$. Writing

Equation:

$$|c_n x^n| = |c_n d^n| \left| \frac{x}{d} \right|^n,$$

we conclude that, for all $n \geq N$,

Equation:

$$|c_n x^n| \leq \left| \frac{x}{d} \right|^n.$$

The series

Equation:

$$\sum_{n=N}^{\infty} \left| \frac{x}{d} \right|^n$$

is a geometric series that converges if $\left|\frac{x}{d}\right| < 1$. Therefore, by the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|$. Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|$.

With this result, we can now prove the theorem. Consider the series
Equation:

$$\sum_{n=0}^{\infty} a_n x^n$$

and let *S* be the set of real numbers for which the series converges. Suppose that the set $S = \{0\}$. Then the series falls under case i. Suppose that the set *S* is the set of all real numbers. Then the series falls under case ii. Suppose that $S \neq \{0\}$ and *S* is not the set of real numbers. Then there exists a real number $x^* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any *x* such that $|x| > |x^*|$. Therefore, the set *S* must be a bounded set, which means that it must have a smallest upper bound. (This fact follows from the Least Upper Bound Property for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.) Call that smallest upper bound *R*. Since $S \neq \{0\}$, the number $R > 0$. Therefore, the series converges for all *x* such that $|x| < R$, and the series falls into case iii.

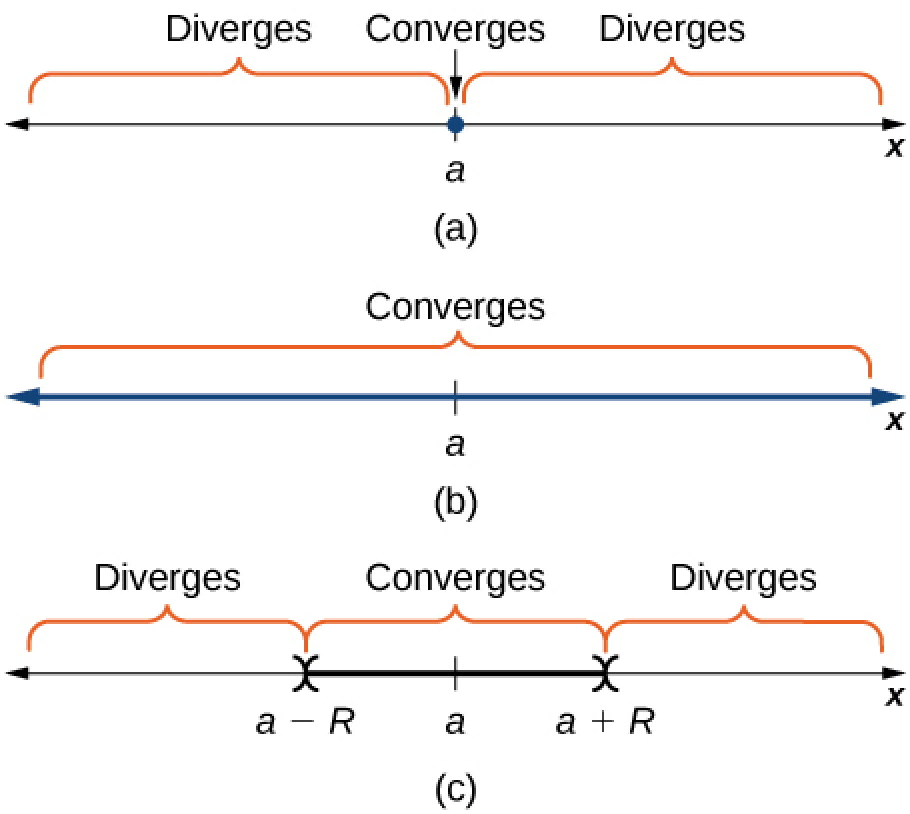
□

If a series $\sum_{n=0}^{\infty} c_n (x-a)^n$ falls into case iii. of [link], then the series converges for all *x* such that $|x-a| < R$ for some $R > 0$, and diverges for all *x* such that $|x-a| > R$. The series may converge or diverge at the values *x* where $|x-a| = R$. The set of values *x* for which the series

$\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is known as the **interval of convergence**. Since the series diverges for all values *x* where $|x-a|>R$, the length of the interval is 2*R*, and therefore, the radius of the interval is *R*. The value *R* is called the **radius of convergence**. For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values *x* in the interval $(-1,1)$ and diverges for all values *x* such that $|x| \geq 1$, the interval of convergence of this series is $(-1,1)$. Since the length of the interval is 2, the radius of convergence is 1.

Note:
Definition

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers *x* where the series converges is the interval of convergence. If there exists a real number $R>0$ such that the series converges for $|x-a|<R$ and diverges for $|x-a|>R$, then *R* is the radius of convergence. If the series converges only at $x=a$, we say the radius of convergence is $R=0$. If the series converges for all real numbers *x*, we say the radius of convergence is $R=\infty$ ([link]).



For a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ graph (a) shows a radius of convergence at $R=0$, graph (b) shows a radius of convergence at $R=\infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x=a+R$ and $x=a-R$.

To determine the interval of convergence for a power series, we typically apply the ratio test. In [link], we show the three different possibilities illustrated in [link].

Example:
Exercise:

Problem:
Finding the Interval and Radius of Convergence

For each of the following series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
b. $\sum_{n=0}^{\infty} n! x^n$
c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solution:

a. To check for convergence, apply the ratio test. We have
Equation:

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \\ &= |x| \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0 < 1 \end{aligned}$$

for all values of *x*. Therefore, the series converges for all real numbers *x*. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

b. Apply the ratio test. For $x \neq 0$, we see that

Equation:

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}|(n+1)x| \\ &= |x|\lim_{n\to\infty}(n+1) \\ &= \infty.\end{aligned}$$

Therefore, the series diverges for all $x \neq 0$. Since the series is centered at $x = 0$, it must converge there, so the series converges only for $x \neq 0$. The interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0$.

c. In order to apply the ratio test, consider
Equation:

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}\cdot\frac{(n+1)3^n}{(x-2)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{3(n+2)}\right| \\ &= \frac{|x-2|}{3}.\end{aligned}$$

The ratio $\rho < 1$ if $|x-2| < 3$. Since $|x-2| < 3$ implies that $-3 < x-2 < 3$, the series converges absolutely if $-1 < x < 5$. The ratio $\rho > 1$ if $|x-2| > 3$. Therefore, the series diverges if $x < -1$ or $x > 5$. The ratio test is inconclusive if $\rho = 1$. The ratio $\rho = 1$ if and only if $x = -1$ or $x = 5$. We need to test these values of *x* separately. For $x = -1$, the series is given by
Equation:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Since this is the alternating harmonic series, it converges. Thus, the series converges at $x = -1$. For $x = 5$, the series is given by
Equation:

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This is the harmonic series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.

Note:
Exercise:

Problem:

Find the interval and radius of convergence for the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

Solution:

The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

Hint

Apply the ratio test to check for absolute convergence.

Representing Functions as Power Series

Being able to represent a function by an “infinite polynomial” is a powerful tool. Polynomial functions are the easiest functions to analyze, since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is, when can we represent a function by a power series?

Consider again the geometric series
Equation:

$$1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty} x^n.$$

Recall that the geometric series
Equation:

$$a+ar+ar^2+ar^3+\cdots$$

converges if and only if $|r|<1$. In that case, it converges to $\frac{a}{1-r}$. Therefore, if $|x|<1$, the series in [link] converges to $\frac{1}{1-x}$ and we write
Equation:

$$1+x+x^2+x^3+\cdots=\frac{1}{1-x} \text{ for } |x|<1.$$

As a result, we are able to represent the function $f(x)=\frac{1}{1-x}$ by the power series
Equation:

$$1+x+x^2+x^3+\cdots \text{ when } |x|<1.$$

We now show graphically how this series provides a representation for the function $f(x) = \frac{1}{1-x}$ by comparing the graph of *f* with the graphs of several of the partial sums of this infinite series.

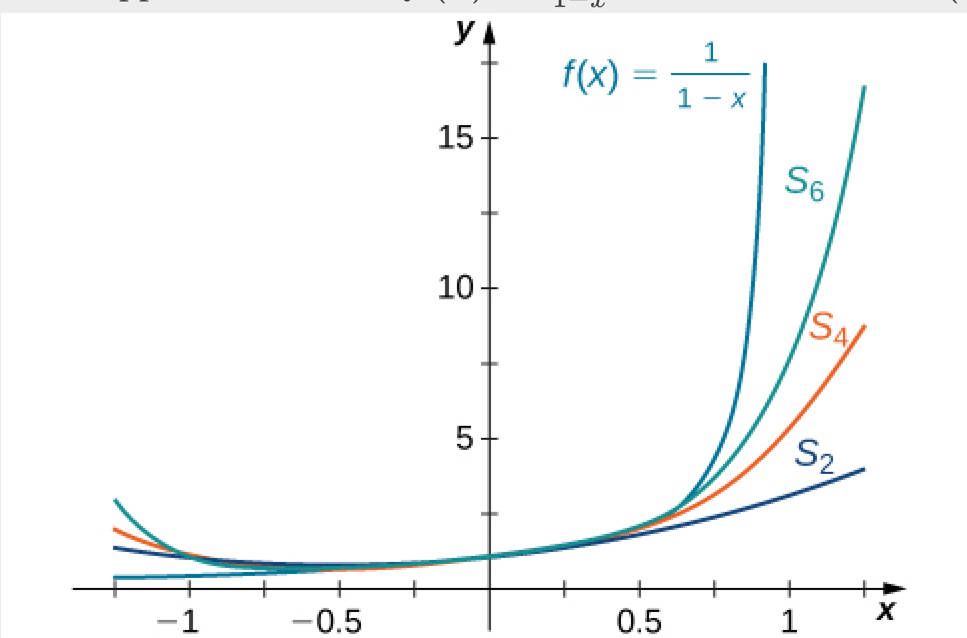
Example:
Exercise:

Problem:
Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x) = \frac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ for $N = 2, 4, 6$ on the interval $(-1, 1)$. Comment on the approximation S_N as *N* increases.

Solution:

From the graph in [link] you see that as *N* increases, S_N becomes a better approximation for $f(x) = \frac{1}{1-x}$ for *x* in the interval $(-1, 1)$.



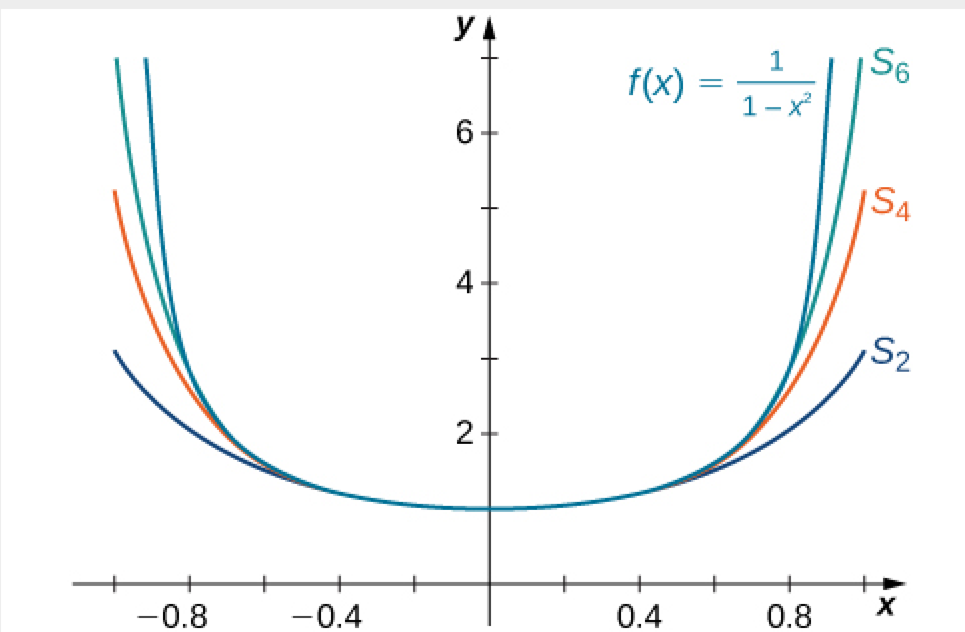
The graph shows a function and three approximations of it by partial sums of a power series.

Note:
Exercise:

Problem:

Sketch a graph of $f(x) = \frac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Solution:



Hint

$S_N(x) = 1 + x^2 + \cdots + x^{2N} = \frac{1-x^{2(N+1)}}{1-x^2}$

Next we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example:
Exercise:

Problem:
Representing a Function with a Power Series

Use a power series to represent each of the following functions f. Find the interval of convergence.

a. $f(x) = \frac{1}{1+x^3}$
b. $f(x) = \frac{x^2}{4-x^2}$

Solution:

a. You should recognize this function *f* as the sum of a geometric series, because
Equation:

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}.$$

Using the fact that, for $|r| < 1$, $\frac{a}{1-r}$ is the sum of the geometric series
Equation:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots,$$

we see that, for $\left|-x^3\right| < 1$,
Equation:

$$\begin{aligned} \frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=0}^{\infty} \left(-x^3\right)^n \\ &= 1 - x^3 + x^6 - x^9 + \cdots. \end{aligned}$$

Since this series converges if and only if $\left|-x^3\right| < 1$, the interval of convergence is $(-1, 1)$, and we have
Equation:

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots \text{ for } |x| < 1.$$

b. This function is not in the exact form of a sum of a geometric series. However, with a little algebraic manipulation, we can relate *f* to a geometric series. By factoring 4 out of the two terms in the denominator, we obtain
Equation:

$$\begin{aligned} \frac{x^2}{4-x^2} &= \frac{x^2}{4\left(\frac{1-x^2}{4}\right)} \\ &= \frac{x^2}{4\left(1-\left(\frac{x}{2}\right)^2\right)}. \end{aligned}$$

Therefore, we have
Equation:

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4\left(1-\left(\frac{x}{2}\right)^2\right)} \\ &= \frac{\frac{x^2}{4}}{1-\left(\frac{x}{2}\right)^2} \\ &= \sum_{n=0}^{\infty} \frac{x^2}{4}\left(\frac{x}{2}\right)^{2n}.\end{aligned}$$

The series converges as long as $\left|\left(\frac{x}{2}\right)^2\right| < 1$ (note that when $\left|\left(\frac{x}{2}\right)^2\right| = 1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2, 2)$ and

Equation:

$$\begin{aligned}\frac{x^2}{4-x^2} &= \sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}} \\ &= \frac{x^2}{4} + \frac{x^4}{4^2} + \frac{x^6}{4^3} + \cdots\end{aligned}$$

for $|x| < 2$.

Note:

Exercise:

Problem:

Represent the function $f(x) = \frac{x^3}{2-x}$ using a power series and find the interval of convergence.

Solution:

$$\sum_{n=0}^{\infty} \frac{x^{n+3}}{2^{n+1}}$$ with interval of convergence $(-2, 2)$

Hint

Rewrite *f* in the form $f(x) = \frac{g(x)}{1-h(x)}$ for some functions *g* and *h*.

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions, and how we can make use of these representations to evaluate, differentiate, and integrate various functions.

Key Concepts

- For a power series centered at $x = a$, one of the following three properties hold:

  i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  ii. The power series converges for all real numbers *x*. In this case, we say that the radius of convergence is $R = \infty$.
  iii. There is a real number *R* such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$. In this case, the radius of convergence is *R*.

- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The ratio test may often be used to determine the radius of convergence.
- The geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

Key Equations

- **Power series centered at $\mathbf{x = 0}$**
  $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$
- **Power series centered at $\mathbf{x = a}$**
  $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$

In the following exercises, state whether each statement is true, or give an example to show that it is false.

Exercise:

Problem: If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

Solution:

True. If a series converges then its terms tend to zero.

Exercise:

Problem: $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.

Exercise:

Problem:

Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

Solution:

False. It would imply that $a_n x^n \to 0$ for $|x| < R$. If $a_n = n^n$, then $a_n x^n = (nx)^n$ does not tend to zero for any $x \neq 0$.

Exercise:

Problem:

If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \leq |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R.

Exercise:

Problem:

Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 1$
b. $x = 2$
c. $x = 3$
d. $x = 0$
e. $x = 5.99$
f. $x = 0.000001$

Solution:

It must converge on $(0, 6]$ and hence at: a. $x = 1$; b. $x = 2$; c. $x = 3$; d. $x = 0$; e. $x = 5.99$; and f. $x = 0.000001$.

Exercise:

Problem:

Suppose that $\sum_{n=0}^{\infty} a_n (x+1)^n$ converges at $x = -2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at *x*, then it converges at any point closer to *c* than *x*.

a. $x = 2$
b. $x = -1$
c. $x = -3$
d. $x = 0$
e. $x = 0.99$
f. $x = 0.000001$

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$. Find the radius of convergence for each series.

Exercise:

Problem: $\sum_{n=0}^{\infty} a_n 2^n x^n$

Solution:

$\left|\frac{a_{n+1} 2^{n+1} x^{n+1}}{a_n 2^n x^n}\right| = 2\,|x|\,\left|\frac{a_{n+1}}{a_n}\right| \to 2\,|x|$ so $R = \frac{1}{2}$

Exercise:

Problem: $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$

Exercise:

Problem: $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

Solution:

$$\left|\frac{a_{n+1}\left(\frac{\pi}{e}\right)^{n+1} x^{n+1}}{a_n\left(\frac{\pi}{e}\right)^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e} \text{ so } R = \frac{e}{\pi}$$

Exercise:

Problem: $\sum_{n=0}^{\infty} \frac{a_n (-1)^n x^n}{10^n}$

Exercise:

Problem: $\sum_{n=0}^{\infty} a_n (-1)^n x^{2n}$

Solution:

$$\left|\frac{a_{n+1}(-1)^{n+1} x^{2n+2}}{a_n(-1)^n x^{2n}}\right| = \left|x^2\right|\left|\frac{a_{n+1}}{a_n}\right| \to \left|x^2\right| \text{ so } R = 1$$

Exercise:

Problem: $\sum_{n=0}^{\infty} a_n (-4)^n x^{2n}$

In the following exercises, find the radius of convergence *R* and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Solution:

$a_n = \frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n} \to 2x$. so $R = \frac{1}{2}$. When $x = \frac{1}{2}$ the series is harmonic and diverges. When $x = -\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I = \left[-\frac{1}{2}, \frac{1}{2}\right)$.

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$

Solution:

$a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Solution:

$a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

Exercise:

Problem: $\displaystyle\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$

Exercise:

Problem: $\displaystyle\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

Solution:

$a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm\frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is $I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Exercise:

Problem: $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Solution:

$a_n = \frac{10^n}{n!}, \frac{a_{n+1}x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all *x* by the ratio test and $I = (-\infty, \infty)$.

Exercise:

Problem: $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\ln(2n)}$

In the following exercises, find the radius of convergence of each series.

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

Solution:

$a_k = \frac{(k!)^2}{(2k)!}$ so $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}$ so $R = 4$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(2n)!x^n}{n^{2n}}$

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} x^k$

Solution:

$a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ so $\frac{a_{k+1}}{a_k} = \frac{k+1}{2k+1} \to \frac{1}{2}$ so $R = 2$

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!} x^k$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Solution:

$a_n = \frac{1}{\binom{2n}{n}}$ so $\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(2n+2)!} \frac{2n!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4}$ so $R = 4$

Exercise:

Problem: $\sum_{n=1}^{\infty} \sin^2 n x^n$

In the following exercises, use the ratio test to determine the radius of convergence of each series.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

Solution:

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27}$ so $R = 27$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!} x^n$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Solution:

$a_n = \frac{n!}{n^n}$ so $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$ so $R = e$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1, 1)$, find the power series for each function with the given center *a*, and identify its interval of convergence.

Exercise:

Problem: $f(x) = \frac{1}{x}; a = 1$ (*Hint:* $\frac{1}{x} = \frac{1}{1-(1-x)}$)

Solution:

$f(x) = \sum_{n=0}^{\infty} (1-x)^n$ on $I = (0, 2)$

Exercise:

Problem: $f\left(x\right)=\frac{1}{1-x^2};a=0$

Exercise:

Problem: $f\left(x\right)=\frac{x}{1-x^2};a=0$

Solution:

$$\sum_{n=0}^{\infty} x^{2n+1} \text{ on } I=\left(-1,1\right)$$

Exercise:

Problem: $f\left(x\right)=\frac{1}{1+x^2};a=0$

Exercise:

Problem: $f\left(x\right)=\frac{x^2}{1+x^2};a=0$

Solution:

$$\sum_{n=0}^{\infty} \left(-1\right)^n x^{2n+2} \text{ on } I=\left(-1,1\right)$$

Exercise:

Problem: $f\left(x\right)=\frac{1}{2-x};a=1$

Exercise:

Problem: $f\left(x\right)=\frac{1}{1-2x};a=0.$

Solution:

$$\sum_{n=0}^{\infty} 2^n x^n \text{ on } \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Exercise:

Problem: $f(x) = \frac{1}{1-4x^2}; a = 0$

Exercise:

Problem: $f(x) = \frac{x^2}{1-4x^2}; a = 0$

Solution:

$$\sum_{n=0}^{\infty} 4^n x^{2n+2} \text{ on } \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Exercise:

Problem: $f(x) = \frac{x^2}{5-4x+x^2}; a = 2$

Use the next exercise to find the radius of convergence of the given series in the subsequent exercises.

Exercise:

Problem:

Explain why, if $|a_n|^{1/n} \to r > 0$, then $|a_n x^n|^{1/n} \to |x|r < 1$ whenever $|x| < \frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{r}$.

Solution:

$|a_n x^n|^{1/n} = |a_n|^{1/n}|x| \to |x|r$ as $n \to \infty$ and $|x|r < 1$ when $|x| < \frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ converges when $|x| < \frac{1}{r}$ by the *n*th root test.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

Exercise:

Problem: $\sum_{k=1}^{\infty} \left(\frac{k-1}{2k+3}\right)^k x^k$

Solution:

$a_k = \left(\frac{k-1}{2k+3}\right)^k$ so $(a_k)^{1/k} \to \frac{1}{2} < 1$ so $R = 2$

Exercise:

Problem: $\sum_{k=1}^{\infty} \left(\frac{2k^2-1}{k^2+3}\right)^k x^k$

Exercise:

Problem: $\sum_{n=1}^{\infty} a_n = \left(n^{1/n} - 1\right)^n x^n$

Solution:

$a_n = \left(n^{1/n} - 1\right)^n$ so $(a_n)^{1/n} \to 0$ so $R = \infty$

Exercise:

Problem:

Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is even. Explain why $p(x) = -p(-x)$.

Exercise:

Problem:

Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is odd. Explain why $p(x) = p(-x)$.

Solution:

We can rewrite $p(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $p(x) = p(-x)$ since $x^{2n+1} = -(-x)^{2n+1}$.

Exercise:

Problem:

Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(Ax)$.

Exercise:

Problem:

Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(2x - 1)$.

Solution:

If $x \in [0,1]$, then $y = 2x - 1 \in [-1,1]$ so $p(2x-1) = p(y) = \sum_{n=0}^{\infty} a_n y^n$ converges.

In the following exercises, suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ where $a_n \geq 0$ for each *n*. State whether each series converges on the full interval $(-1,1)$, or if there is not enough information to draw a conclusion. Use the comparison test when appropriate.

Exercise:

Problem: $\sum_{n=0}^{\infty} a_n x^{2n}$

Exercise:

Problem: $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

Solution:

Converges on $(-1,1)$ by the ratio test

Exercise:

Problem: $\sum_{n=0}^{\infty} a_{2n} x^n \ \left(Hint\colon x = \pm\sqrt{x^2}\right)$

Exercise:

Problem:

$\sum_{n=0}^{\infty} a_{n^2} x^{n^2}$ (*Hint:* Let $b_k = a_k$ if $k = n^2$ for some *n*, otherwise $b_k = 0.$)

Solution:

Consider the series $\sum b_k x^k$ where $b_k = a_k$ if $k = n^2$ and $b_k = 0$ otherwise. Then $b_k \leq a_k$ and so the series converges on $(-1, 1)$ by the comparison test.

Exercise:

Problem:

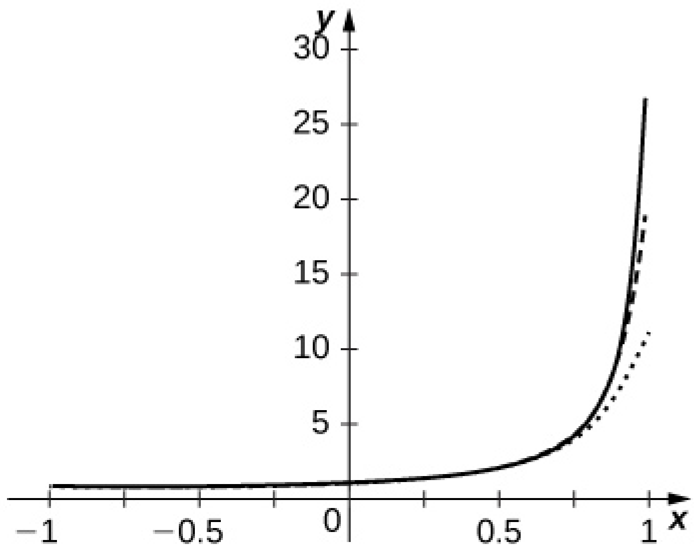
Suppose that $p(x)$ is a polynomial of degree *N*. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n) x^n$.

Exercise:

Problem:

[T] Plot the graphs of $\frac{1}{1-x}$ and of the partial sums $S_N = \sum_{n=0}^{N} x^n$ for $n = 10, 20, 30$ on the interval $[-0.99, 0.99]$. Comment on the approximation of $\frac{1}{1-x}$ by S_N near $x = -1$ and near $x = 1$ as *N* increases.

Solution:



The approximation is more accurate near $x = -1$. The partial sums follow $\frac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

Exercise:

Problem:

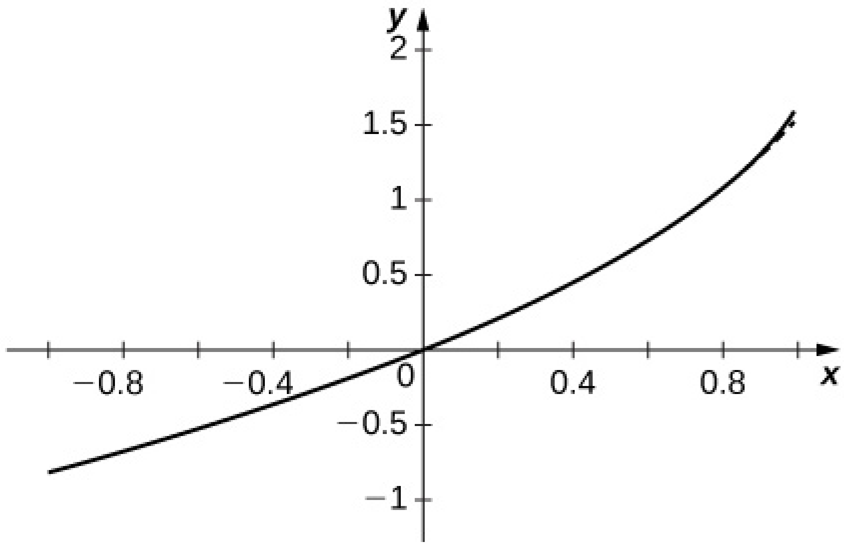
[T] Plot the graphs of $-\ln(1-x)$ and of the partial sums $S_N = \sum_{n=1}^{N} \frac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Exercise:

Problem:

[T] Plot the graphs of the partial sums $S_n = \sum_{n=1}^{N} \frac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Solution:



The approximation appears to stabilize quickly near both $x = \pm 1$.

Exercise:

Problem:

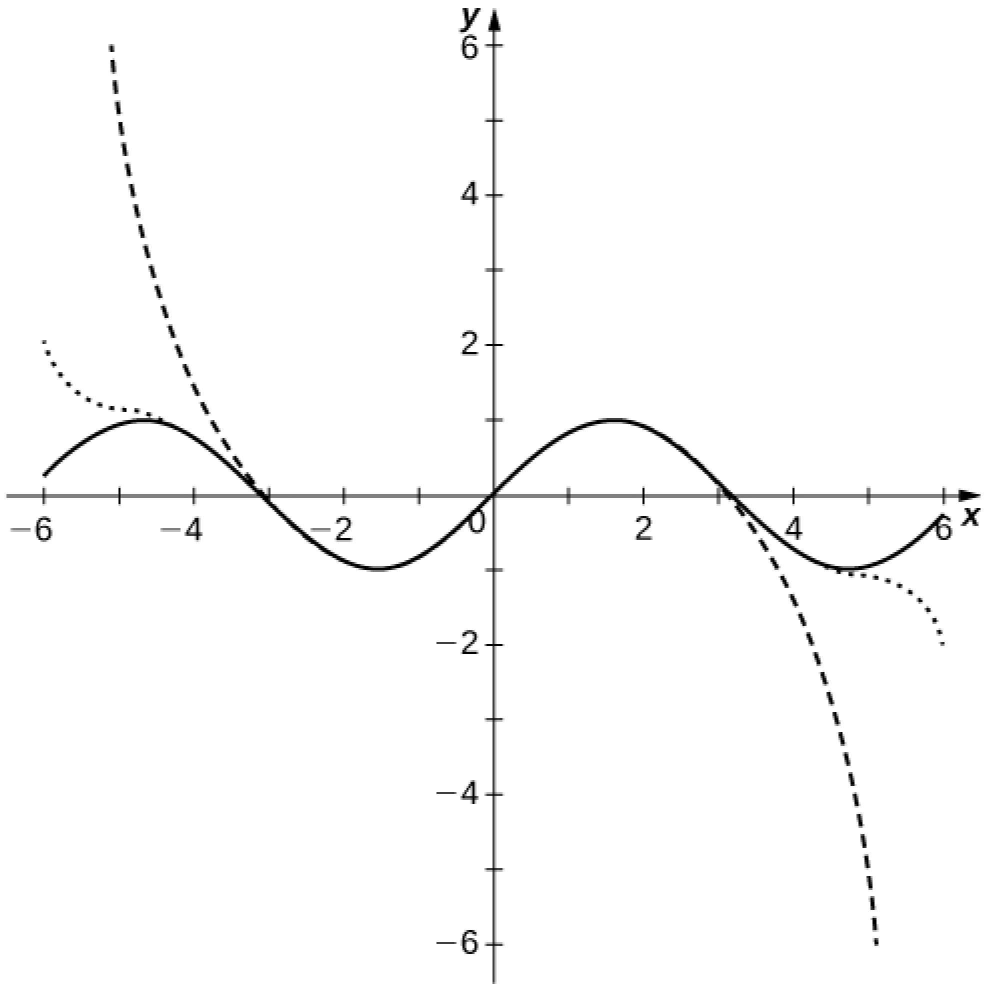
[T] Plot the graphs of the partial sums $S_N = \sum_{n=1}^{N} \sin n x^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as *N* increases.

Exercise:

Problem:

[T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\sin x$ as *N* increases.

Solution:



The polynomial curves have roots close to those of $\sin x$ up to their degree and then the polynomials diverge from $\sin x$.

Exercise:

Problem:

[T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\cos x$ as N increases.

Glossary

interval of convergence
: the set of real numbers x for which a power series converges

power series
: a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

radius of convergence
: if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

Properties of Power Series

- Combine power series by addition or subtraction.
- Create a new power series by multiplication by a power of the variable or a constant, or by substitution.
- Multiply two power series together.
- Differentiate and integrate power series term-by-term.

In the preceding section on power series and functions we showed how to represent certain functions using power series. In this section we discuss how power series can be combined, differentiated, or integrated to create new power series. This capability is particularly useful for a couple of reasons. First, it allows us to find power series representations for certain elementary functions, by writing those functions in terms of functions with known power series. For example, given the power series representation for $f\left(x\right)=\frac{1}{1-x}$, we can find a power series representation for $f'\left(x\right)=\frac{1}{{\left(1-x\right)}^{2}}$. Second, being able to create power series allows us to define new functions that cannot be written in terms of elementary functions. This capability is particularly useful for solving differential equations for which there is no solution in terms of elementary functions.

Combining Power Series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new power series, also with the same interval of convergence. Similarly, we can multiply a power series by a power of *x* or evaluate a power series at x^m for a positive integer *m* to create a new power series. Being able to do this allows us to find power series representations for certain functions by using power series representations of other functions. For example, since we know the power series representation for $f\left(x\right)=\frac{1}{1-x}$, we can find power series representations for related functions, such as

Equation:

$$y=\frac{3x}{1-x^2}\text{ and }y=\frac{1}{\left(x-1\right)\left(x-3\right)}.$$

In [link] we state results regarding addition or subtraction of power series, composition of a power series, and multiplication of a power series by a power of the variable. For simplicity, we state the theorem for power series centered at $x=0$. Similar results hold for power series centered at $x=a$.

Note:

Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty}c_nx^n$ and $\sum_{n=0}^{\infty}d_nx^n$ converge to the functions *f* and *g*, respectively, on a common interval *I*.

i. The power series $\sum_{n=0}^{\infty}\left(c_nx^n\pm d_nx^n\right)$ converges to $f\pm g$ on *I*.
ii. For any integer $m\ge 0$ and any real number *b*, the power series $\sum_{n=0}^{\infty}bx^mc_nx^n$ converges to $bx^mf\left(x\right)$ on *I*.
iii. For any integer $m\ge 0$ and any real number *b*, the series $\sum_{n=0}^{\infty}c_n{\left(bx^m\right)}^n$ converges to $f\left(bx^m\right)$ for all *x* such that bx^m is in *I*.

Proof

We prove i. in the case of the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions *f* and *g*, respectively, on the interval *I*. Let *x* be a point in *I* and let $S_N(x)$ and $T_N(x)$ denote the *N*th partial sums of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$, respectively. Then the sequence $\{S_N(x)\}$ converges to $f(x)$ and the sequence $\{T_N(x)\}$ converges to $g(x)$. Furthermore, the *N*th partial sum of $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ is

Equation:

$$\begin{aligned}\sum_{n=0}^{N}(c_n x^n + d_n x^n) &= \sum_{n=0}^{N} c_n x^n + \sum_{n=0}^{N} d_n x^n \\ &= S_N(x) + T_N(x).\end{aligned}$$

Because
Equation:

$$\begin{aligned}\lim_{N\to\infty}(S_N(x) + T_N(x)) &= \lim_{N\to\infty} S_N(x) + \lim_{N\to\infty} T_N(x) \\ &= f(x) + g(x),\end{aligned}$$

we conclude that the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ converges to $f(x) + g(x)$.

□

We examine products of power series in a later theorem. First, we show several applications of [link] and how to find the interval of convergence of a power series given the interval of convergence of a related power series.

Example:
Exercise:

Problem:
Combining Power Series

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a power series whose interval of convergence is $(-1, 1)$, and suppose that $\sum_{n=0}^{\infty} b_n x^n$ is a power series whose interval of convergence is $(-2, 2)$.

a. Find the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$.

b. Find the interval of convergence of the series $\sum_{n=0}^{\infty} a_n 3^n x^n$.

Solution:

a. Since the interval $(-1, 1)$ is a common interval of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, the interval of convergence of the series $\sum_{n=0}^{\infty} (a_n x^n + b_n x^n)$ is $(-1, 1)$.

b. Since $\sum_{n=0}^{\infty} a_n x^n$ is a power series centered at zero with radius of convergence 1, it converges for all *x* in the interval $(-1, 1)$. By [link], the series

Equation:

$$\sum_{n=0}^{\infty} a_n 3^n x^n = \sum_{n=0}^{\infty} a_n (3x)^n$$

converges if 3*x* is in the interval $(-1, 1)$. Therefore, the series converges for all *x* in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

Note:
Exercise:

Problem:

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has an interval of convergence of $(-1, 1)$. Find the interval of convergence of $\sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^n$.

Solution:

Interval of convergence is $(-2, 2)$.

Hint

Find the values of *x* such that $\frac{x}{2}$ is in the interval $(-1, 1)$.

In the next example, we show how to use [link] and the power series for a function *f* to construct power series for functions related to *f*. Specifically, we consider functions related to the function $f(x) = \frac{1}{1-x}$ and we use the fact that

Equation:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $|x| < 1$.

Example:

Exercise:

Problem:
Constructing Power Series from Known Power Series

Use the power series representation for $f(x) = \frac{1}{1-x}$ combined with [link] to construct a power series for each of the following functions. Find the interval of convergence of the power series.

a. $f(x) = \frac{3x}{1+x^2}$
b. $f(x) = \frac{1}{(x-1)(x-3)}$

Solution:

a. First write $f(x)$ as
Equation:

$$f(x) = 3x\left(\frac{1}{1-(-x^2)}\right).$$

Using the power series representation for $f(x) = \frac{1}{1-x}$ and parts ii. and iii. of [link], we find that a power series representation for *f* is given by
Equation:

$$\sum_{n=0}^{\infty} 3x(-x^2)^n = \sum_{n=0}^{\infty} 3(-1)^n x^{2n+1}.$$

Since the interval of convergence of the series for $\frac{1}{1-x}$ is $(-1,1)$, the interval of convergence for this new series is the set of real numbers *x* such that $|x^2| < 1$. Therefore, the interval of convergence is $(-1,1)$.

b. To find the power series representation, use partial fractions to write $f(x) = \frac{1}{(1-x)(x-3)}$ as the sum of two fractions. We have
Equation:

$$\begin{aligned}\frac{1}{(x-1)(x-3)} &= \frac{-1/2}{x-1} + \frac{1/2}{x-3}\\ &= \frac{1/2}{1-x} - \frac{1/2}{3-x}\\ &= \frac{1/2}{1-x} - \frac{1/6}{1-\frac{x}{3}}.\end{aligned}$$

First, using part ii. of [link], we obtain
Equation:

$$\frac{1/2}{1-x} = \sum_{n=0}^{\infty} \frac{1}{2}x^n \text{ for } |x| < 1.$$

Then, using parts ii. and iii. of [link], we have
Equation:

$$\frac{1/6}{1 - x/3} = \sum_{n=0}^{\infty} \frac{1}{6} \left(\frac{x}{3}\right)^n \text{ for } |x| < 3.$$

Since we are combining these two power series, the interval of convergence of the difference must be the smaller of these two intervals. Using this fact and part i. of [link], we have
Equation:

$$\frac{1}{(x-1)(x-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{2} - \frac{1}{6 \cdot 3^n}\right) x^n$$

where the interval of convergence is $(-1, 1)$.

Note:
Exercise:

Problem:

Use the series for $f(x) = \frac{1}{1-x}$ on $|x| < 1$ to construct a series for $\frac{1}{(1-x)(x-2)}$. Determine the interval of convergence.

Solution:

$\sum_{n=0}^{\infty} \left(-1 + \frac{1}{2^{n+1}}\right) x^n$. The interval of convergence is $(-1, 1)$.

Hint

Use partial fractions to rewrite $\frac{1}{(1-x)(x-2)}$ as the difference of two fractions.

In [link], we showed how to find power series for certain functions. In [link] we show how to do the opposite: given a power series, determine which function it represents.

Example:
Exercise:

Problem:
Finding the Function Represented by a Given Power Series

Consider the power series $\sum_{n=0}^{\infty} 2^n x^n$. Find the function *f* represented by this series. Determine the interval of convergence of the series.

Solution:

Writing the given series as
Equation:

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n,$$

we can recognize this series as the power series for

Equation:

$$f(x) = \frac{1}{1-2x}.$$

Since this is a geometric series, the series converges if and only if $|2x| < 1$. Therefore, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Note:

Exercise:

Problem:

Find the function represented by the power series $\sum_{n=0}^{\infty} \frac{1}{3^n} x^n$. Determine its interval of convergence.

Solution:

$f(x) = \frac{3}{3-x}$. The interval of convergence is $(-3, 3)$.

Hint

Write $\frac{1}{3^n} x^n = \left(\frac{x}{3}\right)^n$.

Recall the questions posed in the chapter opener about which is the better way of receiving payouts from lottery winnings. We now revisit those questions and show how to use series to compare values of payments over time with a lump sum payment today. We will compute how much future payments are worth in terms of today's dollars, assuming we have the ability to invest winnings and earn interest. The value of future payments in terms of today's dollars is known as the *present value* of those payments.

Example:

Exercise:

Problem:

Chapter Opener: Present Value of Future Winnings

(credit: modification of work by Robert Huffstutter, Flickr)

Suppose you win the lottery and are given the following three options: (1) Receive 20 million dollars today; (2) receive 1.5 million dollars per year over the next 20 years; or (3) receive 1 million dollars per year indefinitely (being passed on to your heirs). Which is the best deal, assuming that the annual interest rate is 5%? We answer this by working through the following sequence of questions.

a. How much is the 1.5 million dollars received annually over the course of 20 years worth in terms of today's dollars, assuming an annual interest rate of 5%?
b. Use the answer to part a. to find a general formula for the present value of payments of C dollars received each year over the next n years, assuming an average annual interest rate r.
c. Find a formula for the present value if annual payments of C dollars continue indefinitely, assuming an average annual interest rate r.
d. Use the answer to part c. to determine the present value of 1 million dollars paid annually indefinitely.
e. Use your answers to parts a. and d. to determine which of the three options is best.

Solution:

a. Consider the payment of 1.5 million dollars made at the end of the first year. If you were able to receive that payment today instead of one year from now, you could invest that money and earn 5% interest. Therefore, the present value of that money P_1 satisfies $P_1\,(1+0.05)=1.5$ million dollars. We conclude that
Equation:

$$P_1=\frac{1.5}{1.05}=\$1.429\text{ million dollars.}$$

Similarly, consider the payment of 1.5 million dollars made at the end of the second year. If you were able to receive that payment today, you could invest that money for two years, earning 5% interest, compounded annually. Therefore, the present value of that money P_2 satisfies $P_2(1+0.05)^2=1.5$ million dollars. We conclude that
Equation:

$$P_2=\frac{1.5}{(1.05)^2}=\$1.361\text{ million dollars.}$$

The value of the future payments today is the sum of the present values $P_1,P_2,\ldots,P_{20}$ of each of those annual payments. The present value P_k satisfies
Equation:

$$P_k=\frac{1.5}{(1.05)^k}.$$

Therefore,
Equation:

$$\begin{aligned}P&=\tfrac{1.5}{1.05}+\tfrac{1.5}{(1.05)^2}+\cdots+\tfrac{1.5}{(1.05)^{20}}\\&=\$18.693\text{ million dollars.}\end{aligned}$$

b. Using the result from part a. we see that the present value *P* of *C* dollars paid annually over the course of *n* years, assuming an annual interest rate *r*, is given by
Equation:

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n} \text{ dollars.}$$

c. Using the result from part b. we see that the present value of an annuity that continues indefinitely is given by the infinite series
Equation:

$$P = \sum_{n=0}^{\infty} \frac{C}{(1+r)^{n+1}}.$$

We can view the present value as a power series in *r*, which converges as long as $\left|\frac{1}{1+r}\right| < 1$. Since $r > 0$, this series converges. Rewriting the series as
Equation:

$$P = \frac{C}{(1+r)} \sum_{n=0}^{\infty} \left(\frac{1}{1+r}\right)^n,$$

we recognize this series as the power series for
Equation:

$$f(r) = \frac{1}{1 - \left(\frac{1}{1+r}\right)} = \frac{1}{\left(\frac{r}{1+r}\right)} = \frac{1+r}{r}.$$

We conclude that the present value of this annuity is
Equation:

$$P = \frac{C}{1+r} \cdot \frac{1+r}{r} = \frac{C}{r}.$$

d. From the result to part c. we conclude that the present value *P* of $C = 1$ million dollars paid out every year indefinitely, assuming an annual interest rate $r = 0.05$, is given by
Equation:

$$P = \frac{1}{0.05} = 20 \text{ million dollars.}$$

e. From part a. we see that receiving $1.5 million dollars over the course of 20 years is worth $18.693 million dollars in today’s dollars. From part d. we see that receiving $1 million dollars per year indefinitely is worth $20 million dollars in today’s dollars. Therefore, either receiving a lump-sum payment of $20 million dollars today or receiving $1 million dollars indefinitely have the same present value.

Multiplication of Power Series

We can also create new power series by multiplying power series. Being able to multiply two power series provides another way of finding power series representations for functions.

The way we multiply them is similar to how we multiply polynomials. For example, suppose we want to multiply
Equation:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

and
Equation:

$$\sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + \cdots.$$

It appears that the product should satisfy
Equation:

$$\begin{aligned} \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=-0}^{\infty} d_n x^n\right) &= \left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \cdot \left(d_0 + d_1 x + d_2 x^2 + \cdots\right) \\ &= c_0 d_0 + \left(c_1 d_0 + c_0 d_1\right) x + \left(c_2 d_0 + c_1 d_1 + c_0 d_2\right) x^2 + \cdots. \end{aligned}$$

In [link], we state the main result regarding multiplying power series, showing that if $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge on a common interval *I*, then we can multiply the series in this way, and the resulting series also converges on the interval *I*.

Note:
Multiplying Power Series
Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to *f* and *g*, respectively, on a common interval *I*.
Let
Equation:

$$\begin{aligned} e_n &= c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0 \\ &= \sum_{k=0}^{n} c_k d_{n-k}. \end{aligned}$$

Then
Equation:

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

and
Equation:

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x) \cdot g(x) \text{ on } I.$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the Cauchy product of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

We omit the proof of this theorem, as it is beyond the level of this text and is typically covered in a more advanced course. We now provide an example of this theorem by finding the power series representation for

Equation:

$$f(x) = \frac{1}{(1-x)(1-x^2)}$$

using the power series representations for

Equation:

$$y = \frac{1}{1-x} \text{ and } y = \frac{1}{1-x^2}.$$

Example:

Exercise:

Problem:

Multiplying Power Series

Multiply the power series representation

Equation:

$$\begin{aligned} \tfrac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

for $|x| < 1$ with the power series representation

Equation:

$$\begin{aligned} \tfrac{1}{1-x^2} &= \sum_{n=0}^{\infty} \left(x^2\right)^n \\ &= 1 + x^2 + x^4 + x^6 + \cdots \end{aligned}$$

for $|x| < 1$ to construct a power series for $f(x) = \frac{1}{(1-x)(1-x^2)}$ on the interval $(-1, 1)$.

Solution:

We need to multiply

Equation:

$$\left(1 + x + x^2 + x^3 + \cdots\right)\left(1 + x^2 + x^4 + x^6 + \cdots\right).$$

Writing out the first several terms, we see that the product is given by
Equation:

$$\left(1+x^2+x^4+x^6+\cdots\right)+\left(x+x^3+x^5+x^7+\cdots\right)+\left(x^2+x^4+x^6+x^8+\cdots\right)+\left(x^3+x^5+x^7+x^9\right.$$
$$=1+x+(1+1)x^2+(1+1)x^3+(1+1+1)x^4+(1+1+1)x^5+\cdots$$
$$=1+x+2x^2+2x^3+3x^4+3x^5+\cdots.$$

Since the series for $y=\frac{1}{1-x}$ and $y=\frac{1}{1-x^2}$ both converge on the interval $(-1,1)$, the series for the product also converges on the interval $(-1,1)$.

Note:
Exercise:

Problem: Multiply the series $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ by itself to construct a series for $\frac{1}{(1-x)(1-x)}$.

Solution:

$1+2x+3x^2+4x^3+\cdots$

Hint

Multiply the first few terms of $\left(1+x+x^2+x^3+\cdots\right)\left(1+x+x^2+x^3+\cdots\right)$.

Differentiating and Integrating Power Series

Consider a power series $\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+\cdots$ that converges on some interval *I*, and let f be the function defined by this series. Here we address two questions about f.

- Is f differentiable, and if so, how do we determine the derivative f'?
- How do we evaluate the indefinite integral $\int f(x)\,dx$?

We know that, for a polynomial with a finite number of terms, we can evaluate the derivative by differentiating each term separately. Similarly, we can evaluate the indefinite integral by integrating each term separately. Here we show that we can do the same thing for convergent power series. That is, if
Equation:

$$f(x)=c_nx^n=c_0+c_1x+c_2x^2+\cdots$$

converges on some interval *I*, then
Equation:

$$f'(x)=c_1+2c_2x+3c_3x^2+\cdots$$

and

Equation:

$$\int f\left(x\right)dx = C + c_0x + c_1\frac{x^2}{2} + c_2\frac{x^3}{3} + \cdots.$$

Evaluating the derivative and indefinite integral in this way is called **term-by-term differentiation of a power series** and **term-by-term integration of a power series**, respectively. The ability to differentiate and integrate power series term-by-term also allows us to use known power series representations to find power series representations for other functions. For example, given the power series for $f\left(x\right) = \frac{1}{1-x}$, we can differentiate term-by-term to find the power series for $f'\left(x\right) = \frac{1}{\left(1-x\right)^2}$. Similarly, using the power series for $g\left(x\right) = \frac{1}{1+x}$, we can integrate term-by-term to find the power series for $G\left(x\right) = \ln\left(1+x\right)$, an antiderivative of g. We show how to do this in [link] and [link]. First, we state [link], which provides the main result regarding differentiation and integration of power series.

Note:
Term-by-Term Differentiation and Integration for Power Series
Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the interval $(a-R, a+R)$ for some $R>0$. Let f be the function defined by the series
Equation:

$$\begin{aligned} f\left(x\right) &= \sum_{n=0}^{\infty} c_n(x-a)^n \\ &= c_0 + c_1\left(x-a\right) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \end{aligned}$$

for $|x-a| < R$. Then f is differentiable on the interval $(a-R, a+R)$ and we can find f' by differentiating the series term-by-term:
Equation:

$$\begin{aligned} f'\left(x\right) &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \\ &= c_1 + 2c_2\left(x-a\right) + 3c_3(x-a)^2 + \cdots \end{aligned}$$

for $|x-a| < R$. Also, to find $\int f\left(x\right)dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R, a+R)$, and we have
Equation:

$$\begin{aligned} \int f\left(x\right)dx &= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1} \\ &= C + c_0\left(x-a\right) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots \end{aligned}$$

for $|x-a| < R$.

The proof of this result is beyond the scope of the text and is omitted. Note that although [link] guarantees the same radius of convergence when a power series is differentiated or integrated term-by-term, it says nothing about

what happens at the endpoints. It is possible that the differentiated and integrated power series have different behavior at the endpoints than does the original series. We see this behavior in the next examples.

Example:
Exercise:

Problem:
Differentiating Power Series

a. Use the power series representation
Equation:

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

for $|x| < 1$ to find a power series representation for
Equation:

$$g(x) = \frac{1}{(1-x)^2}$$

on the interval $(-1, 1)$. Determine whether the resulting series converges at the endpoints.

b. Use the result of part a. to evaluate the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{4^n}$.

Solution:

a. Since $g(x) = \frac{1}{(1-x)^2}$ is the derivative of $f(x) = \frac{1}{1-x}$, we can find a power series representation for g by differentiating the power series for *f* term-by-term. The result is
Equation:

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^2} \\ &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) \\ &= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right) \\ &= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1)x^n \end{aligned}$$

for $|x| < 1$. [link] does not guarantee anything about the behavior of this series at the endpoints. Testing the endpoints by using the divergence test, we find that the series diverges at both endpoints $x = \pm 1$. Note that this is the same result found in [link].

b. From part a. we know that

Equation:

$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}.$$

Therefore,

Equation:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n+1}{4^n} &= \sum_{n=0}^{\infty}(n+1)\left(\frac{1}{4}\right)^n \\ &= \frac{1}{\left(1-\frac{1}{4}\right)^2} \\ &= \frac{1}{\left(\frac{3}{4}\right)^2} \\ &= \frac{16}{9}.\end{aligned}$$

Note:

Exercise:

Problem:

Differentiate the series $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$ term-by-term to find a power series representation for $\frac{2}{(1-x)^3}$ on the interval $(-1,1)$.

Solution:

$$\sum_{n=0}^{\infty}(n+2)(n+1)x^n$$

Hint

Write out the first several terms and apply the power rule.

Example:

Exercise:

Problem:

Integrating Power Series

For each of the following functions *f*, find a power series representation for *f* by integrating the power series for f' and find its interval of convergence.

a. $f(x) = \ln(1+x)$

b. $f(x) = \tan^{-1}x$

Solution:

a. For $f(x) = \ln(1+x)$, the derivative is $f'(x) = \frac{1}{1+x}$. We know that
Equation:

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= \sum_{n=0}^{\infty} (-x)^n \\ &= 1 - x + x^2 - x^3 + \cdots \end{aligned}$$

for $|x| < 1$. To find a power series for $f(x) = \ln(1+x)$, we integrate the series term-by-term.
Equation:

$$\begin{aligned} \int f'(x)\,dx &= \int \left(1 - x + x^2 - x^3 + \cdots\right) dx \\ &= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \end{aligned}$$

Since $f(x) = \ln(1+x)$ is an antiderivative of $\frac{1}{1+x}$, it remains to solve for the constant *C*. Since $\ln(1+0) = 0$, we have $C = 0$. Therefore, a power series representation for $f(x) = \ln(1+x)$ is
Equation:

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \end{aligned}$$

for $|x| < 1$. [link] does not guarantee anything about the behavior of this power series at the endpoints. However, checking the endpoints, we find that at $x = 1$ the series is the alternating harmonic series, which converges. Also, at $x = -1$, the series is the harmonic series, which diverges. It is important to note that, even though this series converges at $x = 1$, [link] does not guarantee that the series actually converges to $\ln(2)$. In fact, the series does converge to $\ln(2)$, but showing this fact requires more advanced techniques. (Abel's theorem, covered in more advanced texts, deals with this more technical point.) The interval of convergence is $(-1, 1]$.

b. The derivative of $f(x) = \tan^{-1}x$ is $f'(x) = \frac{1}{1+x^2}$. We know that
Equation:

$$\begin{aligned} \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\ &= \sum_{n=0}^{\infty} \left(-x^2\right)^n \\ &= 1 - x^2 + x^4 - x^6 + \cdots \end{aligned}$$

for $|x| < 1$. To find a power series for $f(x) = \tan^{-1}x$, we integrate this series term-by-term.
Equation:

$$\begin{aligned} \int f'(x)\,dx &= \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \end{aligned}$$

Since $\tan^{-1}(0) = 0$, we have $C = 0$. Therefore, a power series representation for $f(x) = \tan^{-1}x$ is
Equation:

$$\begin{aligned} \tan^{-1}x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

for $|x| < 1$. Again, [link] does not guarantee anything about the convergence of this series at the endpoints. However, checking the endpoints and using the alternating series test, we find that the series converges at $x = 1$ and $x = -1$. As discussed in part a., using Abel's theorem, it can be shown that the series actually converges to $\tan^{-1}(1)$ and $\tan^{-1}(-1)$ at $x = 1$ and $x = -1$, respectively. Thus, the interval of convergence is $[-1, 1]$.

Note:
Exercise:

Problem:

Integrate the power series $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ term-by-term to evaluate $\int \ln(1+x)\,dx$.

Solution:

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}$$

Hint

Use the fact that $\frac{x^{n+1}}{(n+1)n}$ is an antiderivative of $\frac{x^n}{n}$.

Up to this point, we have shown several techniques for finding power series representations for functions. However, how do we know that these power series are unique? That is, given a function *f* and a power series for *f* at *a*, is it possible that there is a different power series for *f* at *a* that we could have found if we had used a different technique? The answer to this question is no. This fact should not seem surprising if we think of power series as polynomials with an infinite number of terms. Intuitively, if
Equation:

$$c_0 + c_1x + c_2x^2 + \cdots = d_0 + d_1x + d_2x^2 + \cdots$$

for all values *x* in some open interval *I* about zero, then the coefficients c_n should equal d_n for $n \geq 0$. We now state this result formally in [link].

Note:
Uniqueness of Power Series

Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ and $\sum_{n=0}^{\infty} d_n (x-a)^n$ be two convergent power series such that
Equation:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} d_n (x-a)^n$$

for all *x* in an open interval containing a. Then $c_n = d_n$ for all $n \geq 0$.

Proof

Let
Equation:

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \\ &= d_0 + d_1(x-a) + d_2(x-a)^2 + d_3(x-a)^3 + \cdots. \end{aligned}$$

Then $f(a) = c_0 = d_0$. By [link], we can differentiate both series term-by-term. Therefore,
Equation:

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \\ &= d_1 + 2d_2(x-a) + 3d_3(x-a)^2 + \cdots, \end{aligned}$$

and thus, $f'(a) = c_1 = d_1$. Similarly,
Equation:

$$\begin{aligned} f''(x) &= 2c_2 + 3 \cdot 2c_3(x-a) + \cdots \\ &= 2d_2 + 3 \cdot 2d_3(x-a) + \cdots \end{aligned}$$

implies that $f''(a) = 2c_2 = 2d_2$, and therefore, $c_2 = d_2$. More generally, for any integer $n \geq 0$, $f^{(n)}(a) = n!c_n = n!d_n$, and consequently, $c_n = d_n$ for all $n \geq 0$.

□

In this section we have shown how to find power series representations for certain functions using various algebraic operations, differentiation, or integration. At this point, however, we are still limited as to the functions for which we can find power series representations. Next, we show how to find power series representations for many more functions by introducing Taylor series.

Key Concepts

- Given two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ that converge to functions *f* and *g* on a common interval *I*, the sum and difference of the two series converge to $f \pm g$, respectively, on *I*. In addition, for any real number *b* and integer $m \geq 0$, the series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ and the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ whenever bx^m is in the interval *I*.

- Given two power series that converge on an interval $(-R, R)$, the Cauchy product of the two power series converges on the interval $(-R, R)$.
- Given a power series that converges to a function *f* on an interval $(-R, R)$, the series can be differentiated term-by-term and the resulting series converges to f' on $(-R, R)$. The series can also be integrated term-by-term and the resulting series converges to $\int f(x)\,dx$ on $(-R, R)$.

Exercise:

Problem:

If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$, find the power series of $\frac{1}{2}(f(x) + g(x))$ and of $\frac{1}{2}(f(x) - g(x))$.

Solution:

$\frac{1}{2}(f(x) + g(x)) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ and $\frac{1}{2}(f(x) - g(x)) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

Exercise:

Problem:

If $C(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ and $S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$, find the power series of $C(x) + S(x)$ and of $C(x) - S(x)$.

In the following exercises, use partial fractions to find the power series of each function.

Exercise:

Problem: $\frac{4}{(x-3)(x+1)}$

Solution:

$$\frac{4}{(x-3)(x+1)} = \frac{1}{x-3} - \frac{1}{x+1} = -\frac{1}{3\left(1-\frac{x}{3}\right)} - \frac{1}{1-(-x)} = -\frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n - \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} \left((-1)^{n+1} - \frac{1}{3^{n+1}}\right.$$

Exercise:

Problem: $\frac{3}{(x+2)(x-1)}$

Exercise:

Problem: $\frac{5}{(x^2+4)(x^2-1)}$

Solution:

$$\frac{5}{(x^2+4)(x^2-1)} = \frac{1}{x^2-1} - \frac{1}{4}\,\frac{1}{1+\left(\frac{x}{2}\right)^2} = -\sum_{n=0}^{\infty} x^{2n} - \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \left((-1) + (-1)^{n+1} \frac{1}{2^{n+2}}\right) x^{2n}$$

Exercise:

Problem: $\frac{30}{(x^2+1)(x^2-9)}$

In the following exercises, express each series as a rational function.

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{x^n}$

Solution:

$\frac{1}{x} \sum_{n=0}^{\infty} \frac{1}{x^n} = \frac{1}{x} \frac{1}{1-\frac{1}{x}} = \frac{1}{x-1}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{x^{2n}}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \frac{1}{(x-3)^{2n-1}}$

Solution:

$\frac{1}{x-3} \frac{1}{1-\frac{1}{(x-3)^2}} = \frac{x-3}{(x-3)^2-1}$

Exercise:

Problem: $\sum_{n=1}^{\infty} \left(\frac{1}{(x-3)^{2n-1}} - \frac{1}{(x-2)^{2n-1}} \right)$

The following exercises explore applications of annuities.

Exercise:

Problem:

Calculate the present values P of an annuity in which \$10,000 is to be paid out annually for a period of 20 years, assuming interest rates of $r = 0.03$, $r = 0.05$, and $r = 0.07$.

Solution:

$P = P_1 + \cdots + P_{20}$ where $P_k = 10{,}000 \frac{1}{(1+r)^k}$. Then $P = 10{,}000 \sum_{k=1}^{20} \frac{1}{(1+r)^k} = 10{,}000 \frac{1-(1+r)^{-20}}{r}$. When $r = 0.03$, $P \approx 10{,}000 \times 14.8775 = 148{,}775$. When $r = 0.05$, $P \approx 10{,}000 \times 12.4622 = 124{,}622$. When $r = 0.07$, $P \approx 105{,}940$.

Exercise:

Problem:

Calculate the present values P of annuities in which \$9,000 is to be paid out annually perpetually, assuming interest rates of $r = 0.03, r = 0.05$ and $r = 0.07$.

Exercise:

Problem:

Calculate the annual payouts C to be given for 20 years on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03, r = 0.05$, and $r = 0.07$.

Solution:

In general, $P = \frac{C\left(1-(1+r)^{-N}\right)}{r}$ for N years of payouts, or $C = \frac{Pr}{1-(1+r)^{-N}}$. For $N = 20$ and $P = 100{,}000$, one has $C = 6721.57$ when $r = 0.03$; $C = 8024.26$ when $r = 0.05$; and $C \approx 9439.29$ when $r = 0.07$.

Exercise:

Problem:

Calculate the annual payouts C to be given perpetually on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03, r = 0.05$, and $r = 0.07$.

Exercise:

Problem:

Suppose that an annuity has a present value $P = 1\text{ million dollars}$. What interest rate r would allow for perpetual annual payouts of \$50,000?

Solution:

In general, $P = \frac{C}{r}$. Thus, $r = \frac{C}{P} = 5 \times \frac{10^4}{10^6} = 0.05$.

Exercise:

Problem:

Suppose that an annuity has a present value $P = 10\text{ million dollars}$. What interest rate r would allow for perpetual annual payouts of \$100,000?

In the following exercises, express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

Exercise:

Problem: $x + x^2 - x^3 + x^4 + x^5 - x^6 + \cdots$ (*Hint:* Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Solution:

$\left(x + x^2 - x^3\right)\left(1 + x^3 + x^6 + \cdots\right) = \frac{x+x^2-x^3}{1-x^3}$

Exercise:

Problem: $x + x^2 - x^3 - x^4 + x^5 + x^6 - x^7 - x^8 + \cdots$ (*Hint:* Group powers x^{4k}, x^{4k-1}, etc.)

Exercise:

Problem: $x - x^2 - x^3 + x^4 - x^5 - x^6 + x^7 - \cdots$ (*Hint:* Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Solution:

$\left(x-x^2-x^3\right)\left(1+x^3+x^6+\cdots\right)=\frac{x-x^2-x^3}{1-x^3}$

Exercise:

Problem: $\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}+\frac{x^5}{32}-\frac{x^6}{64}+\cdots$ (*Hint:* Group powers $\left(\frac{x}{2}\right)^{3k}$, $\left(\frac{x}{2}\right)^{3k-1}$, and $\left(\frac{x}{2}\right)^{3k-2}$.)

In the following exercises, find the power series of $f(x)g(x)$ given f and g as defined.

Exercise:

Problem: $f(x)=2\sum_{n=0}^{\infty}x^n, g(x)=\sum_{n=0}^{\infty}nx^n$

Solution:

$a_n=2, b_n=n$ so $c_n=\sum_{k=0}^{n}b_ka_{n-k}=2\sum_{k=0}^{n}k=(n)(n+1)$ and $f(x)g(x)=\sum_{n=1}^{\infty}n(n+1)x^n$

Exercise:

Problem:

$f(x)=\sum_{n=1}^{\infty}x^n, g(x)=\sum_{n=1}^{\infty}\frac{1}{n}x^n$. Express the coefficients of $f(x)g(x)$ in terms of $H_n=\sum_{k=1}^{n}\frac{1}{k}$.

Exercise:

Problem: $f(x)=g(x)=\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n$

Solution:

$a_n=b_n=2^{-n}$ so $c_n=\sum_{k=1}^{n}b_ka_{n-k}=2^{-n}\sum_{k=1}^{n}1=\frac{n}{2^n}$ and $f(x)g(x)=\sum_{n=1}^{\infty}n\left(\frac{x}{2}\right)^n$

Exercise:

Problem: $f(x)=g(x)=\sum_{n=1}^{\infty}nx^n$

In the following exercises, differentiate the given series expansion of *f* term-by-term to obtain the corresponding series expansion for the derivative of *f*.

Exercise:

Problem: $f(x)=\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^nx^n$

Solution:

The derivative of f is $-\frac{1}{(1+x)^2} = -\sum_{n=0}^{\infty}(-1)^n (n+1)x^n.$

Exercise:

Problem: $f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

In the following exercises, integrate the given series expansion of f term-by-term from zero to *x* to obtain the corresponding series expansion for the indefinite integral of f.

Exercise:

Problem: $f(x) = \frac{2x}{(1+x^2)^2} = \sum_{n=1}^{\infty}(-1)^n (2n)x^{2n-1}$

Solution:

The indefinite integral of f is $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$

Exercise:

Problem: $f(x) = \frac{2x}{1+x^2} = 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$

In the following exercises, evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

Exercise:

Problem: Evaluate $\sum_{n=1}^{\infty}\frac{n}{2^n}$ as $f'\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n.$

Solution:

$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}; f'\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{n}{2^{n-1}} = \frac{d}{dx}(1-x)^{-1}\Big|_{x=1/2} = \frac{1}{(1-x)^2}\Big|_{x=1/2} = 4$ so $\sum_{n=1}^{\infty}\frac{n}{2^n} = 2.$

Exercise:

Problem: Evaluate $\sum_{n=1}^{\infty}\frac{n}{3^n}$ as $f'\left(\frac{1}{3}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n.$

Exercise:

Problem: Evaluate $\sum_{n=2}^{\infty}\frac{n(n-1)}{2^n}$ as $f\prime\prime\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n.$

Solution:

$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; $f\prime\prime\left(\frac{1}{2}\right) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} = \frac{d^2}{dx^2}(1-x)^{-1}\Big|_{x=1/2} = \frac{2}{(1-x)^3}\Big|_{x=1/2} = 16$ so $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} = 4.$

Exercise:

Problem: Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ as $\int_0^1 f(t)\,dt$ where $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$.

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, use term-by-term differentiation or integration to find power series for each function centered at the given point.

Exercise:

Problem: $f(x) = \ln x$ centered at $x = 1$ (*Hint:* $x = 1 - (1-x)$)

Solution:

$$\int \sum (1-x)^n dx = \int \sum (-1)^n (x-1)^n dx = \sum \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

Exercise:

Problem: $\ln(1-x)$ at $x = 0$

Exercise:

Problem: $\ln\left(1-x^2\right)$ at $x = 0$

Solution:

$$-\int_{t=0}^{x^2} \frac{1}{1-t}dt = -\sum_{n=0}^{\infty} \int_0^{x^2} t^n dx - \sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$$

Exercise:

Problem: $f(x) = \frac{2x}{(1-x^2)^2}$ at $x = 0$

Exercise:

Problem: $f(x) = \tan^{-1}\left(x^2\right)$ at $x = 0$

Solution:

$$\int_0^{x^2} \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \int_0^{x^2} t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\Bigg|_{t=0}^{x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

Exercise:

Problem: $f(x) = \ln\left(1+x^2\right)$ at $x = 0$

Exercise:

Problem: $f(x) = \int_0^x \ln t dt$ where $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$

Solution:

Term-by-term integration gives
$\int_0^x \ln t dt = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+1}\right)(x-1)^{n+1} = (x-1)\ln x + \sum_{n=2}^{\infty} (-1)^{\prime}$

Exercise:

Problem:

[T] Evaluate the power series expansion $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ at $x = 1$ to show that $\ln(2)$ is the sum of the alternating harmonic series. Use the alternating series test to determine how many terms of the sum are needed to estimate $\ln(2)$ accurate to within 0.001, and find such an approximation.

Exercise:

Problem:

[T] Subtract the infinite series of $\ln(1-x)$ from $\ln(1+x)$ to get a power series for $\ln\left(\frac{1+x}{1-x}\right)$. Evaluate at $x = \frac{1}{3}$. What is the smallest *N* such that the *N*th partial sum of this series approximates $\ln(2)$ with an error less than 0.001?

Solution:

We have $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ so $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$. Thus,
$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \left(1 + (-1)^{n-1}\right) \frac{x^n}{n} = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$. When $x = \frac{1}{3}$ we obtain
$\ln(2) = 2\sum_{n=1}^{\infty} \frac{1}{3^{2n-1}(2n-1)}$. We have $2\sum_{n=1}^{3} \frac{1}{3^{2n-1}(2n-1)} = 0.69300\ldots$, while
$2\sum_{n=1}^{4} \frac{1}{3^{2n-1}(2n-1)} = 0.69313\ldots$ and $\ln(2) = 0.69314\ldots$; therefore, $N = 4$.

In the following exercises, using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

Exercise:

Problem: $\sum_{k=0}^{\infty} \left(x^k - x^{2k+1}\right)$

Exercise:

Problem: $\sum_{k=1}^{\infty} \frac{x^{3k}}{6k}$

Solution:

$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$ so $\sum_{k=1}^{\infty} \frac{x^{3k}}{6k} = -\frac{1}{6}\ln\left(1-x^3\right)$. The radius of convergence is equal to 1 by the ratio test.

Exercise:

Problem: $\sum_{k=1}^{\infty} \left(1+x^2\right)^{-k}$ using $y = \frac{1}{1+x^2}$

Exercise:

Problem: $\sum_{k=1}^{\infty} 2^{-kx}$ using $y = 2^{-x}$

Solution:

If $y = 2^{-x}$, then $\sum_{k=1}^{\infty} y^k = \frac{y}{1-y} = \frac{2^{-x}}{1-2^{-x}} = \frac{1}{2^x-1}$. If $a_k = 2^{-kx}$, then $\frac{a_{k+1}}{a_k} = 2^{-x} < 1$ when $x > 0$. So the series converges for all $x > 0$.

Exercise:

Problem: Show that, up to powers x^3 and y^3, $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfies $E(x+y) = E(x)E(y)$.

Exercise:

Problem: Differentiate the series $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ term-by-term to show that $E(x)$ is equal to its derivative.

Solution:

Answers will vary.

Exercise:

Problem:

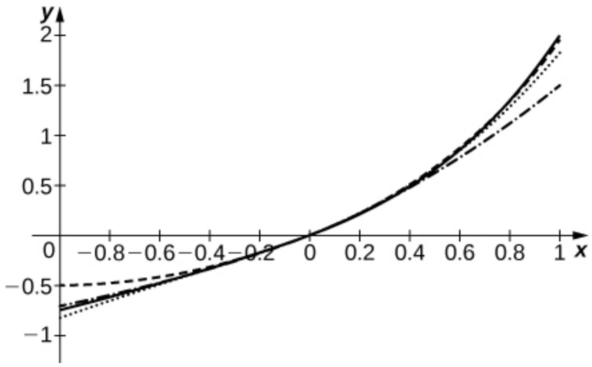
Show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a sum of even powers, that is, $a_n = 0$ if n is odd, then $F = \int_0^x f(t)\,dt$ is a sum of odd powers, while if f is a sum of odd powers, then F is a sum of even powers.

Exercise:

Problem:

[T] Suppose that the coefficients a_n of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{n} + \frac{a_{n-2}}{n(n-1)}$. For $a_0 = 0$ and $a_1 = 1$, compute and plot the sums $S_N = \sum_{n=0}^{N} a_n x^n$ for $N = 2, 3, 4, 5$ on $[-1, 1]$.

Solution:



The solid curve is S_5. The dashed curve is S_2, dotted is S_3, and dash-dotted is S_4

Exercise:

Problem:

[T] Suppose that the coefficients a_n of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{\sqrt{n}} - \frac{a_{n-2}}{\sqrt{n(n-1)}}$. For $a_0 = 1$ and $a_1 = 0$, compute and plot the sums $S_N = \sum_{n=0}^{N} a_n x^n$ for $N = 2, 3, 4, 5$ on $[-1, 1]$.

Exercise:

Problem:

[T] Given the power series expansion $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, determine how many terms *N* of the sum evaluated at $x = -1/2$ are needed to approximate $\ln(2)$ accurate to within 1/1000. Evaluate the corresponding partial sum $\sum_{n=1}^{N} (-1)^{n-1} \frac{x^n}{n}$.

Solution:

When $x = -\frac{1}{2}$, $-\ln(2) = \ln\left(\frac{1}{2}\right) = -\sum_{n=1}^{\infty} \frac{1}{n2^n}$. Since $\sum_{n=11}^{\infty} \frac{1}{n2^n} < \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1}{2^{10}}$, one has $\sum_{n=1}^{10} \frac{1}{n2^n} = 0.69306\ldots$ whereas $\ln(2) = 0.69314\ldots$; therefore, $N = 10$.

Exercise:

Problem:

[T] Given the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$, use the alternating series test to determine how many terms *N* of the sum evaluated at $x = 1$ are needed to approximate $\tan^{-1}(1) = \frac{\pi}{4}$ accurate to within 1/1000. Evaluate the corresponding partial sum $\sum_{k=0}^{N} (-1)^k \frac{x^{2k+1}}{2k+1}$.

Exercise:

Problem:

[T] Recall that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$. Assuming an exact value of $\left(\frac{1}{\sqrt{3}}\right)$, estimate $\frac{\pi}{6}$ by evaluating partial sums $S_N\left(\frac{1}{\sqrt{3}}\right)$ of the power series expansion $\tan^{-1}(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$ at $x=\frac{1}{\sqrt{3}}$. What is the smallest number N such that $6S_N\left(\frac{1}{\sqrt{3}}\right)$ approximates π accurately to within 0.001? How many terms are needed for accuracy to within 0.00001?

Solution:

$6S_N\left(\frac{1}{\sqrt{3}}\right)=2\sqrt{3}\sum_{n=0}^{N}(-1)^n\frac{1}{3^n(2n+1)}$. One has $\pi-6S_4\left(\frac{1}{\sqrt{3}}\right)=0.00101\ldots$ and $\pi-6S_5\left(\frac{1}{\sqrt{3}}\right)=0.00028\ldots$ so $N=5$ is the smallest partial sum with accuracy to within 0.001. Also, $\pi-6S_7\left(\frac{1}{\sqrt{3}}\right)=0.00002\ldots$ while $\pi-6S_8\left(\frac{1}{\sqrt{3}}\right)=-0.000007\ldots$ so $N=8$ is the smallest N to give accuracy to within 0.00001.

Glossary

term-by-term differentiation of a power series

a technique for evaluating the derivative of a power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ by evaluating the derivative of each term separately to create the new power series $\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}$

term-by-term integration of a power series

a technique for integrating a power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ by integrating each term separately to create the new power series $C+\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}$

Taylor and Maclaurin Series

- Describe the procedure for finding a Taylor polynomial of a given order for a function.
- Explain the meaning and significance of Taylor's theorem with remainder.
- Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections we discussed how to find power series representations for certain types of functions—specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

Overview of Taylor/Maclaurin Series

Consider a function f that has a power series representation at $x=a$. Then the series has the form
Equation:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series [link] is a representation for f at $x=a$, we certainly want the series to equal $f(a)$ at $x=a$. Evaluating the series at $x=a$, we see that
Equation:

$$\begin{aligned}\sum_{n=0}^{\infty} c_n(x-a)^n & = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots \\ & = c_0.\end{aligned}$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f'(a)$ at $x=a$. Differentiating [link] term-by-term, we see that
Equation:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots.$$

Therefore, at $x=a$, the derivative is
Equation:

$$\begin{aligned}\tfrac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) & = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots \\ & = c_1.\end{aligned}$$

Therefore, the derivative of the series equals $f'(a)$ if the coefficient $c_1 = f'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series [link] will agree with all the corresponding derivatives of f at $x=a$. The second and third derivatives of [link] are given by
Equation:

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 2c_2 + 3 \cdot 2c_3\,(x-a) + 4 \cdot 3c_4 (x-a)^2 + \cdots$$

and
Equation:

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4\,(x-a) + 5 \cdot 4 \cdot 3c_5 (x-a)^2 + \cdots.$$

Therefore, at $x = a$, the second and third derivatives
Equation:

$$\begin{aligned} \tfrac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) &= 2c_2 + 3 \cdot 2c_3\,(a-a) + 4 \cdot 3c_4 (a-a)^2 + \cdots \\ &= 2c_2 \end{aligned}$$

and
Equation:

$$\begin{aligned} \tfrac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) &= 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4\,(a-a) + 5 \cdot 4 \cdot 3c_5 (a-a)^2 + \cdots \\ &= 3 \cdot 2c_3 \end{aligned}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \frac{f''(a)}{2}$ and $c_3 = \frac{f'''(a)}{3} \cdot 2$. More generally, we see that if f has a power series representation at $x = a$, then the coefficients should be given by $c_n = \frac{f^{(n)}(a)}{n!}$. That is, the series should be
Equation:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

This power series for f is known as the Taylor series for f at a. If $a = 0$, then this series is known as the Maclaurin series for f.

Note:
Definition
If f has derivatives of all orders at $x = a$, then the **Taylor series** for the function f at a is
Equation:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The Taylor series for f at 0 is known as the **Maclaurin series** for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall from [link] that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Note:
Uniqueness of Taylor Series
If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from [link].

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials**.

Note:
Visit the MacTutor History of Mathematics archive to read brief biographies of Brook Taylor and Colin Maclaurin and how they developed the concepts named after them.

Taylor Polynomials

The *n*th partial sum of the Taylor series for a function f at a is known as the *n*th Taylor polynomial. For example, the 0th, 1st, 2nd, and 3rd partial sums of the Taylor series are given by
Equation:

$$\begin{aligned}
p_0(x) &= f(a),\\
p_1(x) &= f(a) + f'(a)(x-a),\\
p_2(x) &= f(a) + f'(a)(x-a) + \tfrac{f''(a)}{2!}(x-a)^2,\\
p_3(x) &= f(a) + f'(a)(x-a) + \tfrac{f''(a)}{2!}(x-a)^2 + \tfrac{f'''(a)}{3!}(x-a)^3,
\end{aligned}$$

respectively. These partial sums are known as the 0th, 1st, 2nd, and 3rd Taylor polynomials of f at a, respectively. If $a = 0$, then these polynomials are known as **Maclaurin polynomials** for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Note:
Definition
If f has n derivatives at $x = a$, then the *n*th Taylor polynomial for f at a is
Equation:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The *n*th Taylor polynomial for f at 0 is known as the *n*th Maclaurin polynomial for f.

We now show how to use this definition to find several Taylor polynomials for $f\left(x\right)=\ln x$ at $x=1$.

Example:
Exercise:

Problem:
Finding Taylor Polynomials

Find the Taylor polynomials p_0,p_1,p_2 and p_3 for $f\left(x\right)=\ln x$ at $x=1$. Use a graphing utility to compare the graph of f with the graphs of p_0,p_1,p_2 and p_3.

Solution:

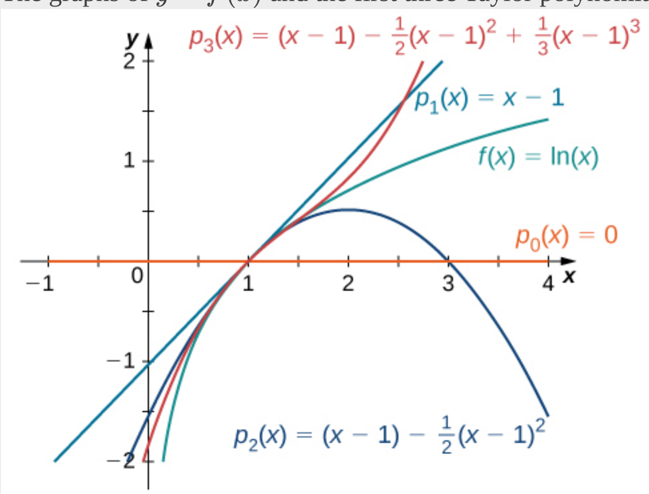
To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x=1$.
Equation:

$$\begin{array}{ccccccc}f\left(x\right)&=&\ln x&&f\left(1\right)&=&0\\ f'\left(x\right)&=&\frac{1}{x}&&f'\left(1\right)&=&1\\ f''\left(x\right)&=&-\frac{1}{x^2}&&f''\left(1\right)&=&-1\\ f'''\left(x\right)&=&\frac{2}{x^3}&&f'''\left(1\right)&=&2\end{array}$$

Therefore,
Equation:

$$\begin{array}{ccl}p_0\left(x\right)&=&f\left(1\right)=0,\\ p_1\left(x\right)&=&f\left(1\right)+f'\left(1\right)\left(x-1\right)=x-1,\\ p_2\left(x\right)&=&f\left(1\right)+f'\left(1\right)\left(x-1\right)+\frac{f''(1)}{2}\left(x-1\right)^2=\left(x-1\right)-\frac{1}{2}\left(x-1\right)^2,\\ p_3\left(x\right)&=&f\left(1\right)+f'\left(1\right)\left(x-1\right)+\frac{f''(1)}{2}\left(x-1\right)^2+\frac{f'''(1)}{3!}\left(x-1\right)^3\\ &=&\left(x-1\right)-\frac{1}{2}\left(x-1\right)^2+\frac{1}{3}\left(x-1\right)^3.\end{array}$$

The graphs of $y=f\left(x\right)$ and the first three Taylor polynomials are shown in [link].



The function $y=\ln x$ and the Taylor polynomials p_0,p_1,p_2 and p_3 at $x=1$ are plotted on this graph.

Note:
Exercise:

Problem: Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f\left(x\right) = \frac{1}{x^2}$ at $x = 1$.

Solution:

$p_0\left(x\right) = 1; p_1\left(x\right) = 1 - 2\left(x - 1\right); p_2\left(x\right) = 1 - 2\left(x - 1\right) + 3(x - 1)^2; p_3\left(x\right) = 1 - 2\left(x - 1\right) + 3(x - 1)^2$

Hint

Find the first three derivatives of f and evaluate them at $x = 1$.

We now show how to find Maclaurin polynomials for e^x, $\sin x$, and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example:
Exercise:

Problem:
Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3. Find a formula for the *n*th Maclaurin polynomial and write it using sigma notation. Use a graphing utilty to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. $f\left(x\right) = e^x$
b. $f\left(x\right) = \sin x$
c. $f\left(x\right) = \cos x$

Solution:

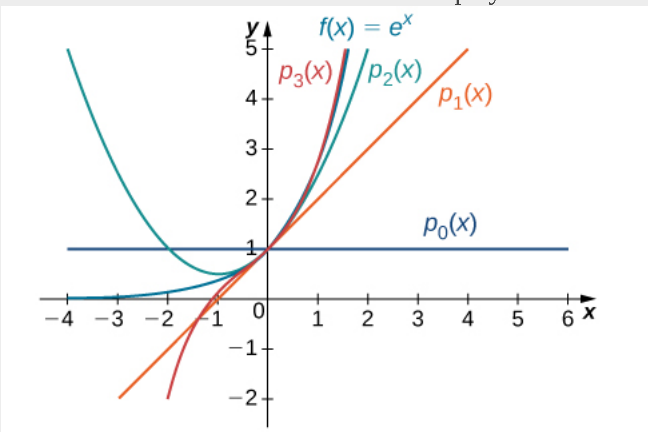
a. Since $f\left(x\right) = e^x$, we know that $f(x) = f'\left(x\right) = f\prime\prime\left(x\right) = \cdots = f^{(n)}\left(x\right) = e^x$ for all positive integers *n*. Therefore,
Equation:

$$f\left(0\right) = f'\left(0\right) = f\prime\prime\left(0\right) = \cdots = f^{(n)}\left(0\right) = 1$$

for all positive integers *n*. Therefore, we have
Equation:

$$\begin{array}{rcl} p_0(x) &=& f(0) = 1, \\ p_1(x) &=& f(0) + f'(0)x = 1 + x, \\ p_2(x) &=& f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2, \\ p_3(x) &=& f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 \\ &=& 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3, \\ p_n(x) &=& f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &=& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\ &=& \sum_{k=0}^{n} \frac{x^k}{k!}. \end{array}$$

The function and the first three Maclaurin polynomials are shown in [link].



The graph shows the function $y = e^x$ and the Maclaurin polynomials p_0, p_1, p_2 and p_3.

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:
Equation:

$$\begin{array}{rclcrcl} f(x) &=& \sin x & & f(0) &=& 0 \\ f'(x) &=& \cos x & & f'(0) &=& 1 \\ f''(x) &=& -\sin x & & f''(0) &=& 0 \\ f'''(x) &=& -\cos x & & f'''(0) &=& -1 \\ f^{(4)}(x) &=& \sin x & & f^{(4)}(0) &=& 0. \end{array}$$

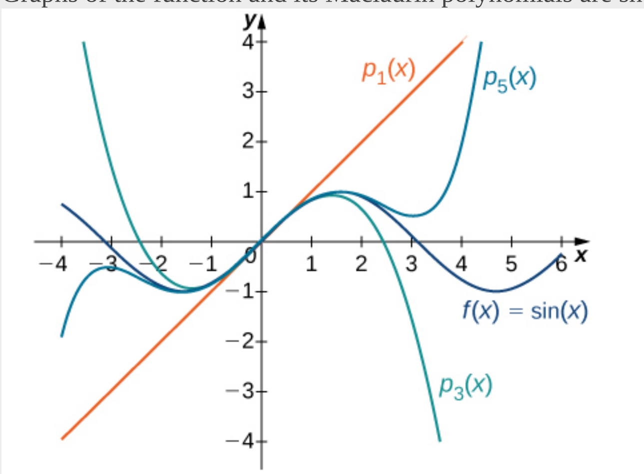
Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have
Equation:

$$p_0(x) = 0,$$
$$p_1(x) = 0 + x = x,$$
$$p_2(x) = 0 + x + 0 = x,$$
$$p_3(x) = 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!},$$
$$p_4(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!},$$
$$p_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

and for $m \geq 0$,
Equation:

$$\begin{aligned} p_{2m+1}(x) &= p_{2m+2}(x) \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

Graphs of the function and its Maclaurin polynomials are shown in [link].



The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5.

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:
Equation:

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1. \end{aligned}$$

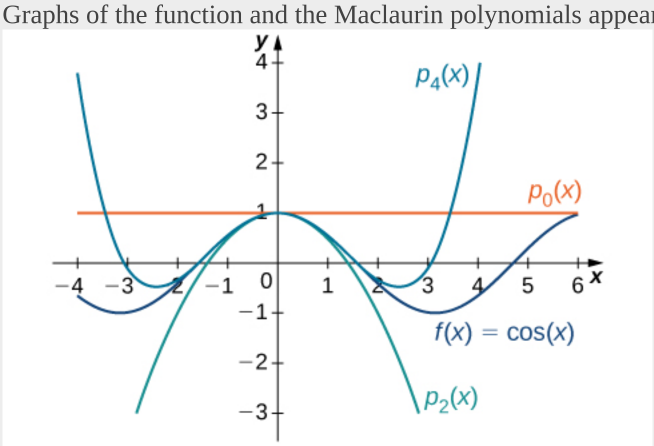
Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,
Equation:

$$p_0(x) = 1,$$
$$p_1(x) = 1 + 0 = 1,$$
$$p_2(x) = 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!},$$
$$p_3(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!},$$
$$p_4(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$
$$p_5(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

and for $n \geq 0$,
Equation:

$$\begin{aligned} p_{2m}(x) &= p_{2m+1}(x) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}. \end{aligned}$$

Graphs of the function and the Maclaurin polynomials appear in [link].



The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.

Note:
Exercise:

Problem:

Find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x) = \frac{1}{1+x}$. Find a formula for the *n*th Maclaurin polynomial. Write your anwer using sigma notation.

Solution:

$$p_0(x) = 1; p_1(x) = 1 - x; p_2(x) = 1 - x + x^2; p_3(x) = 1 - x + x^2 - x^3; p_n(x) = 1 - x + x^2 - x^3 + \cdots$$

Hint

Evaluate the first four derivatives of f and look for a pattern.

Taylor's Theorem with Remainder

Recall that the *n*th Taylor polynomial for a function f at *a* is the *n*th partial sum of the Taylor series for f at *a*. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials $\{p_n\}$ converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

Equation:

$$R_n(x) = f(x) - p_n(x).$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the *n*th Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0th Taylor polynomial at *a* for a function f. The remainder R_0 satisfies

Equation:

$$\begin{aligned} R_0(x) &= f(x) - p_0(x) \\ &= f(x) - f(a). \end{aligned}$$

If f is differentiable on an interval *I* containing *a* and *x*, then by the Mean Value Theorem there exists a real number *c* between *a* and *x* such that $f(x) - f(a) = f'(c)(x - a)$. Therefore,

Equation:

$$R_0(x) = f'(c)(x - a).$$

Using the Mean Value Theorem in a similar argument, we can show that if f is *n* times differentiable on an interval *I* containing *a* and *x*, then the *n*th remainder R_n satisfies

Equation:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$$

for some real number *c* between *a* and *x*. It is important to note that the value *c* in the numerator above is not the center *a*, but rather an unknown value *c* between *a* and *x*. This formula allows us to get a bound on the remainder R_n. If we happen to know that $\left|f^{(n+1)}(x)\right|$ is bounded by some real number *M* on this interval *I*, then

Equation:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1}$$

for all *x* in the interval *I*.

We now state Taylor's theorem, which provides the formal relationship between a function f and its *n*th degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Note:
Taylor's Theorem with Remainder
Let f be a function that can be differentiated $n+1$ times on an interval *I* containing the real number *a*. Let p_n be the *n*th Taylor polynomial of f at *a* and let
Equation:

$$R_n(x) = f(x) - p_n(x)$$

be the *n*th remainder. Then for each *x* in the interval *I*, there exists a real number *c* between *a* and *x* such that
Equation:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

If there exists a real number *M* such that $\left|f^{(n+1)}(x)\right| \leq M$ for all $x \in I$, then
Equation:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all *x* in *I*.

Proof

Fix a point $x \in I$ and introduce the function *g* such that
Equation:

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f\prime\prime(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

We claim that *g* satisfies the criteria of Rolle's theorem. Since *g* is a polynomial function (in *t*), it is a differentiable function. Also, *g* is zero at $t = a$ and $t = x$ because
Equation:

$$\begin{array}{lll} g(a) & = & f(x) - f(a) - f'(a)(x-a) - \frac{f\prime\prime(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ & = & f(x) - p_n(x) - R_n(x) \\ & = & 0, \\ g(x) & = & f(x) - f(x) - 0 - \cdots - 0 \\ & = & 0. \end{array}$$

Therefore, *g* satisfies Rolle's theorem, and consequently, there exists *c* between *a* and *x* such that $g'(c) = 0$. We now calculate g'. Using the product rule, we note that
Equation:

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = \frac{-f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Consequently,
Equation:

$$\begin{aligned} g'(t) &= -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \tfrac{f'''(t)}{2!}(x-t)^2\right] + \cdots \\ &\quad + \left[\tfrac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \tfrac{f^{(n+1)}(t)}{n!}(x-t)^n\right] + (n+1)R_n(x)\tfrac{(x-t)^n}{(x-a)^{n+1}}. \end{aligned}$$

Notice that there is a telescoping effect. Therefore,
Equation:

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}.$$

By Rolle’s theorem, we conclude that there exists a number *c* between *a* and *x* such that $g'(c) = 0$. Since
Equation:

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}$$

we conclude that
Equation:

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} = 0.$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that
Equation:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

as desired. From this fact, it follows that if there exists *M* such that $\left|f^{(n+1)}(x)\right| \le M$ for all *x* in *I*, then
Equation:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

□

Not only does Taylor’s theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x) = \sqrt[3]{x}$ at $x = 8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example:
Exercise:

Problem:
Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f\left(x\right)=\sqrt[3]{x}.$

a. Find the first and second Taylor polynomials for f at $x=8$. Use a graphing utility to compare these polynomials with f near $x=8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor’s theorem to bound the error.

Solution:

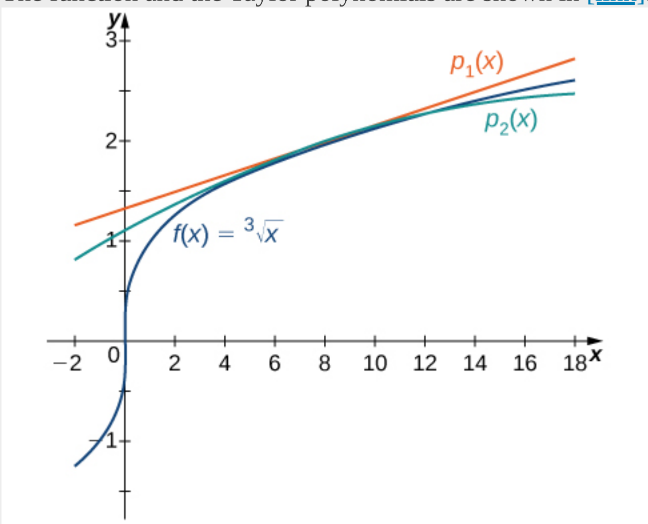
a. For $f\left(x\right)=\sqrt[3]{x}$, the values of the function and its first two derivatives at $x=8$ are as follows:
Equation:

$$\begin{array}{ccccccc}f\left(x\right)&=&\sqrt[3]{x}&&f\left(8\right)&=&2\\ {f}^{\prime }\left(x\right)&=&\frac{1}{3{x}^{2/3}}&&{f}^{\prime }\left(8\right)&=&\frac{1}{12}\\ f\prime\prime \left(x\right)&=&\frac{-2}{9{x}^{5/3}}&&f\prime\prime \left(8\right)&=&-\frac{1}{144}.\end{array}$$

Thus, the first and second Taylor polynomials at $x=8$ are given by
Equation:

$$\begin{array}{ccc}{p}_{1}\left(x\right)&=&f\left(8\right)+{f}^{\prime }\left(8\right)\left(x-8\right)\\ &=&2+\frac{1}{12}\left(x-8\right)\\ {p}_{2}\left(x\right)&=&f\left(8\right)+{f}^{\prime }\left(8\right)\left(x-8\right)+\frac{f\prime\prime \left(8\right)}{2!}{\left(x-8\right)}^{2}\\ &=&2+\frac{1}{12}\left(x-8\right)-\frac{1}{288}{\left(x-8\right)}^{2}.\end{array}$$

The function and the Taylor polynomials are shown in [link].



The graphs of $f\left(x\right)=\sqrt[3]{x}$ and the linear and quadratic approximations ${p}_{1}\left(x\right)$ and ${p}_{2}\left(x\right)$.

b. Using the first Taylor polynomial at $x = 8$, we can estimate
Equation:

$$\sqrt[3]{11} \approx p_1(11) = 2 + \frac{1}{12}(11 - 8) = 2.25.$$

Using the second Taylor polynomial at $x = 8$, we obtain
Equation:

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11 - 8) - \frac{1}{288}(11 - 8)^2 = 2.21875.$$

c. By [link], there exists a *c* in the interval $(8, 11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies
Equation:

$$R_1(11) = \frac{f''(c)}{2!}(11 - 8)^2.$$

We do not know the exact value of *c*, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8, 11)$. Since $f''(x) = -\frac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x = 8$. Using the fact that $f''(8) = -\frac{1}{144}$, we obtain
Equation:

$$|R_1(11)| \le \frac{1}{144 \cdot 2!}(11 - 8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that
Equation:

$$R_2(11) = \frac{f'''(c)}{3!}(11 - 8)^3.$$

Since $f'''(x) = \frac{10}{27x^{8/3}}$, the maximum value of f''' on the interval $(8, 11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have
Equation:

$$|R_2(11)| \le \frac{0.0011468}{3!}(11 - 8)^3 \approx 0.0065104.$$

Note:
Exercise:

Problem:

Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x = 4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Solution:

$p_1(x) = 2 + \frac{1}{4}(x-4); p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2; p_1(6) = 2.5; p_2(6) = 2.4375;$

$|R_1(6)| \leq 0.0625; |R_2(6)| \leq 0.015625$

Hint

Evaluate $f(4)$, $f'(4)$, and $f\prime\prime(4)$.

Example:
Exercise:

Problem:
Approximating sin *x* Using Maclaurin Polynomials

From [link]b., the Maclaurin polynomials for $\sin x$ are given by
Equation:

$$\begin{aligned} p_{2m+1}(x) &= p_{2m+2}(x) \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} \end{aligned}$$

for $m = 0, 1, 2, \ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\frac{\pi}{18}\right)$ and bound the error.
b. For what values of *x* does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution:

a. The fifth Maclaurin polynomial is
Equation:

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Using this polynomial, we can estimate as follows:
Equation:

$$\begin{aligned} \sin\left(\frac{\pi}{18}\right) &\approx p_5\left(\frac{\pi}{18}\right) \\ &= \frac{\pi}{18} - \frac{1}{3!}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 \\ &\approx 0.173648. \end{aligned}$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x) = p_5(x)$ and calculate a bound on $R_6\left(\frac{\pi}{18}\right)$. By [link], the remainder is
Equation:

$$R_6\left(\frac{\pi}{18}\right) = \frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $\left|f^{(7)}(x)\right| \leq 1$ for all x, we find that the magnitude of the error is at most
Equation:

$$\frac{1}{7!} \cdot \left(\frac{\pi}{18}\right)^7 \leq 9.8 \times 10^{-10}.$$

b. We need to find the values of x such that
Equation:

$$\frac{1}{7!}|x|^7 \leq 0.0001.$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x| < 0.907$.

Note:
Exercise:

Problem: Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Solution:

0.96593

Hint

The fourth Maclaurin polynomial is $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

Example:
Exercise:

Problem:
Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution:

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

Equation:

$$\begin{array}{rclcrcl} f(x) & = & \frac{1}{x} & & f(1) & = & 1 \\ f'(x) & = & -\frac{1}{x^2} & & f'(1) & = & -1 \\ f''(x) & = & \frac{2}{x^3} & & f''(1) & = & 2! \\ f'''(x) & = & -\frac{3\cdot 2}{x^4} & & f'''(1) & = & -3! \\ f^{(4)}(x) & = & \frac{4\cdot 3\cdot 2}{x^5} & & f^{(4)}(1) & = & 4!. \end{array}$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by
Equation:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

To find the interval of convergence, we use the ratio test. We find that
Equation:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|(-1)^{n+1}(x-1)^{n+1}\right|}{|(-1)^n (x-1)^n|} = |x-1|.$$

Thus, the series converges if $|x-1| < 1$. That is, the series converges for $0 < x < 2$. Next, we need to check the endpoints. At $x = 2$, we see that
Equation:

$$\sum_{n=0}^{\infty} (-1)^n (2-1)^n = \sum_{n=0}^{\infty} (-1)^n$$

diverges by the divergence test. Similarly, at $x = 0$,
Equation:

$$\sum_{n=0}^{\infty} (-1)^n (0-1)^n = \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} 1$$

diverges. Therefore, the interval of convergence is $(0, 2)$.

Note:
Exercise:

Problem: Find the Taylor series for $f(x) = \frac{1}{2x}$ at $x = 2$ and determine its interval of convergence.

Solution:

$\displaystyle\sum_{n=0}^{\infty} \left(\frac{2-x}{2^{n+2}}\right)^n$. The interval of convergence is $(0, 4)$.

Hint

$f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}$

We know that the Taylor series found in this example converges on the interval $(0, 2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

Equation:

$$f(x) = \frac{1}{x} = \frac{1}{1 - (1 - x)}.$$

That is, f can be represented by the geometric series $\sum_{n=0}^{\infty} (1 - x)^n$. Since this is a geometric series, it converges to $\frac{1}{x}$ as long as $|1 - x| < 1$. Therefore, the Taylor series found in [link] does converge to $f(x) = \frac{1}{x}$ on $(0, 2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

Equation:

$$\lim_{n \to \infty} p_n(x) = f(x).$$

Since the remainder $R_n(x) = f(x) - p_n(x)$, the Taylor series converges to f if and only if

Equation:

$$\lim_{n \to \infty} R_n(x) = 0.$$

We now state this theorem formally.

Note:

Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

Equation:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

converges to $f(x)$ for all x in I if and only if

Equation:

$$\lim_{n \to \infty} R_n(x) = 0$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

Equation:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x) \to 0$ for all real numbers x.

Example:

Exercise:

Problem:

Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use [link] to prove that the Maclaurin series for f converges to f on that interval.

a. e^x
b. $\sin x$

Solution:

a. Using the nth Maclaurin polynomial for e^x found in [link]a., we find that the Maclaurin series for e^x is given by

Equation:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

Equation:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

we have

Equation:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty, \infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x) = e^x$ for all $n \geq 0$ and e^x is an increasing function on $(-\infty, \infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x| \leq b$ is e^b. Thus,

Equation:

$$\left|R_n(x)\right| \le \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that
Equation:

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all *x*, by the divergence test, we know that
Equation:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number *x*. By combining this fact with the squeeze theorem, the result is $\lim_{n\to\infty} R_n(x) = 0$.

b. Using the *n*th Maclaurin polynomial for $\sin x$ found in [link]b., we find that the Maclaurin series for $\sin x$ is given by
Equation:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider
Equation:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+3)(2n+2)}.$$

Since
Equation:

$$\lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all *x*, we obtain the interval of convergence as $(-\infty, \infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each *x* there exists a real number *c* between 0 and *x* such that
Equation:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

Since $\left|f^{(n+1)}(c)\right| \le 1$ for all integers *n* and all real numbers *c*, we have
Equation:

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers *x*. Using the same idea as in part a., the result is $\lim_{n\to\infty} R_n(x) = 0$ for all *x*, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real *x*.

Note:
Exercise:

Problem:

Find the Maclaurin series for $f(x) = \cos x$. Use the ratio test to show that the interval of convergence is $(-\infty, \infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers *x*.

Solution:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

By the ratio test, the interval of convergence is $(-\infty, \infty)$. Since $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real *x*.

Hint

Use the Maclaurin polynomials for $\cos x$.

Note:
Proving that *e* is Irrational
In this project, we use the Maclaurin polynomials for e^x to prove that *e* is irrational. The proof relies on supposing that *e* is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e = r/s$ for some integers *r* and *s* where $s \neq 0$.

1. Write the Maclaurin polynomials $p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)$ for e^x. Evaluate $p_0(1), p_1(1), p_2(1), p_3(1), p_4(1)$ to estimate *e*.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers *r* and *s*, evaluate $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$, we can find an integer *k* such that $kR_n(1)$ is an integer for $n = 0, 1, 2, 3, 4$.
4. Write down the formula for the *n*th Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0$, and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if *n* is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that *e* is rational must be false.

Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.
- The *n*th degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for e^x, $\sin x$, and $\cos x$ converge to the respective functions for all real *x*.

Key Equations

- **Taylor series for the function f at the point $\mathbf{x = a}$**
  $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$

In the following exercises, find the Taylor polynomials of degree two approximating the given function centered at the given point.

Exercise:

Problem: $f(x) = 1 + x + x^2$ at $a = 1$

Exercise:

Problem: $f(x) = 1 + x + x^2$ at $a = -1$

Solution:

$f(-1) = 1;\ f'(-1) = -1;\ f''(-1) = 2;\ f(x) = 1 - (x+1) + (x+1)^2$

Exercise:

Problem: $f(x) = \cos(2x)$ at $a = \pi$

Exercise:

Problem: $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

Solution:

$f'(x) = 2\cos(2x);\ f''(x) = -4\sin(2x);\ p_2(x) = -2\left(x - \frac{\pi}{2}\right)$

Exercise:

Problem: $f(x) = \sqrt{x}$ at $a = 4$

Exercise:

Problem: $f(x) = \ln x$ at $a = 1$

Solution:

$f'(x) = \frac{1}{x};\ f''(x) = -\frac{1}{x^2};\ p_2(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$

Exercise:

Problem: $f(x) = \frac{1}{x}$ at $a = 1$

Exercise:

Problem: $f(x) = e^x$ at $a = 1$

Solution:

$p_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$

In the following exercises, verify that the given choice of n in the remainder estimate $|R_n| \le \frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $\left|f^{(n+1)}(z)\right|$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$. Find the value of the Taylor polynomial p_n of f at the indicated point.

Exercise:

Problem: **[T]** $\sqrt{10}; a = 9, n = 3$

Exercise:

Problem: **[T]** $(28)^{1/3}; a = 27, n = 1$

Solution:

$\frac{d^2}{dx^2}x^{1/3} = -\frac{2}{9x^{5/3}} \ge -0.00092\ldots$ when $x \ge 28$ so the remainder estimate applies to the linear approximation $x^{1/3} \approx p_1(27) = 3 + \frac{x-27}{27}$, which gives $(28)^{1/3} \approx 3 + \frac{1}{27} = 3.\overline{037}$, while $(28)^{1/3} \approx 3.03658.$

Exercise:

Problem: **[T]** $\sin(6); a = 2\pi, n = 5$

Exercise:

Problem: **[T]** $e^2; a = 0, n = 9$

Solution:

Using the estimate $\frac{2^{10}}{10!} < 0.000283$ we can use the Taylor expansion of order 9 to estimate e^x at $x = 2.$ as $e^2 \approx p_9(2) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \cdots + \frac{2^9}{9!} = 7.3887\ldots$ whereas $e^2 \approx 7.3891.$

Exercise:

Problem: **[T]** $\cos\left(\frac{\pi}{5}\right); a = 0, n = 4$

Exercise:

Problem: **[T]** $\ln(2); a = 1, n = 1000$

Solution:

Since $\frac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$, $R_{1000} \approx \frac{1}{1001}$. One has $p_{1000}(1) = \sum_{n=1}^{1000} \frac{(-1)^{n-1}}{n} \approx 0.6936$ whereas $\ln(2) \approx 0.6931\cdots$.

Exercise:

Problem:

Integrate the approximation $\sin t \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$ evaluated at *πt* to approximate $\int_0^1 \frac{\sin \pi t}{\pi t} dt$.

Exercise:

Problem:

Integrate the approximation $e^x \approx 1 + x + \frac{x^2}{2} + \cdots + \frac{x^6}{720}$ evaluated at $-x^2$ to approximate $\int_0^1 e^{-x^2} dx$.

Solution:

$$\int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \frac{x^{12}}{720}\right) dx$$

$= 1 - \frac{1^3}{3} + \frac{1^5}{10} - \frac{1^7}{42} + \frac{1^9}{9\cdot 24} - \frac{1^{11}}{120\cdot 11} + \frac{1^{13}}{720\cdot 13} \approx 0.74683$ whereas $\int_0^1 e^{-x^2} dx \approx 0.74682$.

In the following exercises, find the smallest value of *n* such that the remainder estimate $|R_n| \le \frac{M}{(n+1)!}(x-a)^{n+1}$, where *M* is the maximum value of $\left|f^{(n+1)}(z)\right|$ on the interval between *a* and the indicated point, yields $|R_n| \le \frac{1}{1000}$ on the indicated interval.

Exercise:

Problem: $f(x) = \sin x$ on $[-\pi, \pi]$, $a = 0$

Exercise:

Problem: $f(x) = \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $a = 0$

Solution:

Since $f^{(n+1)}(z)$ is $\sin z$ or $\cos z$, we have $M = 1$. Since $|x - 0| \le \frac{\pi}{2}$, we seek the smallest *n* such that $\frac{\pi^{n+1}}{2^{n+1}(n+1)!} \le 0.001$. The smallest such value is $n = 7$. The remainder estimate is $R_7 \le 0.00092$.

Exercise:

Problem: $f(x) = e^{-2x}$ on $[-1, 1]$, $a = 0$

Exercise:

Problem: $f(x) = e^{-x}$ on $[-3, 3]$, $a = 0$

Solution:

Since $f^{(n+1)}(z) = \pm e^{-z}$ one has $M = e^3$. Since $|x - 0| \le 3$, one seeks the smallest *n* such that $\frac{3^{n+1}e^3}{(n+1)!} \le 0.001$. The smallest such value is $n = 14$. The remainder estimate is $R_{14} \le 0.000220$.

In the following exercises, the maximum of the right-hand side of the remainder estimate $|R_1| \leq \frac{\max|f''(z)|}{2} R^2$ on $[a-R, a+R]$ occurs at a or $a \pm R$. Estimate the maximum value of R such that $\frac{\max|f''(z)|}{2} R^2 \leq 0.1$ on $[a-R, a+R]$ by plotting this maximum as a function of R.

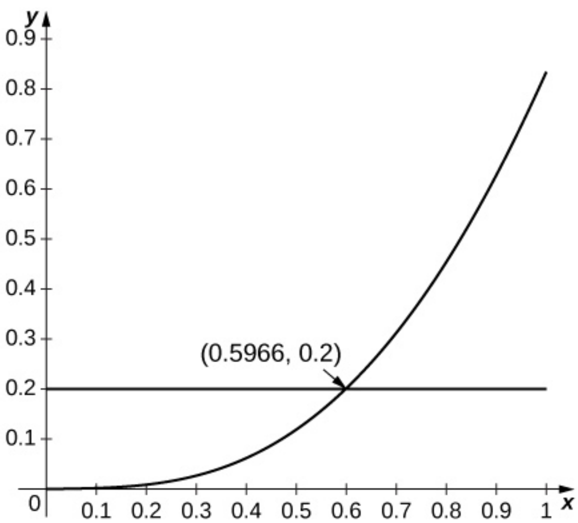
Exercise:

Problem: **[T]** e^x approximated by $1+x, a=0$

Exercise:

Problem: **[T]** $\sin x$ approximated by x, $a=0$

Solution:



Since $\sin x$ is increasing for small x and since $\sin''x = -\sin x$, the estimate applies whenever $R^2 \sin(R) \leq 0.2$, which applies up to $R = 0.596$.

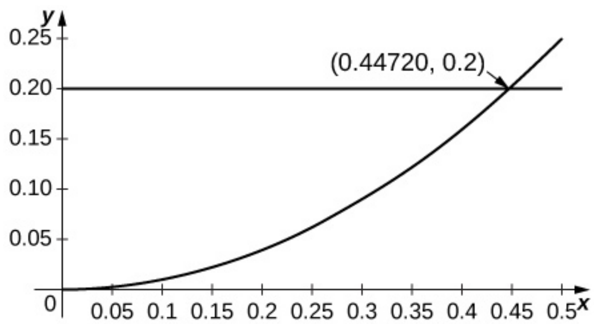
Exercise:

Problem: **[T]** $\ln x$ approximated by $x-1, a=1$

Exercise:

Problem: **[T]** $\cos x$ approximated by $1, a=0$

Solution:



Since the second derivative of $\cos x$ is $-\cos x$ and since $\cos x$ is decreasing away from $x = 0$, the estimate applies when $R^2 \cos R \leq 0.2$ or $R \leq 0.447$.

In the following exercises, find the Taylor series of the given function centered at the indicated point.

Exercise:

Problem: x^4 at $a = -1$

Exercise:

Problem: $1 + x + x^2 + x^3$ at $a = -1$

Solution:

$(x+1)^3 - 2(x+1)^2 + 2\,(x+1)$

Exercise:

Problem: $\sin x$ at $a = \pi$

Exercise:

Problem: $\cos x$ at $a = 2\pi$

Solution:

Values of derivatives are the same as for $x = 0$ so $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-2\pi)^{2n}}{(2n)!}$

Exercise:

Problem: $\sin x$ at $x = \frac{\pi}{2}$

Exercise:

Problem: $\cos x$ at $x = \frac{\pi}{2}$

Solution:

$\cos\left(\frac{\pi}{2}\right) = 0, -\sin\left(\frac{\pi}{2}\right) = -1$ so $\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$, which is also $-\cos\left(x - \frac{\pi}{2}\right)$.

Exercise:

Problem: e^x at $a = -1$

Exercise:

Problem: e^x at $a = 1$

Solution:

The derivatives are $f^{(n)}(1) = e$ so $e^x = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

Exercise:

Problem: $\frac{1}{(x-1)^2}$ at $a = 0$ (*Hint:* Differentiate $\frac{1}{1-x}$.)

Exercise:

Problem: $\frac{1}{(x-1)^3}$ at $a = 0$

Solution:

$$\frac{1}{(x-1)^3} = -\left(\frac{1}{2}\right) \frac{d^2}{dx^2} \frac{1}{1-x} = -\sum_{n=0}^{\infty} \left(\frac{(n+2)(n+1)x^n}{2} \right)$$

Exercise:

Problem:

$$F(x) = \int_0^x \cos\left(\sqrt{t}\right) dt;\, f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!} \text{ at } a = 0 \text{ (}\textit{Note}\text{: } f \text{ is the Taylor series of } \cos\left(\sqrt{t}\right).\text{)}$$

In the following exercises, compute the Taylor series of each function around $x = 1$.

Exercise:

Problem: $f(x) = 2 - x$

Solution:

$2 - x = 1 - (x - 1)$

Exercise:

Problem: $f(x) = x^3$

Exercise:

Problem: $f(x) = (x-2)^2$

Solution:

$((x-1)-1)^2 = (x-1)^2 - 2(x-1) + 1$

Exercise:

Problem: $f(x) = \ln x$

Exercise:

Problem: $f(x) = \frac{1}{x}$

Solution:

$\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$

Exercise:

Problem: $f(x) = \frac{1}{2x-x^2}$

Exercise:

Problem: $f(x) = \frac{x}{4x-2x^2-1}$

Solution:

$x \sum_{n=0}^{\infty} 2^n (1-x)^{2n} = \sum_{n=0}^{\infty} 2^n (x-1)^{2n+1} + \sum_{n=0}^{\infty} 2^n (x-1)^{2n}$

Exercise:

Problem: $f(x) = e^{-x}$

Exercise:

Problem: $f(x) = e^{2x}$

Solution:

$e^{2x} = e^{2(x-1)+2} = e^2 \sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$

[T] In the following exercises, identify the value of *x* such that the given series $\sum_{n=0}^{\infty} a_n$ is the value of the Maclaurin series of $f(x)$ at x. Approximate the value of $f(x)$ using $S_{10} = \sum_{n=0}^{10} a_n$.

Exercise:

Problem: $\sum_{n=0}^{\infty} \frac{1}{n!}$

Exercise:

Problem: $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Solution:

$x = e^2; S_{10} = \frac{34,913}{4725} \approx 7.3889947$

Exercise:

Problem: $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$

Exercise:

Problem: $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}$

Solution:

$\sin(2\pi) = 0; S_{10} = 8.27 \times 10^{-5}$

The following exercises make use of the functions $S_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$ and $C_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ on $[-\pi, \pi]$.

Exercise:

Problem:

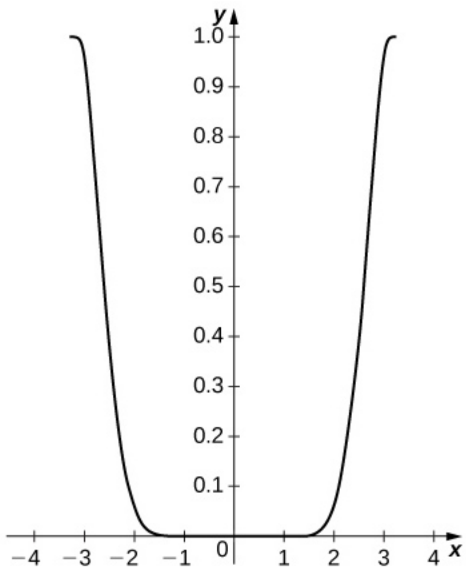
[T] Plot $\sin^2 x - (S_5(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\sin x$.

Exercise:

Problem:

[T] Plot $\cos^2 x - (C_4(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\cos x$.

Solution:



The difference is small on the interior of the interval but approaches 1 near the endpoints. The remainder estimate is $|R_4| = \frac{\pi^5}{120} \approx 2.552.$

Exercise:

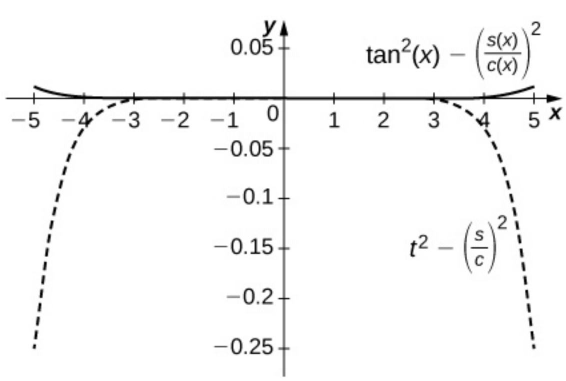
Problem: [T] Plot $|2S_5\,(x)C_4\,(x) - \sin\,(2x)|$ on $[-\pi, \pi]$.

Exercise:

Problem:

[T] Compare $\frac{S_5(x)}{C_4(x)}$ on $[-1, 1]$ to $\tan x$. Compare this with the Taylor remainder estimate for the approximation of $\tan x$ by $x + \frac{x^3}{3} + \frac{2x^5}{15}$.

Solution:



The difference is on the order of 10^{-4} on $[-1, 1]$ while the Taylor approximation error is around 0.1 near ± 1. The top curve is a plot of $\tan^2 x - \left(\frac{S_5(x)}{C_4(x)}\right)^2$ and the lower dashed plot shows $t^2 - \left(\frac{S_5}{C_4}\right)^2$.

Exercise:

Problem:

[T] Plot $e^x - e_4(x)$ where $e_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ on $[0, 2]$. Compare the maximum error with the Taylor remainder estimate.

Exercise:

Problem:

(Taylor approximations and root finding.) Recall that Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}$ approximates solutions of $f(x) = 0$ near the input x_0.

a. If f and g are inverse functions, explain why a solution of $g(x) = a$ is the value $f(a)$ of f.
b. Let $p_N(x)$ be the $N\text{th}$ degree Maclaurin polynomial of e^x. Use Newton's method to approximate solutions of $p_N(x) - 2 = 0$ for $N = 4, 5, 6$.
c. Explain why the approximate roots of $p_N(x) - 2 = 0$ are approximate values of $\ln(2)$.

Solution:

a. Answers will vary. b. The following are the x_n values after 10 iterations of Newton's method to approximation a root of $p_N(x) - 2 = 0$: for $N = 4, x = 0.6939...$; for $N = 5, x = 0.6932...$; for $N = 6, x = 0.69315...$; . (*Note:* $\ln(2) = 0.69314...$) c. Answers will vary.

In the following exercises, use the fact that if $q(x) = \sum_{n=1}^{\infty} a_n(x-c)^n$ converges in an interval containing c, then $\lim_{x \to c} q(x) = a_0$ to evaluate each limit using Taylor series.

Exercise:

Problem: $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$

Exercise:

Problem: $\lim_{x \to 0} \frac{\ln(1-x^2)}{x^2}$

Solution:

$\frac{\ln(1-x^2)}{x^2} \to -1$

Exercise:

Problem: $\lim_{x \to 0} \frac{e^{x^2} - x^2 - 1}{x^4}$

Exercise:

Problem: $\lim_{x \to 0^+} \frac{\cos(\sqrt{x}) - 1}{2x}$

Solution:

$\frac{\cos(\sqrt{x}) - 1}{2x} \approx \frac{\left(1 - \frac{x}{2} + \frac{x^2}{4!} - \cdots\right) - 1}{2x} \to -\frac{1}{4}$

Glossary

Maclaurin polynomial
: a Taylor polynomial centered at 0; the nth Taylor polynomial for f at 0 is the nth Maclaurin polynomial for f

Maclaurin series
: a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

Taylor polynomials
: the nth Taylor polynomial for f at $x = a$ is $p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor series
: a power series at a that converges to a function f on some open interval containing a

Taylor's theorem with remainder
: for a function f and the nth Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$
for some c between x and a; if there exists an interval I containing a and a real number M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all x in I, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

Working with Taylor Series

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to solve differential equations.
- Use Taylor series to evaluate nonelementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}dx$, an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x) = (1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series**. We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r = 0, 1, 2, 3, 4$, $f(x) = (1+x)^r$ can be written as
Equation:

$$\begin{aligned}
f(x) &= (1+x)^0 = 1, \\
f(x) &= (1+x)^1 = 1+x, \\
f(x) &= (1+x)^2 = 1+2x+x^2, \\
f(x) &= (1+x)^3 = 1+3x+3x^2+x^3, \\
f(x) &= (1+x)^4 = 1+4x+6x^2+4x^3+x^4.
\end{aligned}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by
Equation:

$$\binom{r}{n} = \frac{r!}{n!\,(r-n)!}$$

and
Equation:

$$\begin{aligned}
f(x) &= (1+x)^r \\
&= \binom{r}{0}1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots + \binom{r}{r-1}x^{r-1} + \binom{r}{r}x^r \\
&= \sum_{n=0}^{r} \binom{r}{n} x^n.
\end{aligned}$$

For example, using this formula for $r = 5$, we see that
Equation:

$$\begin{aligned} f(x) &= (1+x)^5 \\ &= \binom{5}{0}1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 \\ &= \tfrac{5!}{0!5!}1 + \tfrac{5!}{1!4!}x + \tfrac{5!}{2!3!}x^2 + \tfrac{5!}{3!2!}x^3 + \tfrac{5!}{4!1!}x^4 + \tfrac{5!}{5!0!}x^5 \\ &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5. \end{aligned}$$

We now consider the case when the exponent r is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x) = (1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x = 0$.

Equation:

$$\begin{array}{rclrcl} f(x) &=& (1+x)^r & f(0) &=& 1 \\ f'(x) &=& r(1+x)^{r-1} & f\prime(0) &=& r \\ f\prime\prime(x) &=& r(r-1)(1+x)^{r-2} & f\prime\prime(0) &=& r(r-1) \\ f\prime\prime\prime(x) &=& r(r-1)(r-2)(1+x)^{r-3} & f\prime\prime\prime(0) &=& r(r-1)(r-2) \\ f^{(n)}(x) &=& r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} & f^{(n)}(0) &=& r(r-1)(r-2)\cdots(r-n+1) \end{array}$$

We conclude that the coefficients in the binomial series are given by

Equation:

$$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

We note that if r is a nonnegative integer, then the $(r+1)$st derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then [link] for the coefficients agrees with [link] for the coefficients, and the formula for the binomial series agrees with [link] for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

Equation:

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

With this notation, we can write the binomial series for $(1+x)^r$ as

Equation:

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots.$$

We now need to determine the interval of convergence for the binomial series [link]. We apply the ratio test. Consequently, we consider

Equation:

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)|x||^{n+1}}{(n+1)!} \cdot \frac{n!}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\ &= \frac{|r-n||x|}{|n+1|}. \end{aligned}$$

Since
Equation:

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=|x|<1$$

if and only if $|x|<1$, we conclude that the interval of convergence for the binomial series is $(-1,1)$. The behavior at the endpoints depends on r. It can be shown that for $r\geq 0$ the series converges at both endpoints; for $-1<r<0$, the series converges at $x=1$ and diverges at $x=-1$; and for $r<-1$, the series diverges at both endpoints. The binomial series does converge to $(1+x)^r$ in $(-1,1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x)\to 0$ is difficult.

Note:
Definition
For any real number r, the Maclaurin series for $f(x)=(1+x)^r$ is the binomial series. It converges to f for $|x|<1$, and we write
Equation:

$$\begin{aligned}(1+x)^r &= \sum_{n=0}^{\infty}\binom{r}{n}x^n \\ &= 1+rx+\frac{r(r-1)}{2!}x^2+\cdots+\frac{r(r-1)\cdots(r-n+1)}{n!}x^n+\cdots\end{aligned}$$

for $|x|<1$.

We can use this definition to find the binomial series for $f(x)=\sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example:
Exercise:

Problem:
Finding Binomial Series

a. Find the binomial series for $f(x)=\sqrt{1+x}$.
b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution:

a. Here $r=\frac{1}{2}$. Using the definition for the binomial series, we obtain
Equation:

$$\begin{aligned}\sqrt{1+x} &= 1+\tfrac{1}{2}x+\tfrac{(1/2)(-1/2)}{2!}x^2+\tfrac{(1/2)(-1/2)(-3/2)}{3!}x^3+\cdots \\ &= 1+\tfrac{1}{2}x-\tfrac{1}{2!}\,\tfrac{1}{2^2}x^2+\tfrac{1}{3!}\,\tfrac{1\cdot 3}{2^3}x^3-\cdots+\tfrac{(-1)^{n+1}}{n!}\,\tfrac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n+\cdots \\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\,\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

Equation:

$$p_3(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3.$$

Therefore,

Equation:

$$\begin{aligned}\sqrt{1.5} &= \sqrt{1+0.5} \\ &\approx 1+\tfrac{1}{2}(0.5)-\tfrac{1}{8}(0.5)^2+\tfrac{1}{16}(0.5)^3 \\ &\approx 1.2266.\end{aligned}$$

From Taylor's theorem, the error satisfies

Equation:

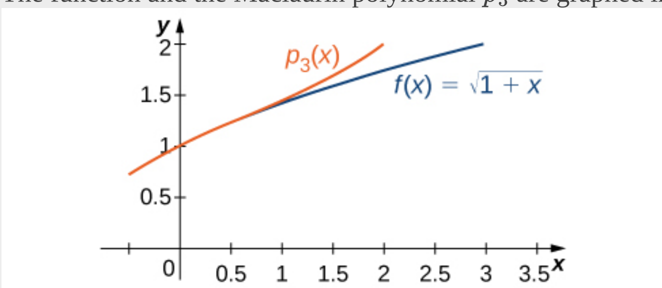
$$R_3(0.5)=\frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x)=-\frac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $\left|f^{(4)}(x)\right|$ on the interval $(0,0.5)$ occurs at $x=0$, we have

Equation:

$$|R_3(0.5)|\le\frac{15}{4!2^4}(0.5)^4\approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in [link].



The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x)=\sqrt{1+x}$ for x near zero.

Note:
Exercise:

Problem: Find the binomial series for $f\left(x\right)=\frac{1}{{\left(1+x\right)}^{2}}$.

Solution:

$$\sum_{n=0}^{\infty}{\left(-1\right)}^{n}\left(n+1\right)x^{n}$$

Hint

Use the definition of binomial series for $r=-2$.

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f\left(x\right)={\left(1+x\right)}^{r}$. In [link], we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f\left(x\right)=\ln\left(1+x\right)$ at the endpoint $x=1$ and the Maclaurin series for $f\left(x\right)=\tan^{-1}x$ at the endpoints $x=1$ and $x=-1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Function	Maclaurin Series	Interval of Convergence
$f\left(x\right)=\frac{1}{1-x}$	$\sum_{n=0}^{\infty}x^{n}$	$-1<x<1$
$f\left(x\right)=e^{x}$	$\sum_{n=0}^{\infty}\frac{x^{n}}{n!}$	$-\infty<x<\infty$
$f\left(x\right)=\sin x$	$\sum_{n=0}^{\infty}{\left(-1\right)}^{n}\frac{x^{2n+1}}{\left(2n+1\right)!}$	$-\infty<x<\infty$
$f\left(x\right)=\cos x$	$\sum_{n=0}^{\infty}{\left(-1\right)}^{n}\frac{x^{2n}}{\left(2n\right)!}$	$-\infty<x<\infty$
$f\left(x\right)=\ln\left(1+x\right)$	$\sum_{n=1}^{\infty}{\left(-1\right)}^{n+1}\frac{x^{n}}{n}$	$-1<x\le 1$

Function	Maclaurin Series	Interval of Convergence
$f\left(x\right)=\tan^{-1}x$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	$-1<x\le 1$
$f\left(x\right)=\left(1+x\right)^r$	$\sum_{n=0}^{\infty}\binom{r}{n}x^n$	$-1<x<1$

Maclaurin Series for Common Functions

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in [link], to create Maclaurin series for other functions.

Example:
Exercise:

Problem:
Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in [link].

a. $f\left(x\right)=\cos\sqrt{x}$
b. $f\left(x\right)=\sinh x$

Solution:

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by
Equation:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{x}\right)^{2n}}{(2n)!} &= \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!} \\ &= 1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\frac{x^4}{8!}-\cdots.\end{aligned}$$

This series converges to $\cos\sqrt{x}$ for all x in the domain of $\cos\sqrt{x}$; that is, for all $x\ge 0$.
b. To find the Maclaurin series for $\sinh x$, we use the fact that
Equation:

$$\sinh x=\frac{e^x-e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $\sinh x$ is given by
Equation:

$$\frac{x^n}{n!}-\frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $\sinh x$ has only odd-order terms and is given by
Equation:

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Note:
Exercise:

Problem: Find the Maclaurin series for $\sin\left(x^2\right)$.

Solution:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Hint

Use the Maclaurin series for $\sin x$.

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In [link], we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example:
Exercise:

Problem:
Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution:

The two functions are related by
Equation:

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by
Equation:

$$\begin{aligned}\frac{1}{\sqrt{1+x}} &= 2\frac{d}{dx}\sqrt{1+x}\\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\ \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n.\end{aligned}$$

Note:
Exercise:

Problem: Find the binomial series for $f(x)=\frac{1}{(1+x)^{3/2}}$

Solution:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\ \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n$$

Hint

Differentiate the series for $\frac{1}{\sqrt{1+x}}$.

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation
Equation:

$$y'(x)=y.$$

Recall that this is a first-order separable equation and its solution is $y=Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y=\sum\limits_{n=0}^{\infty}c_nx^n$ and determine what the coefficients would need to be. In the next example, we consider an initial-value problem involving $y'=y$ to illustrate the technique.

Example:
Exercise:

Problem:
Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem
Equation:

$$y' = y, \ \ y(0) = 3.$$

Solution:

Suppose that there exists a power series solution
Equation:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this series term by term, we obtain
Equation:

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If y satisfies the differential equation, then
Equation:

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots.$$

Using [link] on the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,
Equation:

$$\begin{aligned} c_0 &= c_1, \\ c_1 &= 2c_2, \\ c_2 &= 3c_3, \\ c_3 &= 4c_4, \\ &\vdots \end{aligned}$$

Using the initial condition $y(0) = 3$ combined with the power series representation
Equation:

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have
Equation:

$$\begin{aligned} c_1 &= c_0 = 3 = \tfrac{3}{1!}, \\ c_2 &= \tfrac{c_1}{2} = \tfrac{3}{2} = \tfrac{3}{2!}, \\ c_3 &= \tfrac{c_2}{3} = \tfrac{3}{3 \cdot 2} = \tfrac{3}{3!}, \\ c_4 &= \tfrac{c_3}{4} = \tfrac{3}{4 \cdot 3 \cdot 2} = \tfrac{3}{4!}. \end{aligned}$$

Therefore,
Equation:

$$\begin{aligned} y &= 3\left[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \frac{1}{4!}x^4 + \cdots\right] \\ &= 3\sum_{n=0}^{\infty} \frac{x^n}{n!}. \end{aligned}$$

You might recognize

Equation:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

Note:

Exercise:

Problem: Use power series to solve $y' = 2y, \;\; y(0) = 5$.

Solution:

$y = 5e^{2x}$

Hint

The equations for the first several coefficients c_n will satisfy $c_0 = 2c_1, c_1 = 2 \cdot 2c_2, c_2 = 2 \cdot 3c_3, \ldots$. In general, for all $n \geq 0, c_n = 2(n+1)C_{n+1}$.

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

Equation:

$$y' - xy = 0$$

is known as Airy's equation. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

Example:

Exercise:

Problem:

Power Series Solution of Airy's Equation

Use power series to solve

Equation:

$$y\prime\prime - xy = 0$$

with the initial conditions $y(0) = a$ and $y\prime(0) = b$.

Solution:

We look for a solution of the form
Equation:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this function term by term, we obtain
Equation:

$$\begin{aligned} y' &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots, \\ y\prime\prime &= 2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + 4 \cdot 3 c_4 x^2 + \cdots. \end{aligned}$$

If y satisfies the equation $y\prime\prime = xy$, then
Equation:

$$2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + 4 \cdot 3 c_4 x^2 + \cdots = x\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right).$$

Using [link] on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,
Equation:

$$\begin{aligned} 2 \cdot 1 c_2 &= 0, \\ 3 \cdot 2 c_3 &= c_0, \\ 4 \cdot 3 c_4 &= c_1, \\ 5 \cdot 4 c_5 &= c_2, \\ &\vdots \end{aligned}$$

More generally, for $n \geq 3$, we have $n \cdot (n-1) c_n = c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2 = 0$. Then
Equation:

$$\begin{aligned} c_3 &= \tfrac{c_0}{3 \cdot 2}, \\ c_4 &= \tfrac{c_1}{4 \cdot 3}. \end{aligned}$$

For c_5, c_6, c_7, we see that
Equation:

$$\begin{aligned} c_5 &= \tfrac{c_2}{5 \cdot 4} = 0, \\ c_6 &= \tfrac{c_3}{6 \cdot 5} = \tfrac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \\ c_7 &= \tfrac{c_4}{7 \cdot 6} = \tfrac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}. \end{aligned}$$

Therefore, the series solution of the differential equation is given by

Equation:

$$y = c_0 + c_1 x + 0 \cdot x^2 + \frac{c_0}{3 \cdot 2} x^3 + \frac{c_1}{4 \cdot 3} x^4 + 0 \cdot x^5 + \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots.$$

The initial condition $y(0) = a$ implies $c_0 = a$. Differentiating this series term by term and using the fact that $y'(0) = b$, we conclude that $c_1 = b$. Therefore, the solution of this initial-value problem is

Equation:

$$y = a\left(1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots\right) + b\left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots\right).$$

Note:
Exercise:

Problem: Use power series to solve $y'' + x^2 y = 0$ with the initial condition $y(0) = a$ and $y'(0) = b$.

Solution:

$y = a\left(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \cdots\right) + b\left(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \cdots\right)$

Hint

The coefficients satisfy $c_0 = a, c_1 = b, c_2 = 0, c_3 = 0$, and for $n \geq 4, n(n-1)c_n = -c_{n-4}$.

Evaluating Nonelementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2} dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - \sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)\, dx$ where the antiderivative of f cannot be written as an elementary function is considered a **nonelementary integral**.

Nonelementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2} dx$.

Example:

Exercise:

Problem:
Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2}dx$ as an infinite series.
b. Evaluate $\int_0^1 e^{-x^2}dx$ to within an error of 0.01.

Solution:

a. The Maclaurin series for e^{-x^2} is given by
Equation:

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} \\ &= 1 - x^2 + \tfrac{x^4}{2!} - \tfrac{x^6}{3!} + \cdots + (-1)^n \tfrac{x^{2n}}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}. \end{aligned}$$

Therefore,
Equation:

$$\begin{aligned} \int e^{-x^2}dx &= \int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx \\ &= C + x - \tfrac{x^3}{3} + \tfrac{x^5}{5.2!} - \tfrac{x^7}{7.3!} + \cdots + (-1)^n \tfrac{x^{2n+1}}{(2n+1)n!} + \cdots. \end{aligned}$$

b. Using the result from part a. we have
Equation:

$$\int_0^1 e^{-x^2}dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots.$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$.

Note:
Exercise:

Problem: Express $\int \cos\sqrt{x}dx$ as an infinite series. Evaluate $\int_0^1 \cos\sqrt{x}dx$ to within an error of 0.01.

Solution:

$$C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n\,(2n-2)!}$$ The definite integral is approximately 0.514 to within an error of 0.01.

Hint

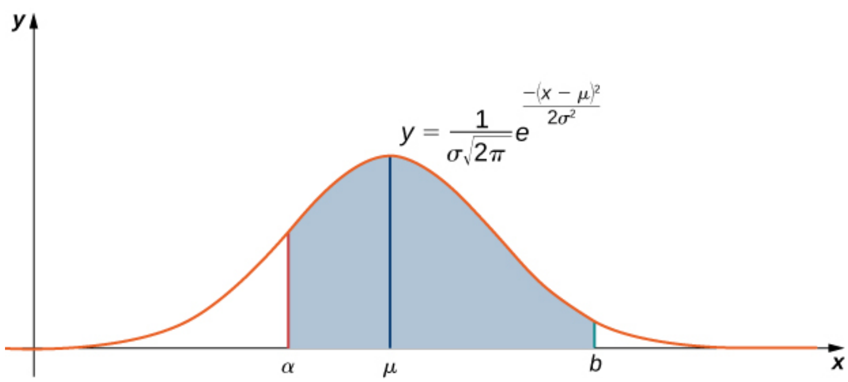
Use the series found in [link].

As mentioned above, the integral $\int e^{-x^2} dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

Equation:

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/(2\sigma^2)} dx.$$

(See [link].)



If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ between $x = a$ and $x = b$.

To simplify this integral, we typically let $z = \frac{x-\mu}{\sigma}$. This quantity z is known as the z score of a data value. With this simplification, integral [link] becomes

Equation:

$$\frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2} dz.$$

In [link], we show how we can use this integral in calculating probabilities.

Example:
Exercise:

Problem:
Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use [link] and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the alternating series test to determine how accurate your approximation is.

Solution:

Since $\mu = 100, \sigma = 50$, and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral [link] becomes
Equation:

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz.$$

The Maclaurin series for $e^{-x^2/2}$ is given by
Equation:

$$\begin{aligned} e^{-x^2/2} &= \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} \\ &= 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}. \end{aligned}$$

Therefore,
Equation:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} dz &= \frac{1}{\sqrt{2\pi}} \int \left(1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots\right) dz \\ &= \frac{1}{\sqrt{2\pi}} \left(C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1) 2^n \cdot n!} + \cdots\right) \\ \frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz &= \frac{1}{\sqrt{2\pi}} \left(2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11 \cdot 2^5 \cdot 5!} + \cdots\right). \end{aligned}$$

Using the first five terms, we estimate that the probability is approximately 0.4922. By the alternating series test, we see that this estimate is accurate to within
Equation:

$$\frac{1}{\sqrt{2\pi}} \frac{2^{13}}{13 \cdot 2^6 \cdot 6!} \approx 0.00546.$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95%. Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5%. The estimate, combined with the bound on the accuracy, falls within this range.

Note:
Exercise:

Problem:

Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Solution:

The estimate is approximately 0.3414. This estimate is accurate to within 0.0000094.

Hint

Evaluate $\int_0^1 e^{-z^2/2} dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$.

Another application in which a nonelementary integral arises involves the period of a pendulum. The integral is
Equation:

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}}.$$

An integral of this form is known as an elliptic integral of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example:
Exercise:

Problem:
Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by
Equation:

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}}$$

where g is the acceleration due to gravity and $k = \sin\left(\frac{\theta_{\text{max}}}{2}\right)$ (see [link]). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized

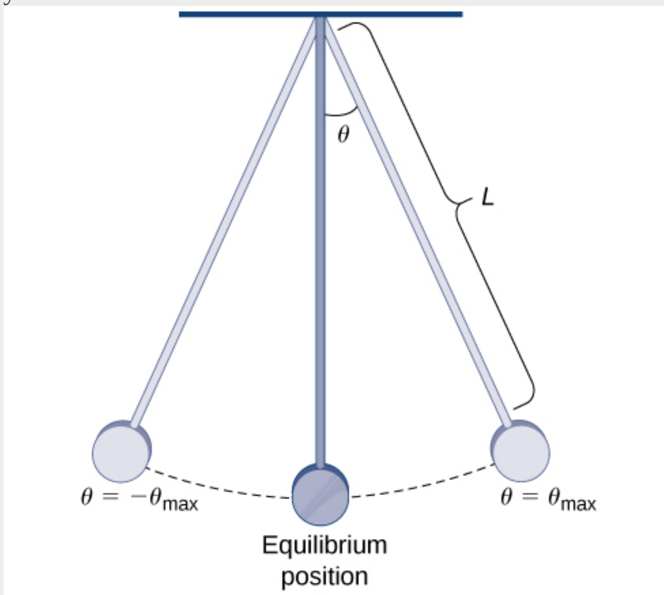
model is used and $\sin\theta$ is approximated by θ.) Use the binomial series

Equation:

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.



This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Solution:

We use the binomial series, replacing x with $-k^2\sin^2\theta$. Then we can write the period as

Equation:

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2\sin^2\theta + \frac{1 \cdot 3}{2!2^2}k^4\sin^4\theta + \cdots\right) d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

Equation:

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If $\theta_{\max}$ is small, then $k = \sin\left(\frac{\theta_{\max}}{2}\right)$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

Equation:

$$\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta+\frac{1\cdot 3}{2!2^2}k^4\sin^4\theta+\cdots\right)d\theta.$$

Since $|\sin x| \le 1$, this integral is bounded by

Equation:

$$\int_0^{\pi/2}\left(\frac{1}{2}k^2+\frac{1.3}{2!2^2}k^4+\cdots\right)d\theta < \frac{\pi}{2}\left(\frac{1}{2}k^2+\frac{1\cdot 3}{2!2^2}k^4+\cdots\right).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

Equation:

$$\frac{\pi k^2}{2}\left(1+k^2+k^4+\cdots\right)=\frac{\pi k^2}{2}\cdot\frac{1}{1-k^2},$$

which is small for k small.

b. For larger values of $\theta_{\max}$, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

Equation:

$$\begin{aligned}T &\approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta\right)d\theta \\ &= 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right).\end{aligned}$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

Key Concepts

- The binomial series is the Maclaurin series for $f(x) = (1+x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

In the following exercises, use appropriate substitutions to write down the Maclaurin series for the given binomial.

Exercise:

Problem: $(1-x)^{1/3}$

Exercise:

Problem: $\left(1+x^2\right)^{-1/3}$

Solution:

$$\left(1+x^2\right)^{-1/3}=\sum_{n=0}^{\infty}\binom{-\frac{1}{3}}{n}x^{2n}$$

Exercise:

Problem: $(1-x)^{1.01}$

Exercise:

Problem: $(1-2x)^{2/3}$

Solution:

$$(1-2x)^{2/3}=\sum_{n=0}^{\infty}(-1)^n 2^n\binom{\frac{2}{3}}{n}x^n$$

In the following exercises, use the substitution $(b+x)^r=(b+a)^r\left(1+\frac{x-a}{b+a}\right)^r$ in the binomial expansion to find the Taylor series of each function with the given center.

Exercise:

Problem: $\sqrt{x+2}$ at $a=0$

Exercise:

Problem: $\sqrt{x^2+2}$ at $a=0$

Solution:

$$\sqrt{2+x^2}=\sum_{n=0}^{\infty}2^{(1/2)-n}\binom{\frac{1}{2}}{n}x^{2n};\ \left(\left|x^2\right|<2\right)$$

Exercise:

Problem: $\sqrt{x+2}$ at $a=1$

Exercise:

Problem: $\sqrt{2x-x^2}$ at $a=1$ (*Hint:* $2x-x^2=1-(x-1)^2$)

Solution:

$$\sqrt{2x-x^2}=\sqrt{1-(x-1)^2} \text{ so } \sqrt{2x-x^2}=\sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n}(x-1)^{2n}$$

Exercise:

Problem: $(x-8)^{1/3}$ at $a=9$

Exercise:

Problem: $\sqrt{x}$ at $a=4$

Solution:

$$\sqrt{x}=2\sqrt{1+\frac{x-4}{4}} \text{ so } \sqrt{x}=\sum_{n=0}^{\infty}2^{1-2n}\binom{\frac{1}{2}}{n}(x-4)^n$$

Exercise:

Problem: $x^{1/3}$ at $a=27$

Exercise:

Problem: $\sqrt{x}$ at $x=9$

Solution:

$$\sqrt{x}=\sum_{n=0}^{\infty}3^{1-3n}\binom{\frac{1}{2}}{n}(x-9)^n$$

In the following exercises, use the binomial theorem to estimate each number, computing enough terms to obtain an estimate accurate to an error of at most $1/1000$.

Exercise:

Problem: **[T]** $(15)^{1/4}$ using $(16-x)^{1/4}$

Exercise:

Problem: **[T]** $(1001)^{1/3}$ using $(1000+x)^{1/3}$

Solution:

$10\left(1+\frac{x}{1000}\right)^{1/3}=\sum_{n=0}^{\infty}10^{1-3n}\binom{\frac{1}{3}}{n}x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives

$$\begin{aligned}(1001)^{1/3} &\approx 10\left(1+\binom{\frac{1}{3}}{1}10^{-3}+\binom{\frac{1}{3}}{2}10^{-6}+\binom{\frac{1}{3}}{3}10^{-9}+\binom{\frac{1}{3}}{4}10^{-12}\right) \text{ whereas} \\ &= 10\left(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}}\right)=10.00333222...\end{aligned}$$

$(1001)^{1/3}=10.00332222839093$ .... Two terms would suffice for three-digit accuracy.

In the following exercises, use the binomial approximation $\sqrt{1-x}\approx 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}$ for $|x|<1$ to approximate each number. Compare this value to the value given by a scientific calculator.

Exercise:

Problem: **[T]** $\frac{1}{\sqrt{2}}$ using $x = \frac{1}{2}$ in $(1-x)^{1/2}$

Exercise:

Problem: **[T]** $\sqrt{5} = 5\, \times\, \frac{1}{\sqrt{5}}$ using $x = \frac{4}{5}$ in $(1-x)^{1/2}$

Solution:

The approximation is 2.3152; the CAS value is $2.23\ldots$.

Exercise:

Problem: **[T]** $\sqrt{3} = \frac{3}{\sqrt{3}}$ using $x = \frac{2}{3}$ in $(1-x)^{1/2}$

Exercise:

Problem: **[T]** $\sqrt{6}$ using $x = \frac{5}{6}$ in $(1-x)^{1/2}$

Solution:

The approximation is $2.583\ldots$; the CAS value is $2.449\ldots$.

Exercise:

Problem: Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x \sqrt{1-t}dt$.

Exercise:

Problem:

[T] Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x=-1$ to $x=1$ to estimate $\frac{\pi}{2}$.

Solution:

$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} + \cdots$. Thus

$\int_{-1}^{1} \sqrt{1-x^2}dx = x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{7\cdot 16} - \frac{5x^9}{9\cdot 128} + \cdots \Big|_{-1}^{1} \approx 2 - \frac{1}{3} - \frac{1}{20} - \frac{1}{56} - \frac{10}{9\cdot 128} + \text{error} = 1.590...$ whereas $\frac{\pi}{2} = 1.570...$

In the following exercises, use the expansion $(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

Exercise:

Problem: $(1+4x)^{1/3}; a = 0$

Exercise:

Problem: $(1+4x)^{4/3}; a=0$

Solution:

$(1+x)^{4/3} = (1+x)\left(1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots\right) = 1+\frac{4x}{3}+\frac{2x^2}{9}-\frac{4x^3}{81}+\frac{5x^4}{243}+\cdots$

Exercise:

Problem: $(3+2x)^{1/3}; a=-1$

Exercise:

Problem: $\left(x^2+6x+10\right)^{1/3}; a=-3$

Solution:

$\left(1+(x+3)^2\right)^{1/3} = 1+\frac{1}{3}(x+3)^2-\frac{1}{9}(x+3)^4+\frac{5}{81}(x+3)^6-\frac{10}{243}(x+3)^8+\cdots$

Exercise:

Problem: Use $(1+x)^{1/3} = 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ with $x=1$ to approximate $2^{1/3}$.

Exercise:

Problem:

Use the approximation $(1-x)^{2/3} = 1-\frac{2x}{3}-\frac{x^2}{9}-\frac{4x^3}{81}-\frac{7x^4}{243}-\frac{14x^5}{729}+\cdots$ for $|x|<1$ to approximate $2^{1/3} = 2.2^{-2/3}$.

Solution:

Twice the approximation is $1.260\ldots$ whereas $2^{1/3} = 1.2599...$.

Exercise:

Problem: Find the 25th derivative of $f(x) = \left(1+x^2\right)^{13}$ at $x=0$.

Exercise:

Problem: Find the 99 th derivative of $f(x) = \left(1+x^4\right)^{25}$.

Solution:

$f^{(99)}(0) = 0$

In the following exercises, find the Maclaurin series of each function.

Exercise:

Problem: $f(x) = xe^{2x}$

Exercise:

Problem: $f(x) = 2^x$

Solution:

$$\sum_{n=0}^{\infty} \frac{(\ln(2)x)^n}{n!}$$

Exercise:

Problem: $f(x) = \frac{\sin x}{x}$

Exercise:

Problem: $f(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}}, \ (x > 0),$

Solution:

For $x > 0, \sin(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)/2}}{\sqrt{x}\,(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n+1)!}.$

Exercise:

Problem: $f(x) = \sin(x^2)$

Exercise:

Problem: $f(x) = e^{x^3}$

Solution:

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

Exercise:

Problem: $f(x) = \cos^2 x$ using the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

Exercise:

Problem: $f(x) = \sin^2 x$ using the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Solution:

$$\sin^2 x = -\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} x^{2k}}{(2k)!}$$

In the following exercises, find the Maclaurin series of $F(x) = \int_0^x f(t)\,dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

Exercise:

Problem: $F\left(x\right)=\int_0^x e^{-t^2}dt; f\left(t\right)=e^{-t^2}=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{t^{2n}}{n!}$

Exercise:

Problem: $F\left(x\right)=\tan^{-1}x;\;\; f\left(t\right)=\frac{1}{1+t^2}=\sum_{n=0}^{\infty}\left(-1\right)^n t^{2n}$

Solution:

$$\tan^{-1}x=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k x^{2k+1}}{2k+1}$$

Exercise:

Problem: $F\left(x\right)=\tanh^{-1}x;\;\; f\left(t\right)=\frac{1}{1-t^2}=\sum_{n=0}^{\infty}t^{2n}$

Exercise:

Problem: $F\left(x\right)=\sin^{-1}x;\;\; f\left(t\right)=\frac{1}{\sqrt{1-t^2}}=\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{t^{2k}}{k!}$

Solution:

$$\sin^{-1}x=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\frac{x^{2n+1}}{\left(2n+1\right)n!}$$

Exercise:

Problem: $F\left(x\right)=\int_0^x\frac{\sin t}{t}dt;\;\; f\left(t\right)=\frac{\sin t}{t}=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{t^{2n}}{\left(2n+1\right)!}$

Exercise:

Problem: $F\left(x\right)=\int_0^x\cos\left(\sqrt{t}\right)dt;\;\; f\left(t\right)=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{x^n}{\left(2n\right)!}$

Solution:

$$F\left(x\right)=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{x^{n+1}}{\left(n+1\right)\left(2n\right)!}$$

Exercise:

Problem: $F\left(x\right)=\int_0^x\frac{1-\cos t}{t^2}dt;\;\; f\left(t\right)=\frac{1-\cos t}{t^2}=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{t^{2n}}{\left(2n+2\right)!}$

Exercise:

Problem: $F(x) = \int_0^x \frac{\ln(1+t)}{t} dt;\ \ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n+1}$

Solution:

$$F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n^2}$$

In the following exercises, compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

Exercise:

Problem: $f(x) = \sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)$

Exercise:

Problem: $f(x) = \tan x$

Solution:

$x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

Exercise:

Problem: $f(x) = \ln(\cos x)$

Exercise:

Problem: $f(x) = e^x \cos x$

Solution:

$1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$

Exercise:

Problem: $f(x) = e^{\sin x}$

Exercise:

Problem: $f(x) = \sec^2 x$

Solution:

$1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \cdots$

Exercise:

Problem: $f(x) = \tanh x$

Exercise:

Problem: $f\left(x\right)=\frac{\tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $\tan x$)

Solution:

Using the expansion for $\tan x$ gives $1+\frac{x}{3}+\frac{2x^2}{15}$.

In the following exercises, find the radius of convergence of the Maclaurin series of each function.
Exercise:

Problem: $\ln\left(1+x\right)$

Exercise:

Problem: $\frac{1}{1+x^2}$

Solution:

$\frac{1}{1+x^2}=\sum_{n=0}^{\infty}\left(-1\right)^n x^{2n}$ so $R=1$ by the ratio test.

Exercise:

Problem: $\tan^{-1}x$

Exercise:

Problem: $\ln\left(1+x^2\right)$

Solution:

$\ln\left(1+x^2\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n}x^{2n}$ so $R=1$ by the ratio test.

Exercise:

Problem: Find the Maclaurin series of $\sinh x=\frac{e^x-e^{-x}}{2}$.

Exercise:

Problem: Find the Maclaurin series of $\cosh x=\frac{e^x+e^{-x}}{2}$.

Solution:

Add series of e^x and e^{-x} term by term. Odd terms cancel and $\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{\left(2n\right)!}$.

Exercise:

Problem:

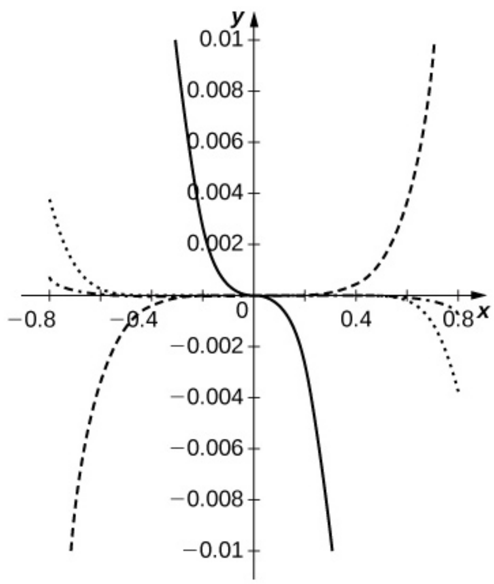
Differentiate term by term the Maclaurin series of $\sinh x$ and compare the result with the Maclaurin series of $\cosh x$.

Exercise:

Problem:

[T] Let $S_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $C_n(x) = \sum_{n=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n+1$ of $\sin x$ and degree $2n$ of $\cos x$. Plot the errors $\frac{S_n(x)}{C_n(x)} - \tan x$ for $n = 1, .., 5$ and compare them to $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} - \tan x$ on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Solution:



The ratio $\frac{S_n(x)}{C_n(x)}$ approximates $\tan x$ better than does $p_7(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}$ for $N \geq 3$. The dashed curves are $\frac{S_n}{C_n} - \tan$ for $n = 1, 2$. The dotted curve corresponds to $n = 3$, and the dash-dotted curve corresponds to $n = 4$. The solid curve is $p_7 - \tan x$.

Exercise:

Problem:

Use the identity $2 \sin x \cos x = \sin(2x)$ to find the power series expansion of $\sin^2 x$ at $x = 0$. (*Hint:* Integrate the Maclaurin series of $\sin(2x)$ term by term.)

Exercise:

Problem: If $y = \sum_{n=0}^{\infty} a_n x^n$, find the power series expansions of xy' and $x^2 y\prime\prime$.

Solution:

By the term-by-term differentiation theorem, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ so $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ $xy' = \sum_{n=1}^{\infty} n a_n x^n$, whereas $y' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ so $xy'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$.

Exercise:

Problem:

[T] Suppose that $y = \sum_{k=0}^{\infty} a_k x^k$ satisfies $y' = -2xy$ and $y(0) = 0$. Show that $a_{2k+1} = 0$ for all k and that $a_{2k+2} = \frac{-a_{2k}}{k+1}$. Plot the partial sum S_{20} of y on the interval $[-4, 4]$.

Exercise:

Problem:

[T] Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

Solution:

The probability is $p = \frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-x^2/2} dx$ where $a = 90$ and $b = 100$, that is,

$p = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sum_{n=0}^{5} (-1)^n \frac{x^{2n}}{2^n n!} dx = \frac{2}{\sqrt{2\pi}} \sum_{n=0}^{5} (-1)^n \frac{1}{(2n+1)2^n n!} \approx 0.6827.$

Exercise:

Problem:

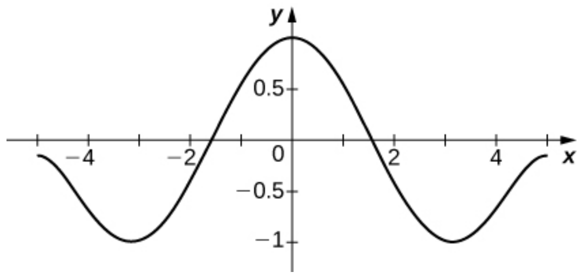
[T] Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

Exercise:

Problem:

[T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 1$, $f'(0) = 0$, and $f''(x) = -f(x)$. Find a formula for a_n and plot the partial sum S_N for $N = 20$ on $[-5, 5]$.

Solution:



As in the previous problem one obtains $a_n = 0$ if n is odd and $a_n = -(n+2)(n+1)a_{n+2}$ if n is even, so $a_0 = 1$ leads to $a_{2n} = \frac{(-1)^n}{(2n)!}$.

Exercise:

Problem:

[T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 0,\;\; f'(0) = 1$, and $f''(x) = -f(x)$. Find a formula for a_n and plot the partial sum S_N for $N = 10$ on $[-5, 5]$.

Exercise:

Problem:

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 1$ and $y'(0) = 0$. Find a formula that relates a_{n+2}, a_{n+1}, and a_n and compute $a_0, ..., a_5$.

Solution:

$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$ and $y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$ so $y'' - y' + y = 0$ implies that $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$ or $a_n = \frac{a_{n-1}}{n} - \frac{a_{n-2}}{n(n-1)}$ for all $n \cdot y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$, so $a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = 0$, and $a_5 = -\frac{1}{120}$.

Exercise:

Problem:

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 0$ and $y'(0) = 1$. Find a formula that relates a_{n+2}, a_{n+1}, and a_n and compute $a_1, ..., a_5$.

The error in approximating the integral $\int_a^b f(t)\,dt$ by that of a Taylor approximation $\int_a^b P_n(t)\,dt$ is at most $\int_a^b R_n(t)\,dt$. In the following exercises, the Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.

b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

Exercise:

Problem:

[T] $\int_0^{\pi} \frac{\sin t}{t} dt; P_s = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{\sin t}{t}$ is bounded by 0.1.)

Solution:

a. (Proof) b. We have $R_s \leq \frac{0.1}{(9)!} \pi^9 \approx 0.0082 < 0.01$. We have $\int_0^{\pi} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}\right) dx = \pi - \frac{\pi^3}{3 \cdot 3!} + \frac{\pi^5}{5 \cdot 5!} - \frac{\pi^7}{7 \cdot 7!} + \frac{\pi^9}{9 \cdot 9!} = 1.852...$, whereas $\int_0^{\pi} \frac{\sin t}{t} dt = 1.85194...$, so the actual error is approximately 0.00006.

Exercise:

Problem:

[T] $\int_0^{2} e^{-x^2} dx; p_{11} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots - \frac{x^{22}}{11!}$ (You may assume that the absolute value of the 23rd derivative of e^{-x^2} is less than 2×10^{14}.)

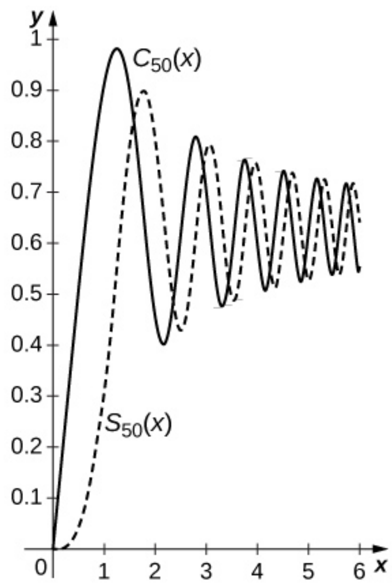
The following exercises deal with Fresnel integrals.

Exercise:

Problem:

The Fresnel integrals are defined by $C(x) = \int_0^{x} \cos\left(t^2\right) dt$ and $S(x) = \int_0^{x} \sin\left(t^2\right) dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N = 50$ nonzero terms on $[0, 2\pi]$.

Solution:



Since cos $\left(t^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n}}{(2n)!}$ and sin $\left(t^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+2}}{(2n+1)!}$, one has $S\left(x\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)\left(2n+1\right)!}$ and $C\left(x\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)\left(2n\right)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}\left(x\right)$ the solid curve and $S_{50}\left(x\right)$ the dashed curve.

Exercise:

Problem:

[T] The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $\left(C\left(t\right), S\left(t\right)\right)$. Plot the curve $\left(C_{50}, S_{50}\right)$ for $0 \le t \le 2\pi$, the coordinates of which were computed in the previous exercise.

Exercise:

Problem:

Estimate $\int_0^{1/4} \sqrt{x - x^2} dx$ by approximating $\sqrt{1-x}$ using the binomial approximation $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{2128} - \frac{7x^5}{256}$.

Solution:

$$\int_0^{1/4} \sqrt{x} \left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}\right) dx$$

$$= \frac{2}{3} 2^{-3} - \frac{1}{2} \frac{2}{5} 2^{-5} - \frac{1}{8} \frac{2}{7} 2^{-7} - \frac{1}{16} \frac{2}{9} 2^{-9} - \frac{5}{128} \frac{2}{11} 2^{-11} - \frac{7}{256} \frac{2}{13} 2^{-13} = 0.0767732...$$

whereas $\int_0^{1/4} \sqrt{x - x^2} dx = 0.076773.$

Exercise:

Problem:

[T] Use Newton's approximation of the binomial $\sqrt{1-x^2}$ to approximate π as follows. The circle centered at $\left(\frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$ has upper semicircle $y = \sqrt{x}\sqrt{1-x}$. The sector of this circle bounded by the x-axis between $x = 0$ and $x = \frac{1}{2}$ and by the line joining $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x = 0$ and $x = \frac{1}{4}$. To find the area of this region you can write $y = \sqrt{x}\sqrt{1-x} = \sqrt{x} \times \left(\text{binomial expansion of} \sqrt{1-x}\right)$ and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

Exercise:

Problem:

Use the approximation $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right)$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{\text{max}} = \frac{\pi}{6}$ where $k = \sin\left(\frac{\theta_{\text{max}}}{2}\right)$. Compare this with the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$.

Solution:

$T \approx 2\pi\sqrt{\frac{10}{9.8}}\left(1 + \frac{\sin^2(\theta/12)}{4}\right) \approx 6.453$ seconds. The small angle estimate is $T \approx 2\pi\sqrt{\frac{10}{9.8} \approx 6.347}$. The relative error is around 2 percent.

Exercise:

Problem:

Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{\text{max}} = \frac{\pi}{6}$. Use $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right)$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$?

Exercise:

Problem:

Evaluate $\int_0^{\pi/2} \sin^4\theta d\theta$ in the approximation $T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1 + \frac{1}{2}k^2\sin^2\theta + \frac{3}{8}k^4\sin^4\theta + \cdots\right)d\theta$ to obtain an improved estimate for T.

Solution:

$\int_0^{\pi/2} \sin^4\theta d\theta = \frac{3\pi}{16}$. Hence $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4} + \frac{9}{256}k^4\right)$.

Exercise:

Problem:

[T] An equivalent formula for the period of a pendulum with amplitude θ_{max} is $T\left(\theta_{\text{max}}\right) = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{\text{max}}} \frac{d\theta}{\sqrt{\cos\theta - \cos\left(\theta_{\text{max}}\right)}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{\text{max}} = \frac{\pi}{3}$ we get $\frac{1}{\sqrt{\cos t - 1/2}} \approx \sqrt{2}\left(1 + \frac{t^2}{2} + \frac{t^4}{3} + \frac{181t^6}{720}\right)$. Integrate this approximation to estimate $T\left(\frac{\pi}{3}\right)$ in terms of L and g. Assuming $g = 9.806$ meters per second squared, find an approximate length L such that $T\left(\frac{\pi}{3}\right) = 2$ seconds.

Chapter Review Exercises

True or False? In the following exercises, justify your answer with a proof or a counterexample.

Exercise:

Problem:

If the radius of convergence for a power series $\sum_{n=0}^{\infty} a_n x^n$ is 5, then the radius of convergence for the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also 5.

Solution:

True

Exercise:

Problem:

Power series can be used to show that the derivative of e^x is e^x. (*Hint:* Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.)

Exercise:

Problem: For small values of $x, \sin x \approx x$.

Solution:

True

Exercise:

Problem: The radius of convergence for the Maclaurin series of $f\left(x\right) = 3^x$ is 3.

In the following exercises, find the radius of convergence and the interval of convergence for the given series.

Exercise:

Problem: $\sum_{n=0}^{\infty} n^2 (x-1)^n$

Solution:

ROC: 1; IOC: $(0,2)$

Exercise:

Problem: $\sum_{n=0}^{\infty}\frac{x^n}{n^n}$

Exercise:

Problem: $\sum_{n=0}^{\infty}\frac{3nx^n}{12^n}$

Solution:

ROC: 12; IOC: $(-16,8)$

Exercise:

Problem: $\sum_{n=0}^{\infty}\frac{2^n}{e^n}(x-e)^n$

In the following exercises, find the power series representation for the given function. Determine the radius of convergence and the interval of convergence for that series.

Exercise:

Problem: $f(x)=\frac{x^2}{x+3}$

Solution:

$\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}x^n$; ROC: 3; IOC: $(-3,3)$

Exercise:

Problem: $f(x)=\frac{8x+2}{2x^2-3x+1}$

In the following exercises, find the power series for the given function using term-by-term differentiation or integration.

Exercise:

Problem: $f(x)=\tan^{-1}(2x)$

Solution:

integration: $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(2x)^{2n+1}$

Exercise:

Problem: $f(x)=\frac{x}{(2+x^2)^2}$

In the following exercises, evaluate the Taylor series expansion of degree four for the given function at the specified point. What is the error in the approximation?
Exercise:

Problem: $f\left(x\right)=x^3-2x^2+4, a=-3$

Solution:

$p_4\left(x\right)=\left(x+3\right)^3-11(x+3)^2+39\left(x+3\right)-41$; exact
Exercise:

Problem: $f\left(x\right)=e^{1/(4x)}, a=4$

In the following exercises, find the Maclaurin series for the given function.
Exercise:

Problem: $f\left(x\right)=\cos\left(3x\right)$

Solution:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n}}{2n!}$$

Exercise:

Problem: $f\left(x\right)=\ln\left(x+1\right)$

In the following exercises, find the Taylor series at the given value.
Exercise:

Problem: $f\left(x\right)=\sin x, a=\frac{\pi}{2}$

Solution:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$$

Exercise:

Problem: $f\left(x\right)=\frac{3}{x}, a=1$

In the following exercises, find the Maclaurin series for the given function.
Exercise:

Problem: $f\left(x\right)=e^{-x^2}-1$

Solution:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^{2n}$$

Exercise:

Problem: $f(x) = \cos x - x \sin x$

In the following exercises, find the Maclaurin series for $F(x) = \int_0^x f(t)dt$ by integrating the Maclaurin series of $f(x)$ term by term.

Exercise:

Problem: $f(x) = \frac{\sin x}{x}$

Solution:

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$$

Exercise:

Problem: $f(x) = 1 - e^x$

Exercise:

Problem: Use power series to prove Euler's formula: $e^{ix} = \cos x + i \sin x$

Solution:

Answers may vary.

The following exercises consider problems of annuity payments.

Exercise:

Problem:

For annuities with a present value of $1 million, calculate the annual payouts given over 25 years assuming interest rates of $1\%, 5\%, \text{ and } 10\%$.

Exercise:

Problem:

A lottery winner has an annuity that has a present value of $10 million. What interest rate would they need to live on perpetual annual payments of $250,000?

Solution:

2.5%

Exercise:

Problem:

Calculate the necessary present value of an annuity in order to support annual payouts of $15,000 given over 25 years assuming interest rates of $1\%, 5\%, \text{ and } 10\%$.

Glossary

binomial series

the Maclaurin series for $f(x) = (1+x)^r$; it is given by

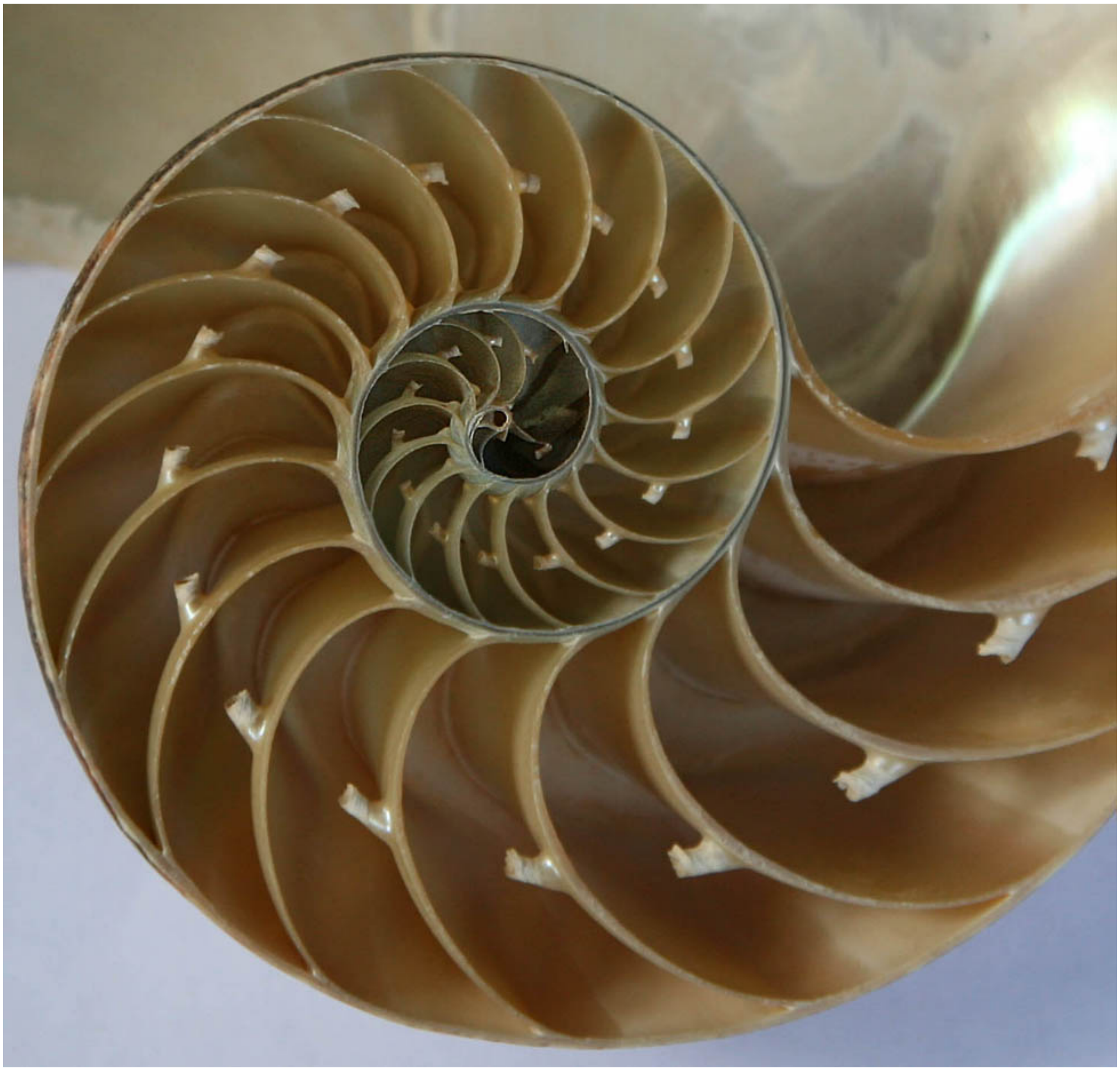
$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \text{ for } |x| < 1$$

nonelementary integral

an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

Introduction
class="introduction"

The chambered nautilus is a marine animal that lives in the tropical Pacific Ocean. Scientists think they have existed mostly unchanged for about 500 million years. (credit: modification of work by Jitze Couperus, Flickr)



The chambered nautilus is a fascinating creature. This animal feeds on hermit crabs, fish, and other crustaceans. It has a hard outer shell with many chambers connected in a spiral fashion, and it can retract into its shell to avoid predators. When part of the shell is cut away, a perfect spiral is revealed, with chambers inside that are somewhat similar to growth rings in a tree.

The mathematical function that describes a spiral can be expressed using rectangular (or Cartesian) coordinates. However, if we change our coordinate system to something that works a bit better with circular patterns, the function becomes much simpler to describe. The polar

coordinate system is well suited for describing curves of this type. How can we use this coordinate system to describe spirals and other radial figures? (See [link].)

In this chapter we also study parametric equations, which give us a convenient way to describe curves, or to study the position of a particle or object in two dimensions as a function of time. We will use parametric equations and polar coordinates for describing many topics later in this text.

Parametric Equations

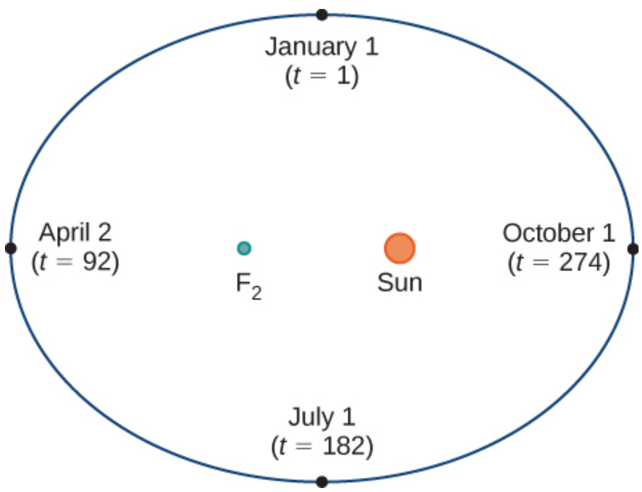
- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both *x* and *y* depend on, and as the parameter increases, the values of *x* and *y* trace out a path along a plane curve. For example, if the parameter is *t* (a common choice), then *t* might represent time. Then *x* and *y* are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 “day 1” of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

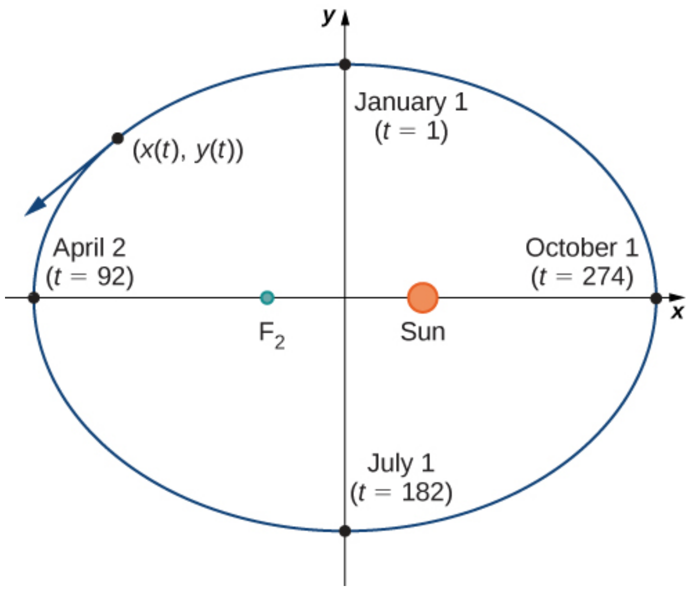
The number of the day in a year can be considered a variable that determines Earth’s position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler’s laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in Conic Sections.



Earth’s orbit around the Sun in one year.

[link] depicts Earth’s orbit around the Sun during one year. The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse ([link]). Then each *x* value on the graph is a value of position

as a function of time, and each *y* value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.



Coordinate axes superimposed on the orbit of Earth.

We can determine the functions for $x\left(t\right)$ and $y\left(t\right)$, thereby parameterizing the orbit of Earth around the Sun. The variable *t* is called an independent parameter and, in this context, represents time relative to the beginning of each year.

A curve in the $\left(x,y\right)$ plane can be represented parametrically. The equations that are used to define the curve are called **parametric equations**.

Note:
Definition
If *x* and *y* are continuous functions of *t* on an interval *I*, then the equations
Equation:

$$x=x\left(t\right)\text{ and }y=y\left(t\right)$$

are called parametric equations and *t* is called the **parameter**. The set of points $\left(x,y\right)$ obtained as *t* varies over the interval *I* is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or *plane curve*, and is denoted by *C*.

Notice in this definition that *x* and *y* are used in two ways. The first is as functions of the independent variable *t*. As *t* varies over the interval *I*, the functions $x\left(t\right)$ and $y\left(t\right)$ generate a set of ordered pairs $\left(x,y\right)$. This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered

pairs, *x* and *y* are variables. It is important to distinguish the variables *x* and *y* from the functions $x\left(t\right)$ and $y\left(t\right)$.

Example:
Exercise:

Problem:
Graphing a Parametrically Defined Curve

Sketch the curves described by the following parametric equations:

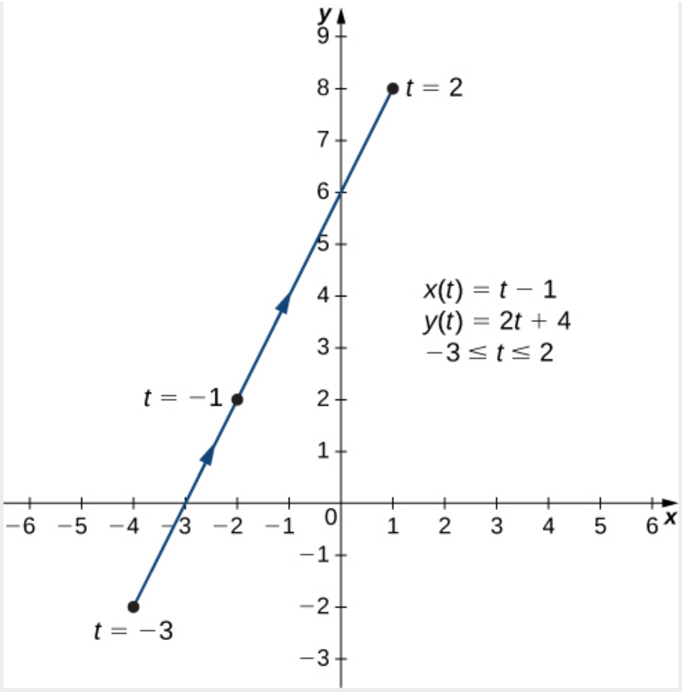
a. $x\left(t\right)=t-1,\quad y\left(t\right)=2t+4,\quad -3\le t\le 2$
b. $x\left(t\right)={t}^{2}-3,\quad y\left(t\right)=2t+1,\quad -2\le t\le 3$
c. $x\left(t\right)=4\cos t,\quad y\left(t\right)=4\sin t,\quad 0\le t\le 2\pi$

Solution:

a. To create a graph of this curve, first set up a table of values. Since the independent variable in both $x\left(t\right)$ and $y\left(t\right)$ is *t*, let *t* appear in the first column. Then $x\left(t\right)$ and $y\left(t\right)$ will appear in the second and third columns of the table.

t	$x\left(t\right)$	$y\left(t\right)$
−3	−4	−2
−2	−3	0
−1	−2	2
0	−1	4
1	0	6
2	1	8

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in [link]. The arrows on the graph indicate the **orientation** of the graph, that is, the direction that a point moves on the graph as *t* varies from −3 to 2.

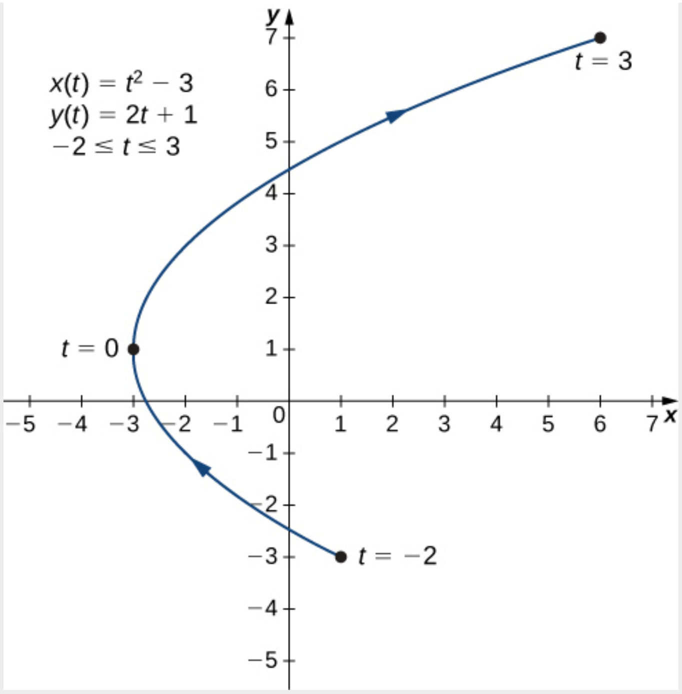


Graph of the plane curve described by the parametric equations in part a.

b. To create a graph of this curve, again set up a table of values.

t	$x\left(t\right)$	$y\left(t\right)$
−2	1	−3
−1	−2	−1
0	−3	1
1	−2	3
2	1	5
3	6	7

The second and third columns in this table give a set of points to be plotted ([link]). The first point on the graph (corresponding to $t=-2$) has coordinates $\left(1,-3\right)$, and the last point (corresponding to $t=3$) has coordinates $\left(6,7\right)$. As t progresses from −2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

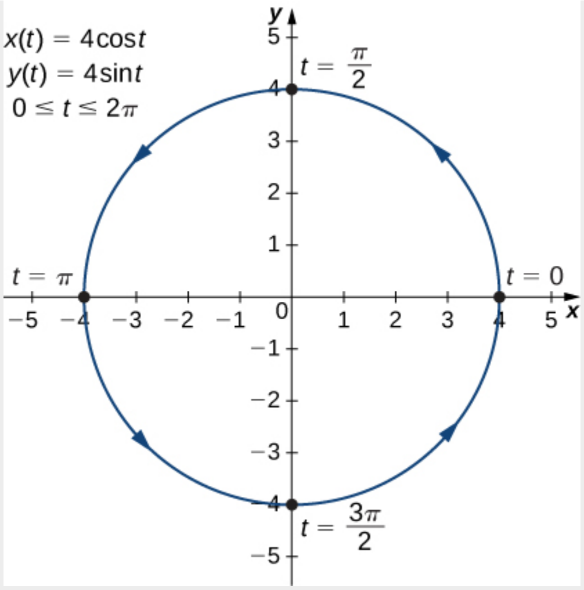


Graph of the plane curve described by the parametric equations in part b.

c. In this case, use multiples of $\pi/6$ for *t* and create another table of values:

t	$x(t)$	$y(t)$		***t***	$x(t)$	$y(t)$
0	4	0		$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2		$\frac{4\pi}{3}$	−2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$		$\frac{3\pi}{2}$	0	−4
$\frac{\pi}{2}$	0	4		$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	−2	$2\sqrt{3} \approx 3.5$		$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2		2π	4	0
π	−4	0				

The graph of this plane curve appears in the following graph.



Graph of the plane curve described by the parametric equations in part c.

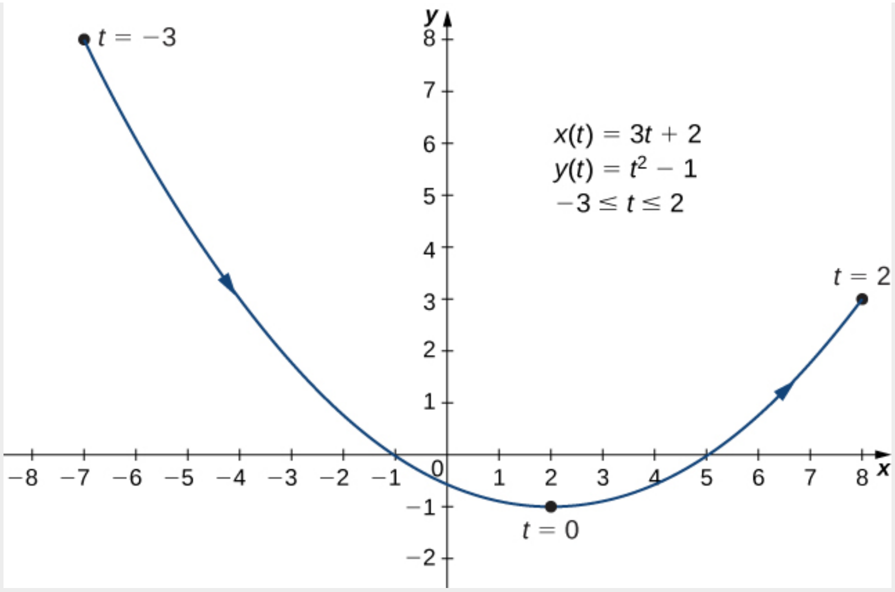
This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending points of the curve both have coordinates $(4, 0)$.

Note:
Exercise:

Problem: Sketch the curve described by the parametric equations
Equation:

$$x\left(t\right) = 3t + 2, \quad y\left(t\right) = t^2 - 1, \quad -3 \le t \le 2.$$

Solution:



Hint

Make a table of values for $x(t)$ and $y(t)$ using t values from −3 to 2.

Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables *x* and *y*. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in [link]b. are

Equation:

$$x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \leq t \leq 3.$$

Solving the second equation for *t* gives

Equation:

$$t = \frac{y-1}{2}.$$

This can be substituted into the first equation:

Equation:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 = \frac{y^2 - 2y + 1}{4} - 3 = \frac{y^2 - 2y - 11}{4}.$$

This equation describes *x* as a function of *y*. These steps give an example of *eliminating the parameter*. The graph of this function is a parabola opening to the right. Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter *t*.

Example:
Exercise:

Problem:
Eliminating the Parameter

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad -2 \le t \le 6$
b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad 0 \le t \le 2\pi$

Solution:

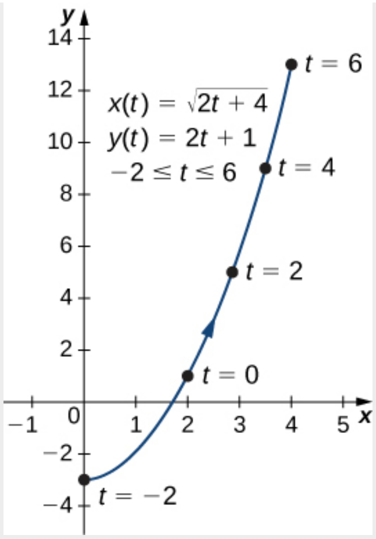
a. To eliminate the parameter, we can solve either of the equations for *t*. For example, solving the first equation for *t* gives
Equation:

$$\begin{aligned} x &= \sqrt{2t+4} \\ x^2 &= 2t+4 \\ x^2-4 &= 2t \\ t &= \tfrac{x^2-4}{2}. \end{aligned}$$

Note that when we square both sides it is important to observe that $x \ge 0$. Substituting $t = \frac{x^2-4}{2}$ this into $y(t)$ yields
Equation:

$$\begin{aligned} y(t) &= 2t+1 \\ y &= 2\left(\tfrac{x^2-4}{2}\right)+1 \\ y &= x^2-4+1 \\ y &= x^2-3. \end{aligned}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter *t*. When $t=-2$, $x = \sqrt{2(-2)+4} = 0$, and when $t=6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.



Graph of the plane curve described by the parametric equations in part a.

b. Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

Equation:

$$x\left(t\right)=4\cos t\text{ and }y\left(t\right)=3\sin t.$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

Equation:

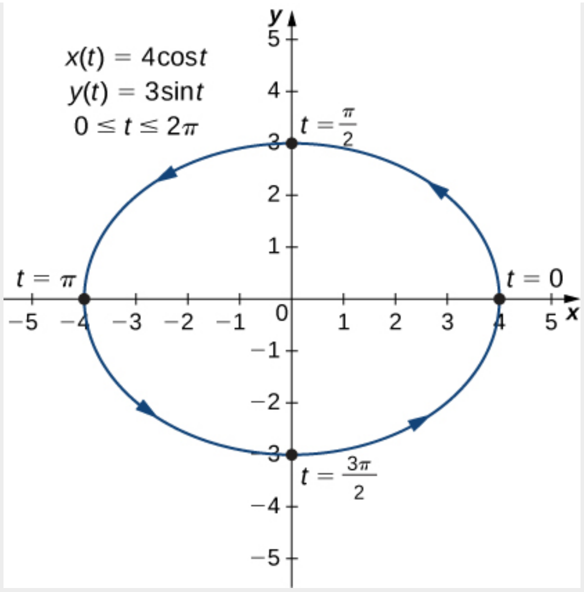
$$\cos t=\frac{x}{4}\text{ and }\sin t=\frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t+\sin^2 t=1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

Equation:

$$\begin{array}{rcl}\left(\frac{x}{4}\right)^2+\left(\frac{y}{3}\right)^2 & = & 1\\ \frac{x^2}{16}+\frac{y^2}{9} & = & 1.\end{array}$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and semiminor axis 3 as shown in the following graph.



Graph of the plane curve described by the parametric equations in part b.

As *t* progresses from 0 to 2π, a point on the curve traverses the ellipse once, in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parameterized curves to model a real-world phenomenon.

Note:
Exercise:

Problem:

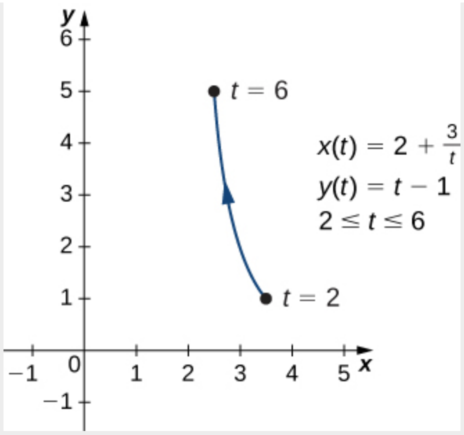
Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph.

Equation:

$$x\left(t\right)=2+\frac{3}{t},\quad y\left(t\right)=t-1,\quad 2\le t\le 6$$

Solution:

$x=2+\frac{3}{y+1}$, or $y=-1+\frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $\left(2,-1\right)$.



Hint

Solve one of the equations for *t* and substitute into the other equation.

So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve**.

Example:
Exercise:

Problem:
Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

Solution:

First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing *x* with *t* in the equation for $y(t)$. This gives the parameterization
Equation:

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of *t*.

We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on *x*; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for *x*. This gives
Equation:

$$\begin{aligned} y(t) &= 2(3t-2)^2 - 2 \\ &= 2\left(9t^2 - 12t + 4\right) - 2 \\ &= 18t^2 - 24t + 8 - 2 \\ &= 18t^2 - 24t + 6. \end{aligned}$$

Therefore, a second parameterization of the curve can be written as

Equation:

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$

Note:

Exercise:

Problem: Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$.

Solution:

One possibility is $x(t) = t, \quad y(t) = t^2 + 2t$. Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3$.

There are, in fact, an infinite number of possibilities.

Hint

Follow the steps in [link]. Remember we have freedom in choosing the parameterization for $x(t)$.

Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a **cycloid** ([link]). A cycloid generated by a circle (or bicycle wheel) of radius *a* is given by the parametric equations

Equation:

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the *x*-axis at a constant height equal to the radius of the wheel. If the radius is *a*, then the coordinates of the center can be given by the equations

Equation:

$$x(t) = at, \quad y(t) = a$$

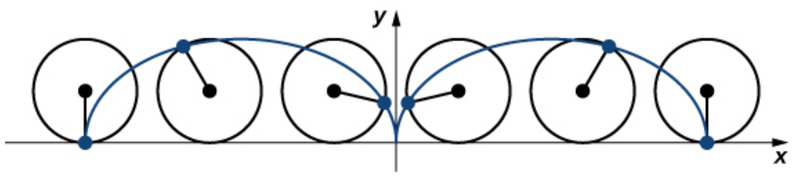
for any value of t. Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of

the circular motion of the ant (relative to the center of the wheel) is given by
Equation:

$$x\left(t\right)=-a\sin t,\quad y\left(t\right)=-a\cos t.$$

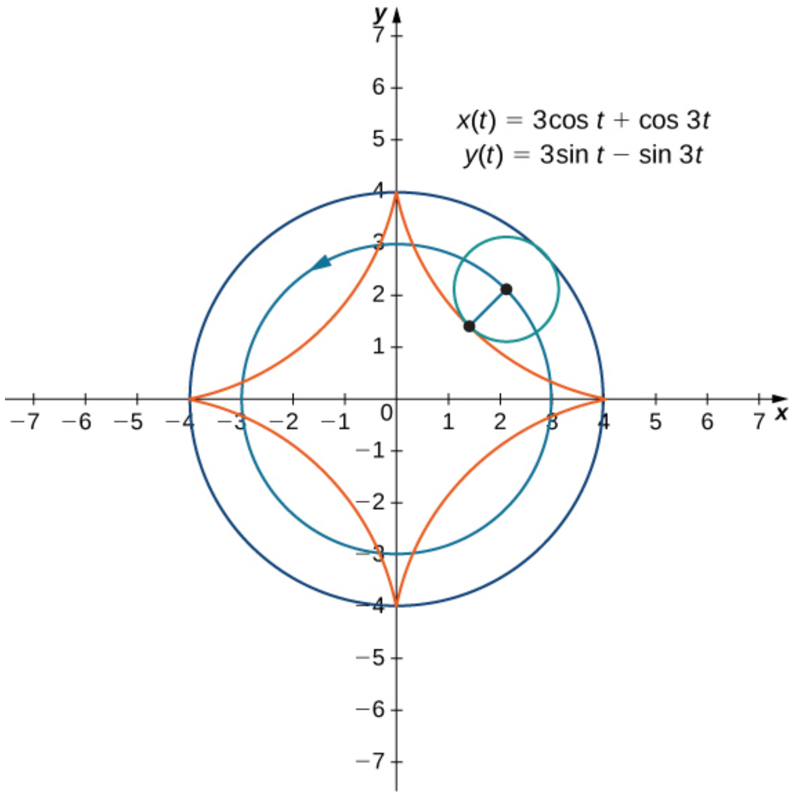
(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.
Equation:

$$x\left(t\right)=a\left(t-\sin t\right),\quad y\left(t\right)=a\left(1-\cos t\right).$$



A wheel traveling along a road without slipping; the point on the edge of the wheel traces out a cycloid.

Now suppose that the bicycle wheel doesn’t travel along a straight road but instead moves along the inside of a larger wheel, as in [link]. In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a hypocycloid.



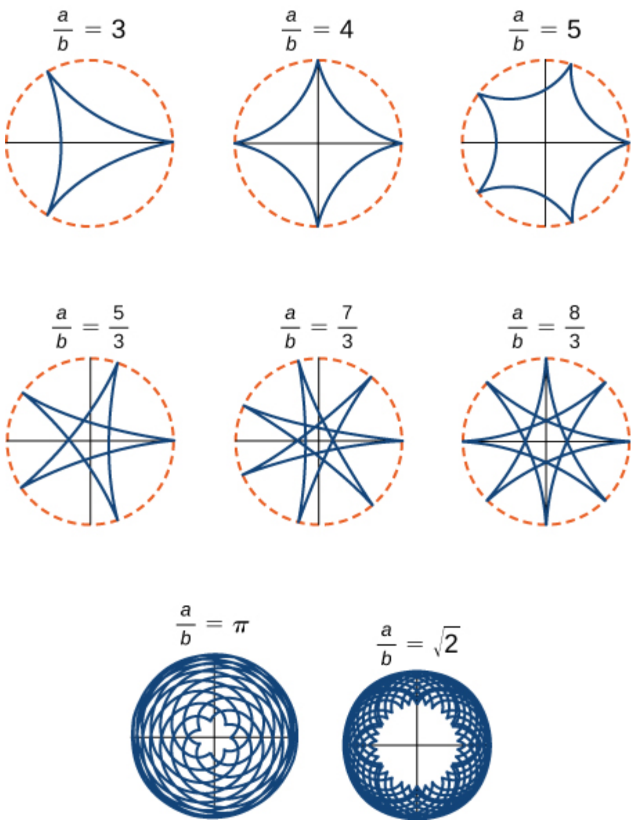
Graph of the hypocycloid described by the parametric equations shown.

The general parametric equations for a hypocycloid are
Equation:

$$x(t) = (a-b)\cos t + b\cos\left(\frac{a-b}{b}\right)t$$
$$y(t) = (a-b)\sin t - b\sin\left(\frac{a-b}{b}\right)t.$$

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is *a* and the radius of the smaller circle is *b*. Then the center of the wheel travels along a circle of radius $a-b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a-b}$.

The ratio $\frac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in [link]. This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. [link] corresponds to $a=4$ and $b=1$. The result is a hypocycloid with four cusps. [link] shows some other possibilities. The last two hypocycloids have irrational values for $\frac{a}{b}$. In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as *space-filling curves*.



Graph of various hypocycloids corresponding to different values of a/b.

Note:

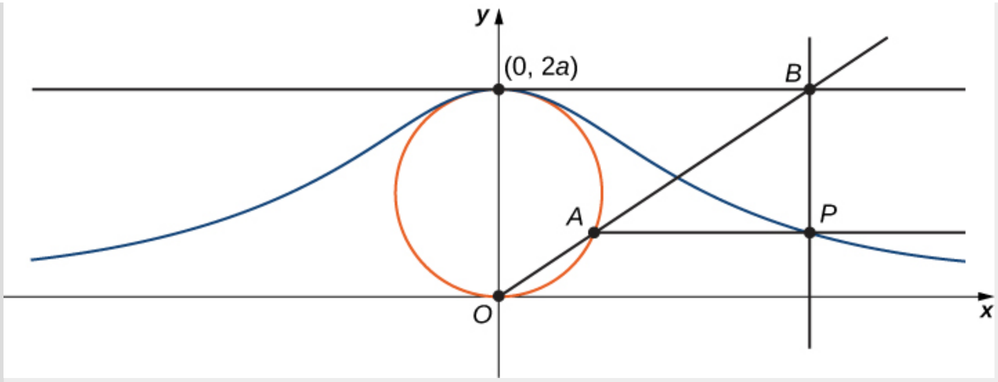
The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the folium of Descartes or the spiral of Archimedes. However, perhaps the strangest name for a curve is the witch of Agnesi. Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of the few recognized women mathematicians of eighteenth-century Italy. She wrote a popular book on analytic geometry, published in 1748, which included an interesting curve that had been studied by Fermat in 1630. The mathematician Guido Grandi showed in 1703 how to construct this curve, which he later called the “versoria,” a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, “versiera,” but in Latin, this same word means a “female goblin.” When Agnesi’s book was translated into English in 1801, the translator used the term “witch” for the curve, instead of rope. The name “witch of Agnesi” has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius a so that the points $(0, 0)$ and $(0, 2a)$ are points on the circle ([link]). Let O denote the origin. Choose any other point A on the circle, and draw the secant line OA. Let B denote the point at which the line OA intersects the horizontal line through $(0, 2a)$. The vertical line through B intersects the horizontal line through A at the point P. As the point A varies, the path that the point P travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the Cauchy distribution. In this project you will parameterize these curves.



As the point A moves around the circle, the point P traces out the witch of Agnesi curve for the given circle.

1. On the figure, label the following points, lengths, and angle:
   a. C is the point on the x-axis with the same x-coordinate as A.
   b. x is the x-coordinate of P, and y is the y-coordinate of P.
   c. E is the point $(0, a)$.
   d. F is the point on the line segment OA such that the line segment EF is perpendicular to the line segment OA.
   e. b is the distance from O to F.
   f. c is the distance from F to A.
   g. d is the distance from O to B.
   h. θ is the measure of angle $\angle COA$.

   The goal of this project is to parameterize the witch using θ as a parameter. To do this, write equations for x and y in terms of only θ.
2. Show that $d = \frac{2a}{\sin\theta}$.
3. Note that $x = d\cos\theta$. Show that $x = 2a\cot\theta$. When you do this, you will have parameterized the x-coordinate of the curve with respect to θ. If you can get a similar equation for y, you will have parameterized the curve.
4. In terms of θ, what is the angle $\angle EOA$?
5. Show that $b + c = 2a\cos\left(\frac{\pi}{2} - \theta\right)$.
6. Show that $y = 2a\cos\left(\frac{\pi}{2} - \theta\right)\sin\theta$.
7. Show that $y = 2a\sin^2\theta$. You have now parameterized the y-coordinate of the curve with respect to θ.
8. Conclude that a parameterization of the given witch curve is
   Equation:
   $$x = 2a\cot\theta, y = 2a\sin^2\theta, -\infty < \theta < \infty.$$
9. Use your parameterization to show that the given witch curve is the graph of the function $f(x) = \frac{8a^3}{x^2 + 4a^2}$.

Note:
Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, which is the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project we look at two different variations of the cycloid, called the curtate and prolate cycloids.
First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We have assumed the ant climbed onto the tire at the very edge, where the tire touches the ground. As the wheel rolls, the ant moves with the edge of the tire ([link]).
As we have discussed, we have a lot of flexibility when parameterizing a curve. In this case we let our parameter *t* represent the angle the tire has rotated through. Looking at [link], we see that after the tire has rotated through an angle of *t*, the position of the center of the wheel, $C=(x_C,y_C)$, is given by
Equation:

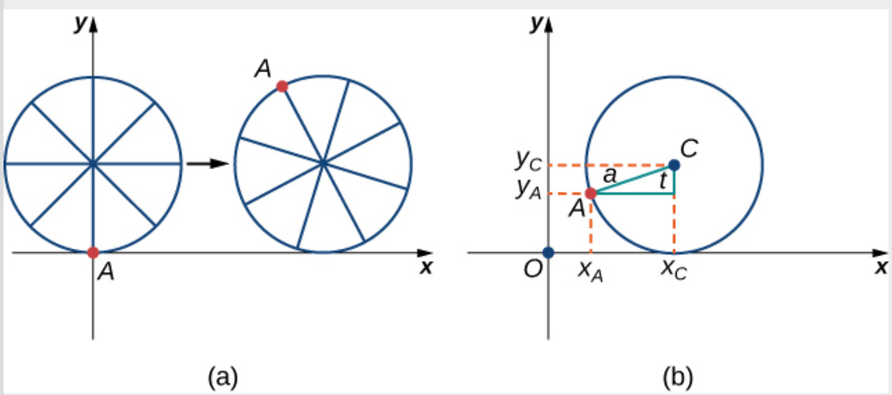
$$x_C=at \text{ and } y_C=a.$$

Furthermore, letting $A=(x_A,y_A)$ denote the position of the ant, we note that
Equation:

$$x_C-x_A=a\sin t \text{ and } y_C-y_A=a\cos t.$$

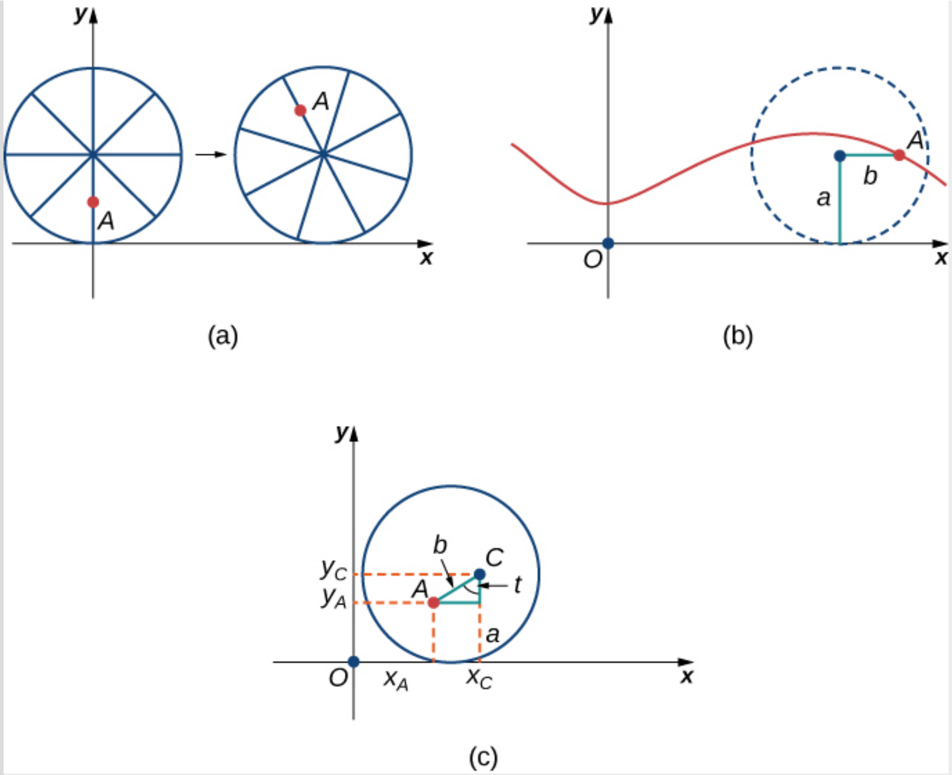
Then
Equation:

$$\begin{aligned}x_A&=x_C-a\sin t=at-a\sin t=a(t-\sin t)\\ y_A&=y_C-a\cos t=a-a\cos t=a(1-\cos t).\end{aligned}$$



(a) The ant clings to the edge of the bicycle tire as the tire rolls along the ground. (b) Using geometry to determine the position of the ant after the tire has rotated through an angle of *t*.

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable *t*.
After a while the ant is getting dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a curtate cycloid ([link]). As shown in the figure, we let *b* denote the distance along the spoke from the center of the wheel to the ant. As before, we let *t* represent the angle the tire has rotated through. Additionally, we let $C=(x_C,y_C)$ represent the position of the center of the wheel and $A=(x_A,y_A)$ represent the position of the ant.



(a) The ant climbs up one of the spokes toward the center of the wheel. (b) The ant's path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid. (c) The new setup, now that the ant has moved closer to the center of the wheel.

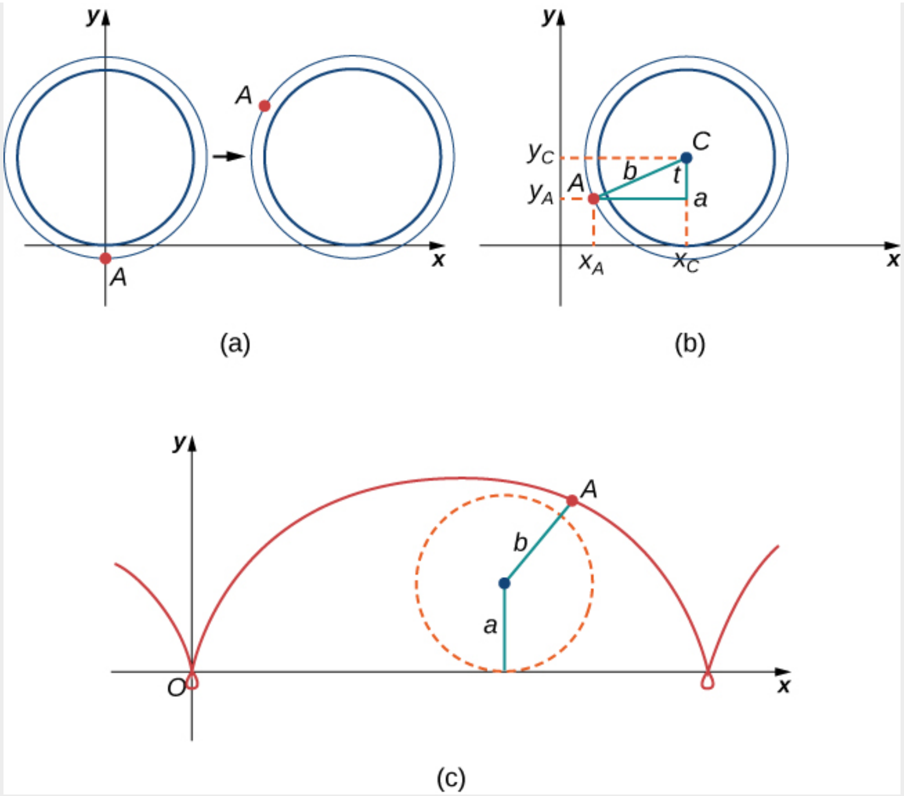
1. What is the position of the center of the wheel after the tire has rotated through an angle of t?
2. Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.
3. On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?

   Once the ant's head clears, he realizes that the bicyclist has made a turn, and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, there is a set of train tracks nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and just catch the edge of the train wheel (without getting squished!).

   The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to just hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is actually greater than the radius of the wheel ([link]).

   The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant ([link]).

   When the distance from the center of the wheel to the ant is greater than the radius of the wheel, his path of motion is called a prolate cycloid. A graph of a prolate cycloid is shown in the figure.



(a) The ant is hanging onto the flange of the train wheel. (b) The new setup, now that the ant has jumped onto the train wheel. (c) The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.
5. What do you notice about your answer to part 3 and your answer to part 4?
   Notice that the ant is actually traveling backward at times (the "loops" in the graph), even though the train continues to move forward. He is probably going to be *really* dizzy by the time he gets home!

Key Concepts

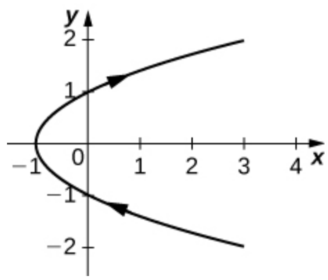
- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

For the following exercises, sketch the curves below by eliminating the parameter *t*. Give the orientation of the curve.

Exercise:

Problem: $x = t^2 + 2t, y = t + 1$

Solution:



orientation: bottom to top

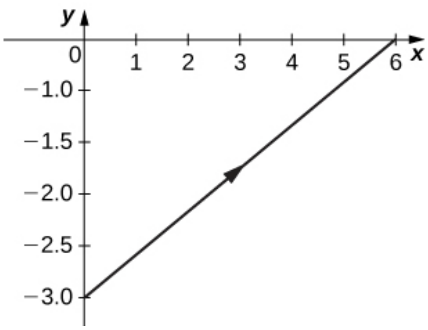
Exercise:

Problem: $x = \cos(t), y = \sin(t), (0, 2\pi]$

Exercise:

Problem: $x = 2t + 4, y = t - 1$

Solution:



orientation: left to right

Exercise:

Problem: $x = 3 - t, y = 2t - 3, 1.5 \leq t \leq 3$

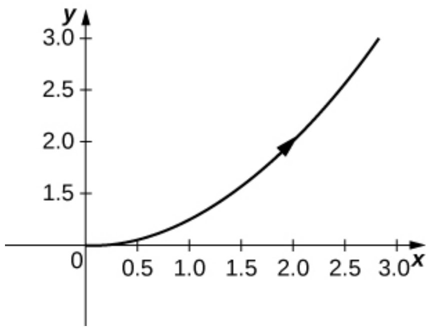
For the following exercises, eliminate the parameter and sketch the graphs.

Exercise:

Problem: $x = 2t^2, \quad y = t^4 + 1$

Solution:

$y = \frac{x^2}{4} + 1$



For the following exercises, use technology (CAS or calculator) to sketch the parametric equations.

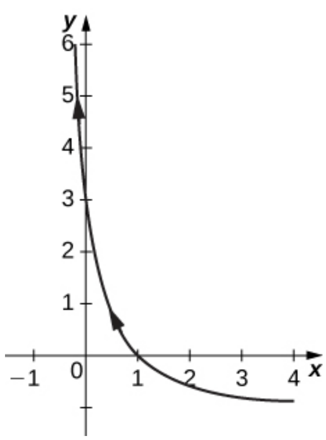
Exercise:

Problem: **[T]** $x = t^2 + t, \quad y = t^2 - 1$

Exercise:

Problem: **[T]** $x = e^{-t}, \quad y = e^{2t} - 1$

Solution:



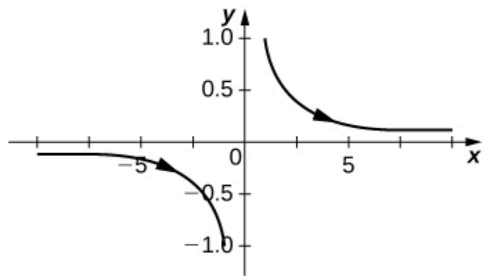
Exercise:

Problem: **[T]** $x = 3\cos t, \quad y = 4\sin t$

Exercise:

Problem: **[T]** $x = \sec t, \quad y = \cos t$

Solution:



For the following exercises, sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.

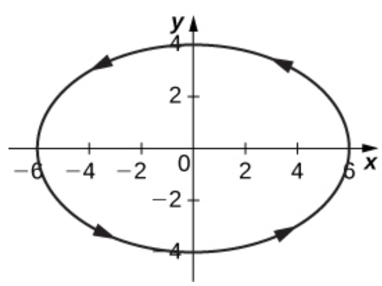
Exercise:

Problem: $x = e^t, \quad y = e^{2t} + 1$

Exercise:

Problem: $x = 6\sin(2\theta), y = 4\cos(2\theta)$

Solution:



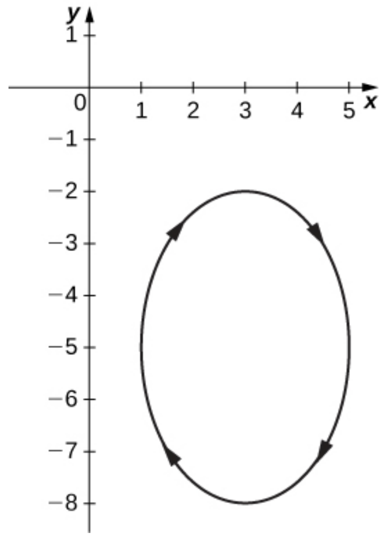
Exercise:

Problem: $x = \cos\theta, \quad y = 2\sin(2\theta)$

Exercise:

Problem: $x = 3 - 2\cos\theta, \quad y = -5 + 3\sin\theta$

Solution:



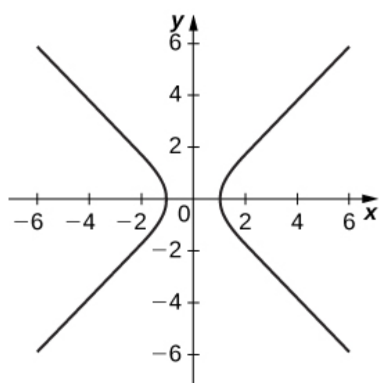
Exercise:

Problem: $x = 4 + 2\cos\theta, \quad y = -1 + \sin\theta$

Exercise:

Problem: $x = \sec t, \quad y = \tan t$

Solution:



Asymptotes are $y = x$ and $y = -x$

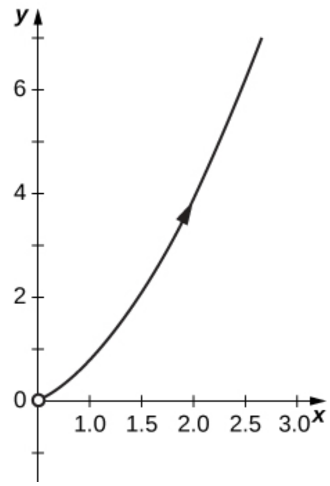
Exercise:

Problem: $x = \ln(2t), \quad y = t^2$

Exercise:

Problem: $x = e^t, \quad y = e^{2t}$

Solution:



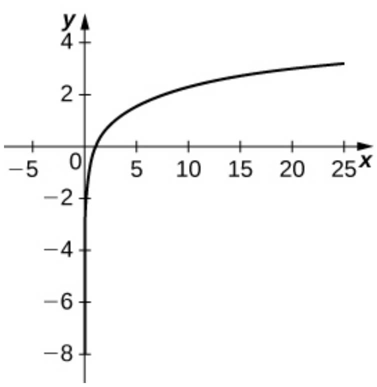
Exercise:

Problem: $x = e^{-2t}, \quad y = e^{3t}$

Exercise:

Problem: $x = t^3, \quad y = 3 \ln t$

Solution:



Exercise:

Problem: $x = 4 \sec \theta, \quad y = 3 \tan \theta$

For the following exercises, convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.

Exercise:

Problem: $x = t^2 - 1, \quad y = \frac{t}{2}$

Solution:

$x = 4y^2 - 1$; domain: $x \in [1, \infty)$.

Exercise:

Problem: $x = \frac{1}{\sqrt{t+1}}, \quad y = \frac{t}{1+t}, t > -1$

Exercise:

Problem: $x = 4\cos\theta, y = 3\sin\theta, t \in (0, 2\pi]$

Solution:

$\frac{x^2}{16} + \frac{y^2}{9} = 1$; domain $x \in [-4, 4]$.

Exercise:

Problem: $x = \cosh t, \quad y = \sinh t$

Exercise:

Problem: $x = 2t - 3, \quad y = 6t - 7$

Solution:

$y = 3x + 2$; domain: all real numbers.

Exercise:

Problem: $x = t^2, \quad y = t^3$

Exercise:

Problem: $x = 1 + \cos t, \quad y = 3 - \sin t$

Solution:

$(x-1)^2 + (y-3)^2 = 1$; domain: $x \in [0, 2]$.

Exercise:

Problem: $x = \sqrt{t}, \quad y = 2t + 4$

Exercise:

Problem: $x = \sec t, \quad y = \tan t, \pi \leq t < \frac{3\pi}{2}$

Solution:

$y = \sqrt{x^2 - 1}$; domain: $x \in [-1, 1]$.

Exercise:

Problem: $x = 2\cosh t, \quad y = 4\sinh t$

Exercise:

Problem: $x = \cos(2t), \quad y = \sin t$

Solution:

$y^2 = \frac{1-x}{2}$; domain: $x \in [2, \infty) \cup (-\infty, -2]$.

Exercise:

Problem: $x = 4t + 3, y = 16t^2 - 9$

Exercise:

Problem: $x = t^2, \quad y = 2 \ln t, t \geq 1$

Solution:

$y = \ln x$; domain: $x \in (0, \infty)$.

Exercise:

Problem: $x = t^3, \quad y = 3 \ln t, t \geq 1$

Exercise:

Problem: $x = t^n, \quad y = n \ln t, t \geq 1,$ where *n* is a natural number

---

Solution:

$y = \ln x$; domain: $x \in (0, \infty)$.

Exercise:

Problem: $\begin{array}{l} x = \ln(5t) \\ y = \ln(t^2) \end{array}$ where $1 \leq t \leq e$

Exercise:

Problem: $\begin{array}{l} x = 2\sin(8t) \\ y = 2\cos(8t) \end{array}$

Solution:

$x^2 + y^2 = 4$; domain: $x \in [-2, 2]$.

Exercise:

Problem: $\begin{array}{l} x = \tan t \\ y = \sec^2 t - 1 \end{array}$

For the following exercises, the pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.

Exercise:

Problem: $x = 3t + 4$
$y = 5t - 2$

Solution:

line

Exercise:

Problem: $x - 4 = 5t$
$y + 2 = t$

Exercise:

Problem: $x = 2t + 1$
$y = t^2 - 3$

Solution:

parabola

Exercise:

Problem: $x = 3\cos t$
$y = 3\sin t$

Exercise:

Problem: $x = 2\cos(3t)$
$y = 2\sin(3t)$

Solution:

circle

Exercise:

Problem: $x = \cosh t$
$y = \sinh t$

Exercise:

Problem: $x = 3\cos t$
$y = 4\sin t$

Solution:

ellipse

Exercise:

Problem: $x = 2\cos(3t)$
$y = 5\sin(3t)$

Exercise:

Problem: $x = 3\cosh(4t)$
$y = 4\sinh(4t)$

Solution:

hyperbola

Exercise:

Problem: $x = 2\cosh t$
$y = 2\sinh t$

Exercise:

Problem: Show that $\begin{array}{l} x = h + r\cos\theta \\ y = k + r\sin\theta \end{array}$ represents the equation of a circle.

Exercise:

Problem:

Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is 5 and whose center is $(-2,\ 3)$.

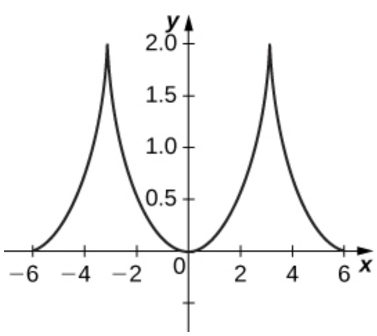
For the following exercises, use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.

Exercise:

Problem: [T] $x = \theta + \sin\theta$
$y = 1 - \cos\theta$

Solution:

The equations represent a cycloid.



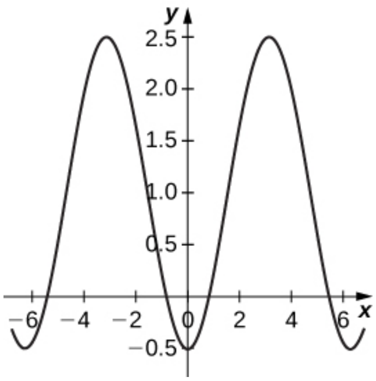
Exercise:

Problem: [T] $x = 2t - 2\sin t$
$y = 2 - 2\cos t$

Exercise:

Problem: **[T]** $x = t - 0.5 \sin t$
$y = 1 - 1.5 \cos t$

Solution:



Exercise:

Problem:

An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t, y = -4.9t^2 + 4000, t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?

Exercise:

Problem:

The trajectory of a bullet is given by $x = v_0 (\cos \alpha) ty = v_0 (\sin \alpha) t - \frac{1}{2}gt^2$ where $v_0 = 500 \text{ m/s}$, $g = 9.8 = 9.8 \text{ m/s}^2$, and $\alpha = 30$ degrees. When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

Solution:

22,092 meters at approximately 51 seconds.

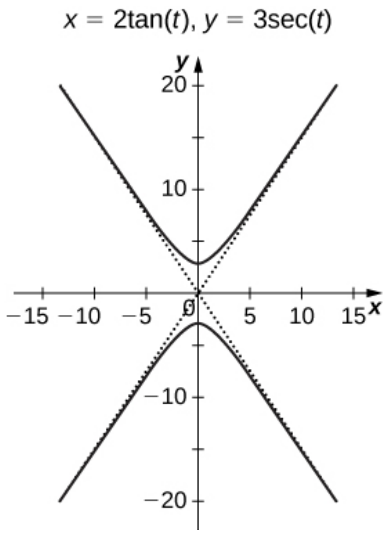
Exercise:

Problem: **[T]** Use technology to sketch the curve represented by $x = \sin(4t), y = \sin(3t), 0 \leq t \leq 2\pi$.

Exercise:

Problem: **[T]** Use technology to sketch $x = 2\tan(t), y = 3\sec(t), -\pi < t < \pi$.

Solution:



Exercise:

Problem:

Sketch the curve known as an *epitrochoid*, which gives the path of a point on a circle of radius *b* as it rolls on the outside of a circle of radius *a*. The equations are

$$x = (a+b)\cos t - c \cdot \cos\left[\frac{(a+b)t}{b}\right]$$
$$y = (a+b)\sin t - c \cdot \sin\left[\frac{(a+b)t}{b}\right].$$

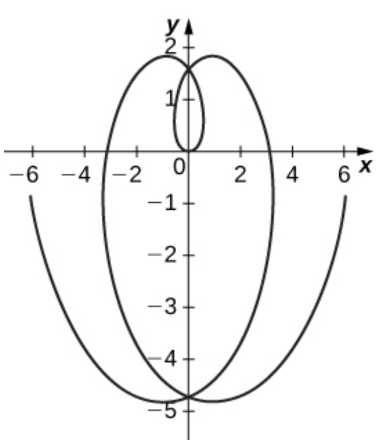
Let $a = 1, b = 2, c = 1$.

Exercise:

Problem:

[T] Use technology to sketch the spiral curve given by $x = t\cos(t), y = t\sin(t)$ from $-2\pi \le t \le 2\pi$.

Solution:



Exercise:

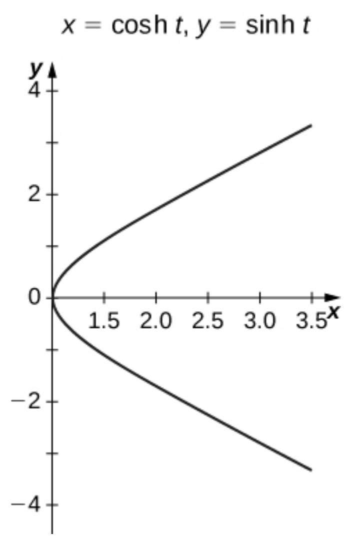
Problem:

[T] Use technology to graph the curve given by the parametric equations $x = 2\cot(t), y = 1 - \cos(2t), -\pi/2 \le t \le \pi/2$. This curve is known as the witch of Agnesi.

Exercise:

Problem: **[T]** Sketch the curve given by parametric equations $\begin{matrix} x = \cosh(t) \\ y = \sinh(t), \end{matrix}$ where $-2 \le t \le 2$.

Solution:



Glossary

cycloid
: the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

cusp
: a pointed end or part where two curves meet

orientation
: the direction that a point moves on a graph as the parameter increases

parameter
: an independent variable that both *x* and *y* depend on in a parametric curve; usually represented by the variable *t*

parametric curve
: the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \le t \le b$ combined with the equations

parametric equations
: the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

parameterization of a curve
: rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

Calculus of Parametric Curves

- Determine derivatives and equations of tangents for parametric curves.
- Find the area under a parametric curve.
- Use the equation for arc length of a parametric curve.
- Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

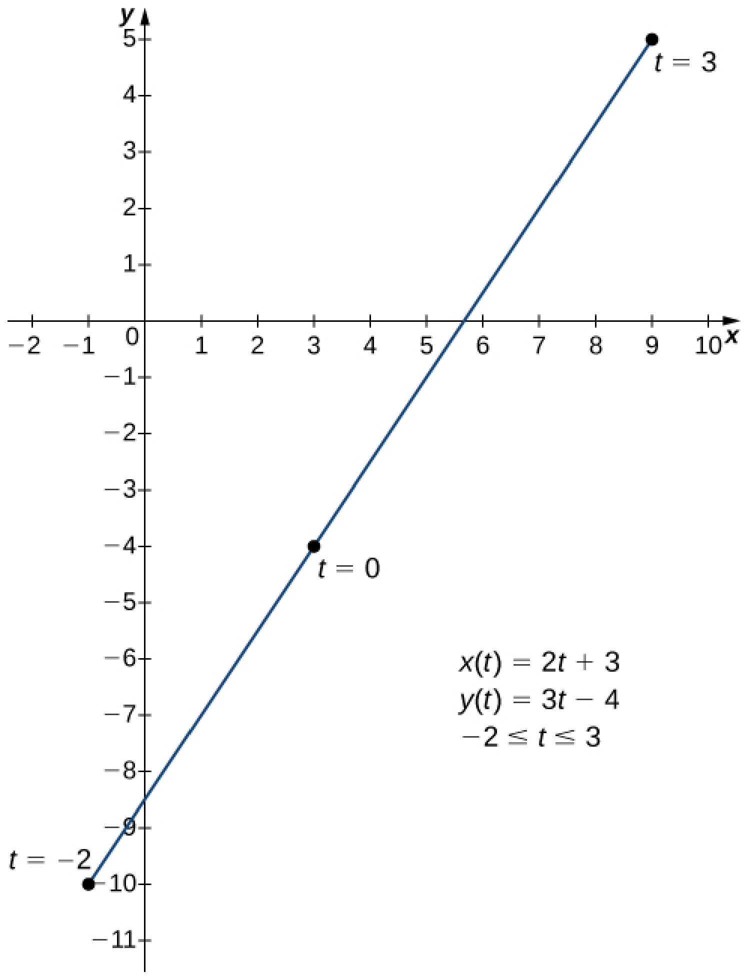
Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher’s hand. If the position of the baseball is represented by the plane curve $\left(x\left(t\right),y\left(t\right)\right)$, then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations
Equation:

$$x\left(t\right)=2t+3,\quad y\left(t\right)=3t-4,\quad -2\le t\le 3.$$

The graph of this curve appears in [link]. It is a line segment starting at $\left(-1,-10\right)$ and ending at $\left(9,5\right)$.



Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving the equation $x(t) = 2t + 3$ for t:
Equation:

$$\begin{array}{rcl} x\left(t\right) & = & 2t+3 \\ x-3 & = & 2t \\ t & = & \frac{x-3}{2}. \end{array}$$

Substituting this into $y(t)$, we obtain
Equation:

$$\begin{aligned}
y(t) &= 3t-4 \\
y &= 3\left(\frac{x-3}{2}\right)-4 \\
y &= \frac{3x}{2}-\frac{9}{2}-4 \\
y &= \frac{3x}{2}-\frac{17}{2}.
\end{aligned}$$

The slope of this line is given by $\frac{dy}{dx}=\frac{3}{2}$. Next we calculate $x'(t)$ and $y'(t)$. This gives $x'(t)=2$ and $y'(t)=3$. Notice that $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3}{2}$. This is no coincidence, as outlined in the following theorem.

Note:
Derivative of Parametric Equations
Consider the plane curve defined by the parametric equations $x=x(t)$ and $y=y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t)\neq 0$. Then the derivative $\frac{dy}{dx}$ is given by
Equation:

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{y'(t)}{x'(t)}.$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter *t* can be eliminated, yielding a differentiable function $y=F(x)$. Then $y(t)=F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields
Equation:

$$y'(t)=F'(x(t))x'(t),$$

so
Equation:

$$F'(x(t))=\frac{y'(t)}{x'(t)}.$$

But $F'(x(t))=\frac{dy}{dx}$, which proves the theorem.

□

[link] can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y=f(x)$ is any point $x=x_0$ such that either $f'(x_0)=0$

or $f'(x_0)$ does not exist. [link] gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Example:
Exercise:

Problem:
Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically defined plane curves, and locate any critical points on their respective graphs.

a. $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad -3 \le t \le 4$
b. $x(t) = 2t + 1, \quad y(t) = t^3 - 3t + 4, \quad -2 \le t \le 5$
c. $x(t) = 5\cos t, \quad y(t) = 5\sin t, \quad 0 \le t \le 2\pi$

Solution:

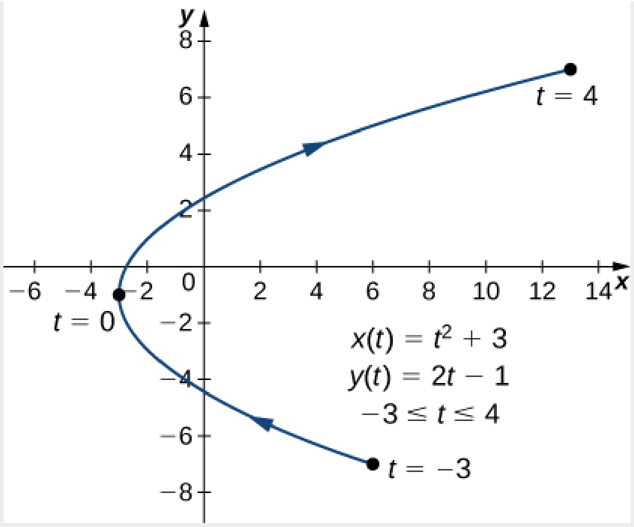
a. To apply [link], first calculate $x'(t)$ and $y'(t)$:
Equation:

$$x'(t) = 2t$$
$$y'(t) = 2.$$

Next substitute these into the equation:
Equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2}{2t}$$
$$\frac{dy}{dx} = \frac{1}{t}.$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex as shown.



Graph of the parabola described by parametric equations in part a.

b. To apply [link], first calculate $x'(t)$ and $y'(t)$:

Equation:

$$x'(t) = 2$$
$$y'(t) = 3t^2 - 3.$$

Next substitute these into the equation:

Equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{3t^2-3}{2}.$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

Equation:

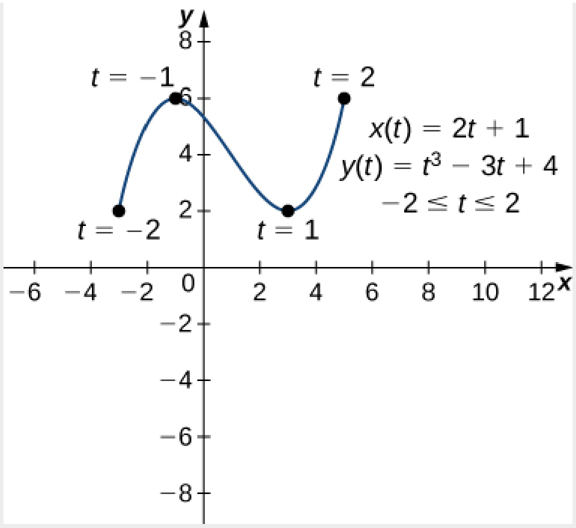
$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

Equation:

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.



Graph of the curve described by parametric equations in part b.

c. To apply [link], first calculate $x'(t)$ and $y'(t)$:
Equation:

$$x'(t) = -5 \sin t$$
$$y'(t) = 5 \cos t.$$

Next substitute these into the equation:
Equation:

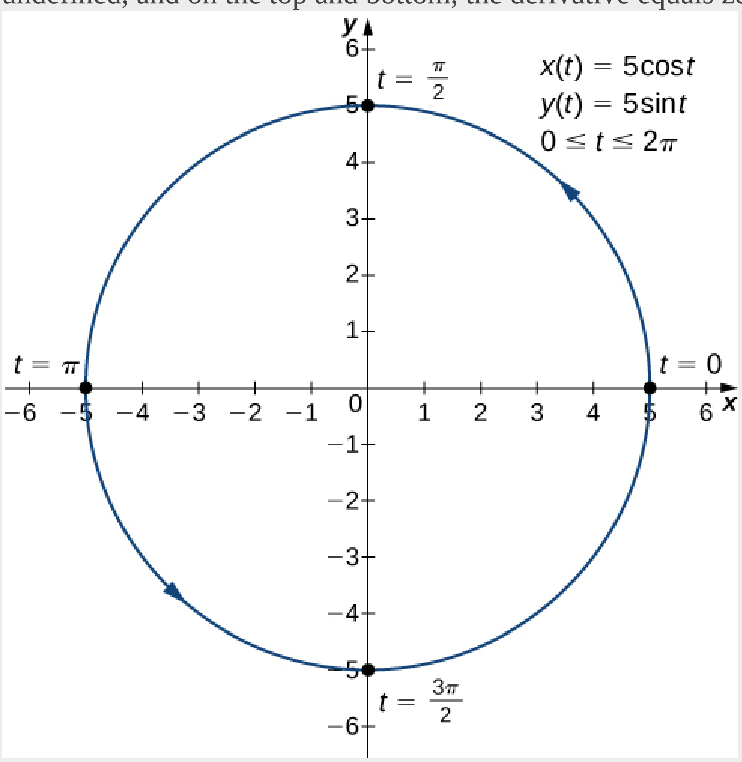
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{5 \cos t}{-5 \sin t}$$
$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π as critical points for t. Substituting each of these into $x(t)$ and $y(t)$, we obtain

t	$x(t)$	$y(t)$
0	5	0

t	$x(t)$	$y(t)$
$\frac{\pi}{2}$	0	5
π	−5	0
$\frac{3\pi}{2}$	0	−5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations ([link]). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.



Graph of the curve described by parametric equations in part c.

Note:
Exercise:

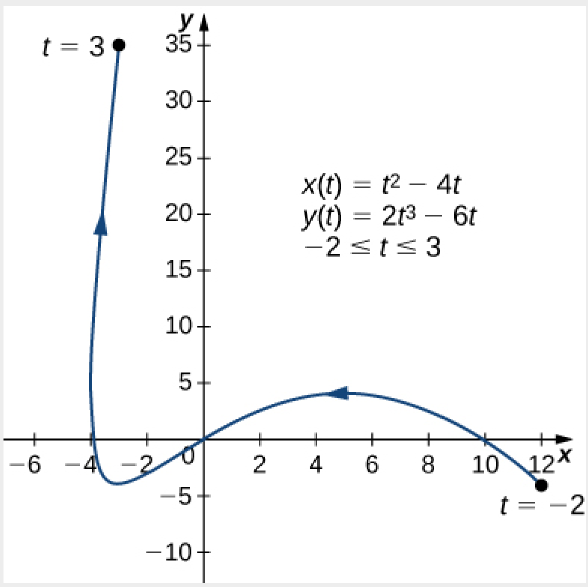
Problem: Calculate the derivative dy/dx for the plane curve defined by the equations
Equation:

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \le t \le 3$$

and locate any critical points on its graph.

Solution:

$x'(t) = 2t - 4$ and $y'(t) = 6t^2 - 6$, so $\frac{dy}{dx} = \frac{6t^2-6}{2t-4} = \frac{3t^2-3}{t-2}$.
This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1$.



Hint

Calculate $x'(t)$ and $y'(t)$ and use [link].

Example:
Exercise:

Problem:
Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations
Equation:

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad -3 \le t \le 4 \text{ when } t = 2.$$

Solution:

First find the slope of the tangent line using [link], which means calculating $x'(t)$ and $y'(t)$:
Equation:

$$x'(t) = 2t$$
$$y'(t) = 2.$$

Next substitute these into the equation:
Equation:

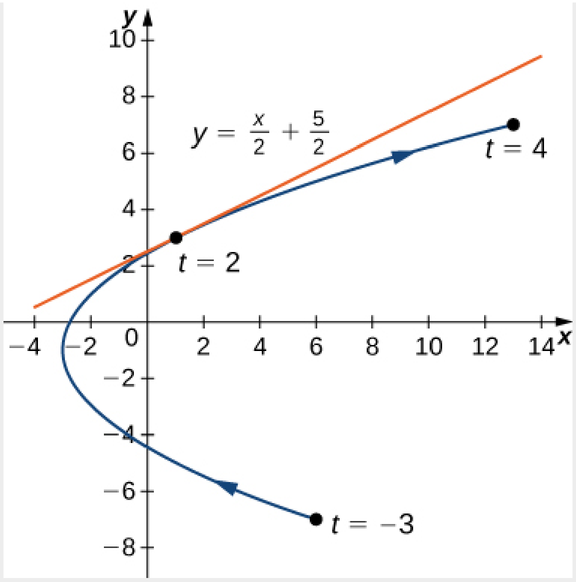
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2}{2t}$$
$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives
Equation:

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph ([link]). Now use the point-slope form of the equation of a line to find the equation of the tangent line:
Equation:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= \frac{1}{2}(x - 1) \\ y - 3 &= \frac{1}{2}x - \frac{1}{2} \\ y &= \frac{1}{2}x + \frac{5}{2}. \end{aligned}$$



Tangent line to the parabola described by the given parametric equations when $t = 2.$

Note:
Exercise:

Problem: Find the equation of the tangent line to the curve defined by the equations
Equation:

$$x\left(t\right) = t^2 - 4t, \quad y\left(t\right) = 2t^3 - 6t, \quad -2 \le t \le 3 \text{ when } t = 5.$$

Solution:

The equation of the tangent line is $y = 24x + 100.$

Hint

Calculate $x'\left(t\right)$ and $y'\left(t\right)$ and use [link].

Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f\left(x\right)$ is defined to be the derivative of the first derivative; that is,
Equation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right].$$

Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, we can replace the y on both sides of this equation with $\frac{dy}{dx}$. This gives us

Equation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)\,(dy/dx)}{dx/dt}.$$

If we know dy/dx as a function of t, then this formula is straightforward to apply.

Example:

Exercise:

Problem:

Finding a Second Derivative

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the parametric equations $x\,(t) = t^2 - 3, y\,(t) = 2t - 1, -3 \le t \le 4$.

Solution:

From [link] we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using [link], we obtain

Equation:

$$\frac{d^2y}{dx^2} = \frac{(d/dt)\,(dy/dx)}{dx/dt} = \frac{(d/dt)\,(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

Note:

Exercise:

Problem: Calculate the second derivative d^2y/dx^2 for the plane curve defined by the equations

Equation:

$$x\,(t) = t^2 - 4t, \quad y\,(t) = 2t^3 - 6t, \quad -2 \le t \le 3$$

and locate any critical points on its graph.

Solution:

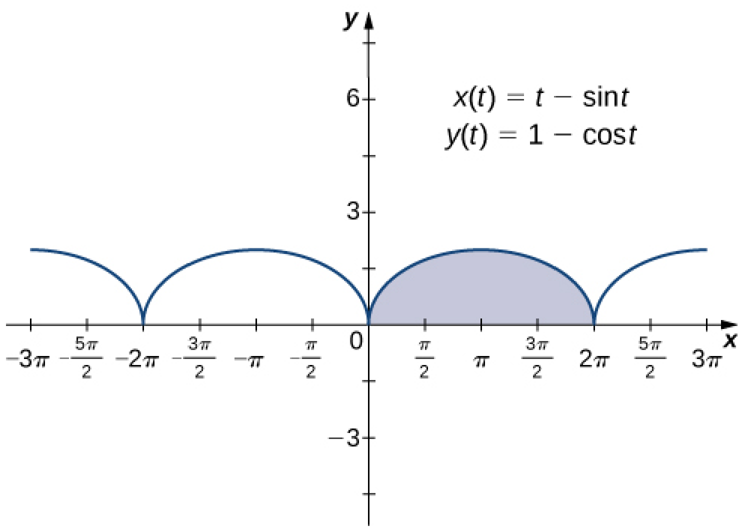
$\frac{d^2y}{dx^2} = \frac{3t^2-12t+3}{2(t-2)^3}$. Critical points $(5, 4)$, $(-3, -4)$, and $(-4, 6)$.

Hint

Start with the solution from the previous checkpoint, and use [link].

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by the equations $x(t) = t - \sin t, \quad y(t) = 1 - \cos t$. Suppose we want to find the area of the shaded region in the following graph.

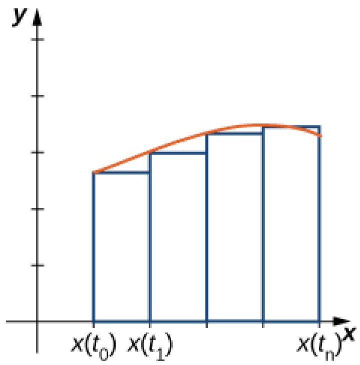


Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions
Equation:

$$x = x(t), \quad y = y(t), \quad a \le t \le b,$$

we assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \le t \le b$. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and consider the following graph.



Approximating the area under a parametrically defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y\left(x\left(\bar{t}_i\right)\right)$ for some value $\bar{t}_i$ in the ith subinterval, and the width can be calculated as $x\left(t_i\right) - x\left(t_{i-1}\right)$. Thus the area of the ith rectangle is given by

Equation:

$$A_i = y\left(x\left(\bar{t}_i\right)\right)\ \left(x\left(t_i\right) - x\left(t_{i-1}\right)\right).$$

Then a Riemann sum for the area is

Equation:

$$A_n = \sum_{i=1}^{n} y\left(x\left(\bar{t}_i\right)\right)\ \left(x\left(t_i\right) - x\left(t_{i-1}\right)\right).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

Equation:

$$A_n = \sum_{i=1}^{n} y\left(x\left(\bar{t}_i\right)\right)\ \left(\frac{x\left(t_i\right) - x\left(t_{i-1}\right)}{t_i - t_{i-1}}\right)\left(t_i - t_{i-1}\right) = \sum_{i=1}^{n} y\left(x\left(\bar{t}_i\right)\right)\ \left(\frac{x\left(t_i\right) - x\left(t_{i-1}\right)}{\Delta t}\right)\Delta t.$$

Taking the limit as n approaches infinity gives

Equation:

$$A = \lim_{n\to\infty} A_n = \int_a^b y\left(t\right)x'\left(t\right)dt.$$

This leads to the following theorem.

Note:
Area under a Parametric Curve
Consider the non-self-intersecting plane curve defined by the parametric equations
Equation:

$$x = x\,(t), \quad y = y\,(t), \quad a \le t \le b$$

and assume that $x\,(t)$ is differentiable. The area under this curve is given by
Equation:

$$A = \int_a^b y\,(t) x'\,(t)\,dt.$$

Example:
Exercise:

Problem:
Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations
Equation:

$$x\,(t) = t - \sin t, \quad y\,(t) = 1 - \cos t, \quad 0 \le t \le 2\pi.$$

Solution:

Using [link], we have
Equation:

$$\begin{aligned} A &= \int_a^b y\,(t) x'\,(t)\,dt \\ &= \int_0^{2\pi} (1 - \cos t)\,(1 - \cos t)\,dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right) dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos 2t}{2}\right) dt \\ &= \tfrac{3t}{2} - 2\sin t + \tfrac{\sin 2t}{4} \Big|_0^{2\pi} \\ &= 3\pi. \end{aligned}$$

Note:
Exercise:

Problem: Find the area under the curve of the hypocycloid defined by the equations
Equation:

$$x\left(t\right)=3\cos t+\cos 3t,\quad y\left(t\right)=3\sin t-\sin 3t,\quad 0\le t\le \pi .$$

Solution:

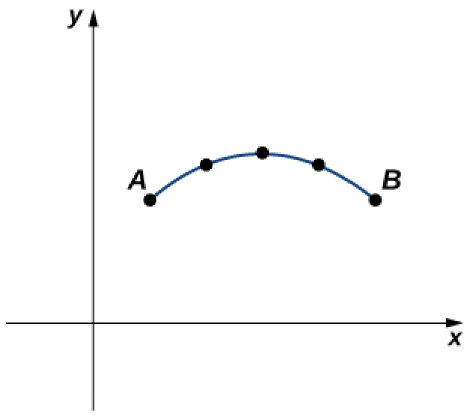
$A=3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x\left(t\right)$ is a decreasing function over the interval $\left[0,2\pi \right]$; that is, the curve is traced from right to left.)

Hint

Use [link], along with the identities $\sin \alpha \sin \beta =\frac{1}{2}\left[\cos\left(\alpha -\beta \right)-\cos\left(\alpha +\beta \right)\right]$ and $\sin^2 t=\frac{1-\cos 2t}{2}$.

Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point *A* to point *B* along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.



Approximation of a curve by line segments.

Given a plane curve defined by the functions $x = x(t), y = y(t), a \le t \le b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b$. The width of each subinterval is given by $\Delta t = (b - a)/n$. We can calculate the length of each line segment:

Equation:

$$\begin{aligned} d_1 &= \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2} \\ d_2 &= \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2} \text{ etc.} \end{aligned}$$

Then add these up. We let *s* denote the exact arc length and s_n denote the approximation by *n* line segments:

Equation:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}.$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of *t*, then the Mean Value Theorem (Introduction to the Applications of Derivatives) applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t}_k$ and $\tilde{t}_k$ such that

Equation:

$$\begin{aligned} x(t_k) - x(t_{k-1}) &= x'\left(\hat{t}_k\right)(t_k - t_{k-1}) = x'\left(\hat{t}_k\right)\Delta t \\ y(t_k) - y(t_{k-1}) &= y'\left(\tilde{t}_k\right)(t_k - t_{k-1}) = y'\left(\tilde{t}_k\right)\Delta t. \end{aligned}$$

Therefore [link] becomes

Equation:

$$\begin{aligned} s &\approx \sum_{k=1}^{n} s_k \\ &= \sum_{k=1}^{n} \sqrt{\left(x'\left(\hat{t}_k\right)\Delta t\right)^2 + \left(y'\left(\tilde{t}_k\right)\Delta t\right)^2} \\ &= \sum_{k=1}^{n} \sqrt{\left(x'\left(\hat{t}_k\right)\right)^2 (\Delta t)^2 + \left(y'\left(\tilde{t}_k\right)\right)^2 (\Delta t)^2} \\ &= \left(\sum_{k=1}^{n} \sqrt{\left(x'\left(\hat{t}_k\right)\right)^2 + \left(y'\left(\tilde{t}_k\right)\right)^2}\right)\Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

Equation:

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \sqrt{\left(x'\left(\hat{t}_k\right)\right)^2 + \left(y'\left(\tilde{t}_k\right)\right)^2} \right) \Delta t \\ &= \int_a^b \sqrt{\left(x'\left(t\right)\right)^2 + \left(y'\left(t\right)\right)^2} dt. \end{aligned}$$

When taking the limit, the values of $\hat{t}_k$ and $\tilde{t}_k$ are both contained within the same ever-shrinking interval of width Δt, so they must converge to the same value.

We can summarize this method in the following theorem.

Note:
Arc Length of a Parametric Curve
Consider the plane curve defined by the parametric equations
Equation:

$$x = x\left(t\right), \quad y = y\left(t\right), \quad t_1 \le t \le t_2$$

and assume that $x\left(t\right)$ and $y\left(t\right)$ are differentiable functions of *t*. Then the arc length of this curve is given by
Equation:

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F\left(x\right)$. Then $y\left(t\right) = F\left(x\left(t\right)\right)$ and the Chain Rule gives $y'\left(t\right) = F'\left(x\left(t\right)\right)x'\left(t\right)$. Substituting this into [link] gives
Equation:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + (F'(x))^2\right)}\, dt \\ &= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\, dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula
Equation:

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is the formula for arc length obtained in the Introduction to the Applications of Integration.

Example:
Exercise:

Problem:
Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations
Equation:

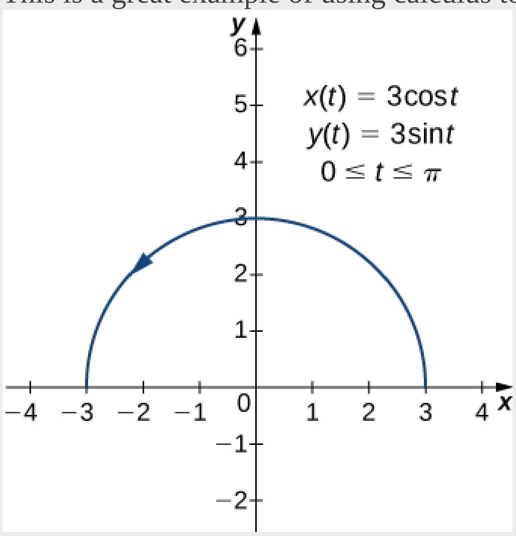
$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad 0 \le t \le \pi.$$

Solution:

The values $t = 0$ to $t = \pi$ trace out the red curve in [link]. To determine its length, use [link]:
Equation:

$$
\begin{aligned}
s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\
&= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\
&= \int_0^{\pi} \sqrt{9\left(\sin^2 t + \cos^2 t\right)}\, dt \\
&= \int_0^{\pi} 3dt = 3t|_0^{\pi} = 3\pi.
\end{aligned}
$$

Note that the formula for the arc length of a semicircle is πr and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.



The arc length of the semicircle is equal to its radius times π.

Note:
Exercise:

Problem: Find the arc length of the curve defined by the equations
Equation:

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad 1 \le t \le 3.$$

Solution:

$s = 2\left(10^{3/2} - 2^{3/2}\right) \approx 57.589$

Hint

Use [link].

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive *x*-axis, the parametric equations for this curve can be written as

Equation:

$$x\left(t\right) = 140t, \quad y\left(t\right) = -16t^2 + 2t$$

where *t* represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x\left(t\right)$ and $y\left(t\right)$ using *v* as an independent variable, so as to eliminate any confusion with the parameter *t*:

Equation:

$$x\left(v\right) = 140v, \quad y\left(v\right) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

Equation:

$$\begin{aligned} s\left(t\right) &= \int_0^t \sqrt{\left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2} dv \\ &= \int_0^t \sqrt{140^2 + \left(-32v + 2\right)^2} dv. \end{aligned}$$

The variable *v* acts as a dummy variable that disappears after integration, leaving the arc length as a function of time *t*. To integrate this expression we can use a formula from Appendix A,

Equation:

$$\int \sqrt{a^2 + u^2} du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2} \ln \left| u + \sqrt{a^2 + u^2} \right| + C.$$

We set $a = 140$ and $u = -32v + 2$. This gives $du = -32dv$, so $dv = -\frac{1}{32}du$. Therefore

Equation:

$$
\begin{aligned}
\int \sqrt{140^2 + (-32v+2)^2} dv &= -\tfrac{1}{32} \int \sqrt{a^2+u^2} du \\
&= -\tfrac{1}{32} \left[\begin{array}{l} \frac{(-32v+2)}{2} \sqrt{140^2 + (-32v+2)^2} \\ + \frac{140^2}{2} \ln \left| (-32v+2) + \sqrt{140^2 + (-32v+2)^2} \right| \end{array} \right] + C
\end{aligned}
$$

and

Equation:

$$
\begin{aligned}
s(t) &= -\tfrac{1}{32} \left[\tfrac{(-32t+2)}{2} \sqrt{140^2 + (-32t+2)^2} + \tfrac{140^2}{2} \ln \left| (-32t+2) + \sqrt{140^2 + (-32t+2)^2} \right| \right] \\
&\quad + \tfrac{1}{32} \left[\sqrt{140^2 + 2^2} + \tfrac{140^2}{2} \ln \left| 2 + \sqrt{140^2 + 2^2} \right| \right] \\
&= \left(\tfrac{t}{2} - \tfrac{1}{32} \right) \sqrt{1024t^2 - 128t + 19604} - \tfrac{1225}{4} \ln \left| (-32t+2) + \sqrt{1024t^2 - 128t + 19604} \right| \\
&\quad + \tfrac{\sqrt{19604}}{32} + \tfrac{1225}{4} \ln \left(2 + \sqrt{19604} \right).
\end{aligned}
$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to *t*. While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

Equation:

$$
\frac{d}{dx} \int_a^x f(u)\, du = f(x).
$$

Therefore

Equation:

$$
\begin{aligned}
s'(t) &= \tfrac{d}{dt} [s(t)] \\
&= \tfrac{d}{dt} \left[\int_0^t \sqrt{140^2 + (-32v+2)^2} dv \right] \\
&= \sqrt{140^2 + (-32t+2)^2} \\
&= \sqrt{1024t^2 - 128t + 19604} \\
&= 2\sqrt{256t^2 - 32t + 4901}.
\end{aligned}
$$

One third of a second after the ball leaves the pitcher’s hand, the distance it travels is equal to

Equation:

$$\begin{aligned} s\left(\tfrac{1}{3}\right) &= \left(\tfrac{1/3}{2} - \tfrac{1}{32}\right)\sqrt{1024\left(\tfrac{1}{3}\right)^2 - 128\left(\tfrac{1}{3}\right) + 19604} \\ &\quad -\tfrac{1225}{4}\ln\left|\left(-32\left(\tfrac{1}{3}\right) + 2\right) + \sqrt{1024\left(\tfrac{1}{3}\right)^2 - 128\left(\tfrac{1}{3}\right) + 19604}\right| \\ &\quad +\tfrac{\sqrt{19604}}{32} + \tfrac{1225}{4}\ln\left(2 + \sqrt{19604}\right) \\ &\approx 46.69\,\text{feet.} \end{aligned}$$

This value is just over three quarters of the way to home plate. The speed of the ball is
Equation:

$$s'\left(\frac{1}{3}\right) = 2\sqrt{256\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 4901} \approx 140.34\,\text{ft/s}.$$

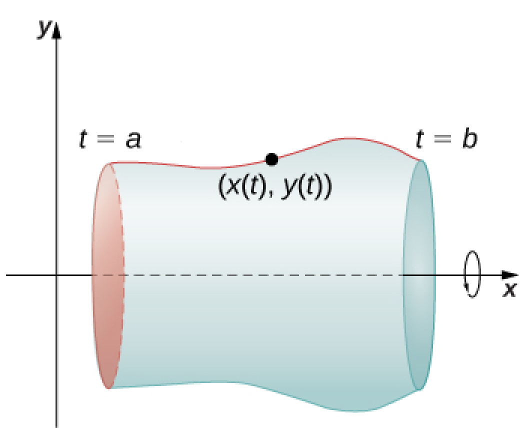
This speed translates to approximately 95 mph—a major-league fastball.

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In Curve Length and Surface Area, we derived a formula for finding the surface area of a volume generated by a function $y = f(x)$ from $x = a$ to $x = b$, revolved around the *x*-axis:
Equation:

$$S = 2\pi\int_a^b f(x)\sqrt{1 + \left(f'(x)\right)^2}dx.$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x = x(t), y = y(t), a \le t \le b$ around the *x*-axis as shown in the following figure.



A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is
Equation:

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

provided that $y(t)$ is not negative on $[a, b]$.

Example:
Exercise:

Problem:
Finding Surface Area

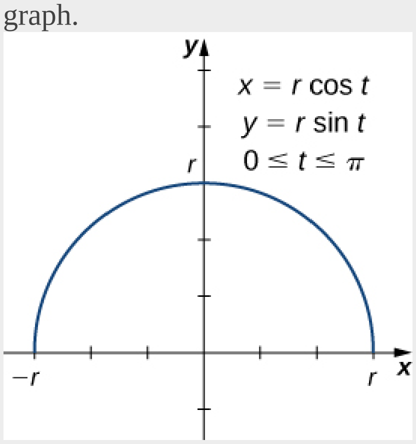
Find the surface area of a sphere of radius *r* centered at the origin.

Solution:

We start with the curve defined by the equations
Equation:

$$x(t) = r\cos t, \quad y(t) = r\sin t, \quad 0 \le t \le \pi.$$

This generates an upper semicircle of radius *r* centered at the origin as shown in the following graph.



A semicircle generated by parametric equations.

When this curve is revolved around the *x*-axis, it generates a sphere of radius *r*. To calculate the surface area of the sphere, we use [link]:

Equation:

$$\begin{aligned} S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}dt \\ &= 2\pi \int_0^\pi r^2 \sin t\, dt \\ &= 2\pi r^2 (-\cos t|_0^\pi) \\ &= 2\pi r^2 \left(-\cos\pi + \cos 0\right) \\ &= 4\pi r^2. \end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.

Note:
Exercise:

Problem: Find the surface area generated when the plane curve defined by the equations

Equation:

$$x(t) = t^3, \quad y(t) = t^2, \quad 0 \le t \le 1$$

is revolved around the *x*-axis.

Solution:

$A = \frac{\pi\left(494\sqrt{13}+128\right)}{1215}$

Hint

Use [link]. When evaluating the integral, use a *u*-substitution.

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- The area between a parametric curve and the *x*-axis can be determined by using the formula $A = \int_{t_1}^{t_2} y(t) x'(t)\, dt$.
- The arc length of a parametric curve can be calculated by using the formula $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.
- The surface area of a volume of revolution revolved around the *x*-axis is given by $S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$. If the curve is revolved around the *y*-axis, then the formula is $S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Key Equations

- **Derivative of parametric equations**
  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$
- **Second-order derivative of parametric equations**
  $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}$
- **Area under a parametric curve**
  $A = \int_a^b y(t) x'(t)\, dt$
- **Arc length of a parametric curve**
  $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
- **Surface area generated by a parametric curve**
  $S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

For the following exercises, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

Exercise:

Problem: $x = 3 + t, \quad y = 1 - t$

Exercise:

Problem: $x = 8 + 2t, \quad y = 1$

Solution:

0

Exercise:

Problem: $x = 4 - 3t, \quad y = -2 + 6t$

Exercise:

Problem: $x = -5t + 7, \quad y = 3t - 1$

Solution:

$\frac{-3}{5}$

For the following exercises, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

Exercise:

Problem: $x = 3\sin t, \quad y = 3\cos t, \quad t = \frac{\pi}{4}$

Exercise:

Problem: $x = \cos t, \quad y = 8\sin t, t = \frac{\pi}{2}$

Solution:

Slope $= 0$; $y = 8$.

Exercise:

Problem: $x = 2t, \quad y = t^3, \quad t = -1$

Exercise:

Problem: $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t = 1$

Solution:

Slope is undefined; $x = 2$.

Exercise:

Problem: $x = \sqrt{t}, \quad y = 2t, \quad t = 4$

For the following exercises, find all points on the curve that have the given slope.

Exercise:

Problem: $x = 4\cos t, \quad y = 4\sin t$, slope = 0.5

Solution:

$t = \arctan(-2)$; $\left(\frac{4}{\sqrt{5}}, \frac{-8}{\sqrt{5}}\right)$.

Exercise:

Problem: $x = 2\cos t, \quad y = 8\sin t, \text{ slope} = -1$

Exercise:

Problem: $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \text{ slope} = 1$

Solution:

No points possible; undefined expression.

Exercise:

Problem: $x = 2 + \sqrt{t}, \quad y = 2 - 4t, \text{ slope} = 0$

For the following exercises, write the equation of the tangent line in Cartesian coordinates for the given parameter *t*.

Exercise:

Problem: $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad t = 1$

Solution:

$y = -\left(\frac{2}{e}\right)x + 3$

Exercise:

Problem: $x = t\ln t, \quad y = \sin^2 t, t = \frac{\pi}{4}$

Exercise:

Problem: $x = e^t, \quad y = (t-1)^2, \quad \text{at}(1,1)$

Solution:

$y = 2x - 7$

Exercise:

Problem:

For $x = \sin(2t), y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of *t* at which a horizontal tangent line exists.

Exercise:

Problem:

For $x = \sin(2t), y = 2\sin t$ where $0 \leq t < 2\pi$. Find all values of t at which a vertical tangent line exists.

Solution:

$\frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

Exercise:

Problem: Find all points on the curve $x = 4\cos(t), y = 4\sin(t)$ that have the slope of $\frac{1}{2}$.

Exercise:

Problem: Find $\frac{dy}{dx}$ for $x = \sin(t), y = \cos(t)$.

Solution:

$\frac{dy}{dx} = -\tan(t)$

Exercise:

Problem: Find the equation of the tangent line to $x = \sin(t), y = \cos(t)$ at $t = \frac{\pi}{4}$.

Exercise:

Problem: For the curve $x = 4t, y = 3t - 2$, find the slope and concavity of the curve at $t = 3$.

Solution:

$\frac{dy}{dx} = \frac{3}{4}$ and $\frac{d^2y}{dx^2} = 0$, so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

Exercise:

Problem:

For the parametric curve whose equation is $x = 4\cos\theta, y = 4\sin\theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

Exercise:

Problem:

Find the slope and concavity for the curve whose equation is $x = 2 + \sec\theta, y = 1 + 2\tan\theta$ at $\theta = \frac{\pi}{6}$.

Solution:

$\frac{dy}{dx} = 4$, $\frac{d^2y}{dx^2} = -6\sqrt{3}$; the curve is concave down at $\theta = \frac{\pi}{6}$.

Exercise:

Problem:

Find all points on the curve $x = t + 4, y = t^3 - 3t$ at which there are vertical and horizontal tangents.

Exercise:

Problem:

Find all points on the curve $x = \sec\theta, y = \tan\theta$ at which horizontal and vertical tangents exist.

Solution:

No horizontal tangents. Vertical tangents at $(1, 0), (-1, 0)$.

For the following exercises, find d^2y/dx^2.

Exercise:

Problem: $x = t^4 - 1, \quad y = t - t^2$

Exercise:

Problem: $x = \sin(\pi t), \quad y = \cos(\pi t)$

Solution:

$-\sec^3(\pi t)$

Exercise:

Problem: $x = e^{-t}, \quad y = te^{2t}$

For the following exercises, find points on the curve at which tangent line is horizontal or vertical.

Exercise:

Problem: $x = t(t^2 - 3), \quad y = 3(t^2 - 3)$

Solution:

Horizontal $(0, -9)$; vertical $(\pm 2, -6)$.

Exercise:

Problem: $x = \frac{3t}{1+t^3}, \quad y = \frac{3t^2}{1+t^3}$

For the following exercises, find dy/dx at the value of the parameter.

Exercise:

Problem: $x = \cos t, \quad y = \sin t, \quad t = \frac{3\pi}{4}$

Solution:

1

Exercise:

Problem: $x = \sqrt{t}, \quad y = 2t + 4, \quad t = 9$

Exercise:

Problem: $x = 4\cos(2\pi s), \quad y = 3\sin(2\pi s), \quad s = -\frac{1}{4}$

Solution:

0

For the following exercises, find d^2y/dx^2 at the given point without eliminating the parameter.

Exercise:

Problem: $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad t = 2$

Exercise:

Problem: $x = \sqrt{t}, \quad y = 2t + 4, \quad t = 1$

Solution:

4

Exercise:

Problem:

Find t intervals on which the curve $x = 3t^2, y = t^3 - t$ is concave up as well as concave down.

Exercise:

Problem: Determine the concavity of the curve $x = 2t + \ln t, y = 2t - \ln t$.

Solution:

Concave up on $t > 0$.

Exercise:

Problem:

Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$.

Exercise:

Problem:

Find the area bounded by the curve $x = \cos t, y = e^t, 0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

Solution:

1

Exercise:

Problem: Find the area enclosed by the ellipse $x = a\cos\theta, y = b\sin\theta, 0 \le \theta < 2\pi$.

Exercise:

Problem:

Find the area of the region bounded by $x = 2\sin^2\theta, y = 2\sin^2\theta\tan\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

Solution:

$\frac{3\pi}{2}$

For the following exercises, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

Exercise:

Problem: $x = 2\cot\theta, y = 2\sin^2\theta, 0 \le \theta \le \pi$

Exercise:

Problem: **[T]** $x = 2a\cos t - a\cos(2t), y = 2a\sin t - a\sin(2t), 0 \le t < 2\pi$

Solution:

$6\pi a^2$

Exercise:

Problem: **[T]** $x = a\sin(2t), y = b\sin(t), 0 \le t < 2\pi$ (the “hourglass”)

Exercise:

Problem: **[T]** $x = 2a\cos t - a\sin(2t), y = b\sin t, 0 \le t < 2\pi$ (the “teardrop”)

Solution:

$2\pi ab$

For the following exercises, find the arc length of the curve on the indicated interval of the parameter.
Exercise:

Problem: $x = 4t + 3, \quad y = 3t - 2, \quad 0 \le t \le 2$

Exercise:

Problem: $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad 0 \le t \le 1$

Solution:

$\frac{1}{3}\left(2\sqrt{2} - 1\right)$

Exercise:

Problem: $x = \cos(2t), \quad y = \sin(2t), \quad 0 \le t \le \frac{\pi}{2}$

Exercise:

Problem: $x = 1 + t^2, \quad y = (1 + t)^3, \quad 0 \le t \le 1$

Solution:

7.075

Exercise:

Problem:

$x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

Exercise:

Problem: $x = a\cos^3\theta, y = a\sin^3\theta$ on the interval $[0, 2\pi)$ (the hypocycloid)

Solution:

$6a$

Exercise:

Problem: Find the length of one arch of the cycloid $x = 4\,(t - \sin t), y = 4\,(1 - \cos t)$.

Exercise:

Problem:

Find the distance traveled by a particle with position (x, y) as t varies in the given time interval: $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \le t \le 3\pi$.

Solution:

$6\sqrt{2}$

Exercise:

Problem: Find the length of one arch of the cycloid $x = \theta - \sin\theta, y = 1 - \cos\theta$.

Exercise:

Problem:

Show that the total length of the ellipse $x = 4\sin\theta, y = 3\cos\theta$ is $L = 16\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$, where $e = \frac{c}{a}$ and $c = \sqrt{a^2 - b^2}$.

Exercise:

Problem: Find the length of the curve $x = e^t - t, y = 4e^{t/2}, -8 \le t \le 3$.

For the following exercises, find the area of the surface obtained by rotating the given curve about the *x*-axis.

Exercise:

Problem: $x = t^3, \quad y = t^2, \quad 0 \le t \le 1$

Solution:

$\frac{2\pi\left(247\sqrt{13}+64\right)}{1215}$

Exercise:

Problem: $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta \le \frac{\pi}{2}$

Exercise:

Problem:

[T] Use a CAS to find the area of the surface generated by rotating $x = t + t^3, y = t - \frac{1}{t^2}, 1 \le t \le 2$ about the *x*-axis. (Answer to three decimal places.)

Solution:

59.101

Exercise:

Problem:

Find the surface area obtained by rotating $x = 3t^2, y = 2t^3, 0 \le t \le 5$ about the *y*-axis.

Exercise:

Problem:

Find the area of the surface generated by revolving $x = t^2, y = 2t, 0 \le t \le 4$ about the *x*-axis.

Solution:

$\frac{8\pi}{3}\left(17\sqrt{17} - 1\right)$

Exercise:

Problem:

Find the surface area generated by revolving $x = t^2, y = 2t^2, 0 \le t \le 1$ about the *y*-axis.

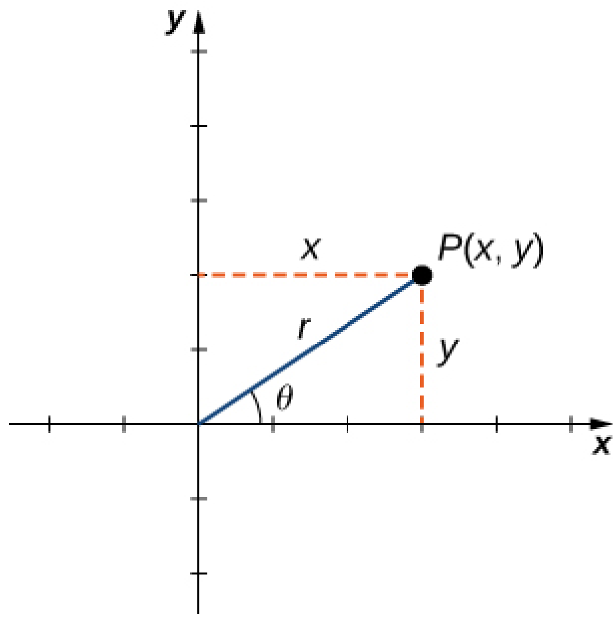
Polar Coordinates

- Locate points in a plane by using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves from given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a *one-to-one mapping* from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. In this section we see that in some circumstances, polar coordinates can be more useful than rectangular coordinates.

Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider [link]. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the **polar coordinate system**. Note that every point in the Cartesian plane has two values (hence the term *ordered pair*) associated with it. In the polar coordinate system, each point also two values associated with it: r and θ.



An arbitrary point in the Cartesian plane.

Using right-triangle trigonometry, the following equations are true for the point P:

Equation:

$$\cos\theta = \frac{x}{r} \text{ so } x = r\cos\theta$$

Equation:

$$\sin\theta = \frac{y}{r} \text{ so } y = r\sin\theta.$$

Furthermore,

Equation:

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}.$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (r, θ) in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

Note that the equation $\tan \theta = y/x$ has an infinite number of solutions for any ordered pair (x, y). However, if we restrict the solutions to values between 0 and 2π then we can assign a unique solution to the quadrant in which the original point (x, y) is located. Then the corresponding value of r is positive, so $r^2 = x^2 + y^2$.

Note:
Converting Points between Coordinate Systems
Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:
Equation:

$$x = r \cos \theta \text{ and } y = r \sin \theta,$$

Equation:

$$r^2 = x^2 + y^2 \text{ and } \tan \theta = \frac{y}{x}.$$

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates.

Example:
Exercise:

Problem:
Converting between Rectangular and Polar Coordinates

Convert each of the following points into polar coordinates.

a. $(1, 1)$
b. $(-3, 4)$
c. $(0, 3)$
d. $(5\sqrt{3}, -5)$

Convert each of the following points into rectangular coordinates.

d. $(3, \pi/3)$
e. $(2, 3\pi/2)$
f. $(6, -5\pi/6)$

Solution:

a. Use $x = 1$ and $y = 1$ in [link]:
Equation:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{1}{1} = 1 \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $\left(\sqrt{2}, \frac{\pi}{4}\right)$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in [link]:
Equation:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (-3)^2 + (4)^2 \\ r &= 5 \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -\frac{4}{3} \\ \theta &= -\arctan\left(\frac{4}{3}\right) \\ &\approx 2.21. \end{aligned}$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

c. Use $x=0$ and $y=3$ in [link]:

Equation:

$$\begin{array}{rcl} r^2 & = & x^2+y^2 \\ & = & (3)^2+(0)^2 \\ & = & 9+0 \\ r & = & 3 \end{array} \quad \text{and} \quad \begin{array}{rcl} \tan\theta & = & \frac{y}{x} \\ & = & \frac{3}{0}. \end{array}$$

Direct application of the second equation leads to division by zero. Graphing the point $(0,3)$ on the rectangular coordinate system reveals that the point is located on the positive *y*-axis. The angle between the positive *x*-axis and the positive *y*-axis is $\frac{\pi}{2}$. Therefore this point can be represented as $\left(3,\frac{\pi}{2}\right)$ in polar coordinates.

d. Use $x=5\sqrt{3}$ and $y=-5$ in [link]:

Equation:

$$\begin{array}{rcl} r^2 & = & x^2+y^2 \\ & = & \left(5\sqrt{3}\right)^2+(-5)^2 \\ & = & 75+25 \\ r & = & 10 \end{array} \quad \text{and} \quad \begin{array}{rcl} \tan\theta & = & \frac{y}{x} \\ & = & \frac{-5}{5\sqrt{3}}=-\frac{\sqrt{3}}{3} \\ \theta & = & -\frac{\pi}{6}. \end{array}$$

Therefore this point can be represented as $\left(10,-\frac{\pi}{6}\right)$ in polar coordinates.

e. Use $r=3$ and $\theta=\frac{\pi}{3}$ in [link]:

Equation:

$$\begin{array}{rcl} x & = & r\cos\theta \\ & = & 3\cos\left(\frac{\pi}{3}\right) \\ & = & 3\left(\frac{1}{2}\right)=\frac{3}{2} \end{array} \quad \text{and} \quad \begin{array}{rcl} y & = & r\sin\theta \\ & = & 3\sin\left(\frac{\pi}{3}\right) \\ & = & 3\left(\frac{\sqrt{3}}{2}\right)=\frac{3\sqrt{3}}{2}. \end{array}$$

Therefore this point can be represented as $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = \frac{3\pi}{2}$ in [link]:

Equation:

$$\begin{array}{rcl} x & = & r\cos\theta \\ & = & 2\cos\left(\frac{3\pi}{2}\right) \\ & = & 2(0) = 0 \end{array} \quad \text{and} \quad \begin{array}{rcl} y & = & r\sin\theta \\ & = & 2\sin\left(\frac{3\pi}{2}\right) \\ & = & 2(-1) = -2. \end{array}$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -\frac{5\pi}{6}$ in [link]:

Equation:

$$\begin{array}{rcl} x & = & r\cos\theta \\ & = & 6\cos\left(-\frac{5\pi}{6}\right) \\ & = & 6\left(-\frac{\sqrt{3}}{2}\right) \\ & = & -3\sqrt{3} \end{array} \quad \text{and} \quad \begin{array}{rcl} y & = & r\sin\theta \\ & = & 6\sin\left(-\frac{5\pi}{6}\right) \\ & = & 6\left(-\frac{1}{2}\right) \\ & = & -3. \end{array}$$

Therefore this point can be represented as $\left(-3\sqrt{3}, -3\right)$ in rectangular coordinates.

Note:
Exercise:

Problem:

Convert $(-8, -8)$ into polar coordinates and $\left(4, \frac{2\pi}{3}\right)$ into rectangular coordinates.

Solution:

$\left(8\sqrt{2}, \frac{5\pi}{4}\right)$ and $\left(-2, 2\sqrt{3}\right)$

Hint

Use [link] and [link]. Make sure to check the quadrant when calculating θ.

The polar representation of a point is not unique. For example, the polar coordinates $\left(2, \frac{\pi}{3}\right)$ and $\left(2, \frac{7\pi}{3}\right)$ both represent the point $\left(1, \sqrt{3}\right)$ in the rectangular system. Also, the value of r can be negative. Therefore, the point with polar coordinates $\left(-2, \frac{4\pi}{3}\right)$ also represents the point $\left(1, \sqrt{3}\right)$ in the rectangular system, as we can see by using [link]:

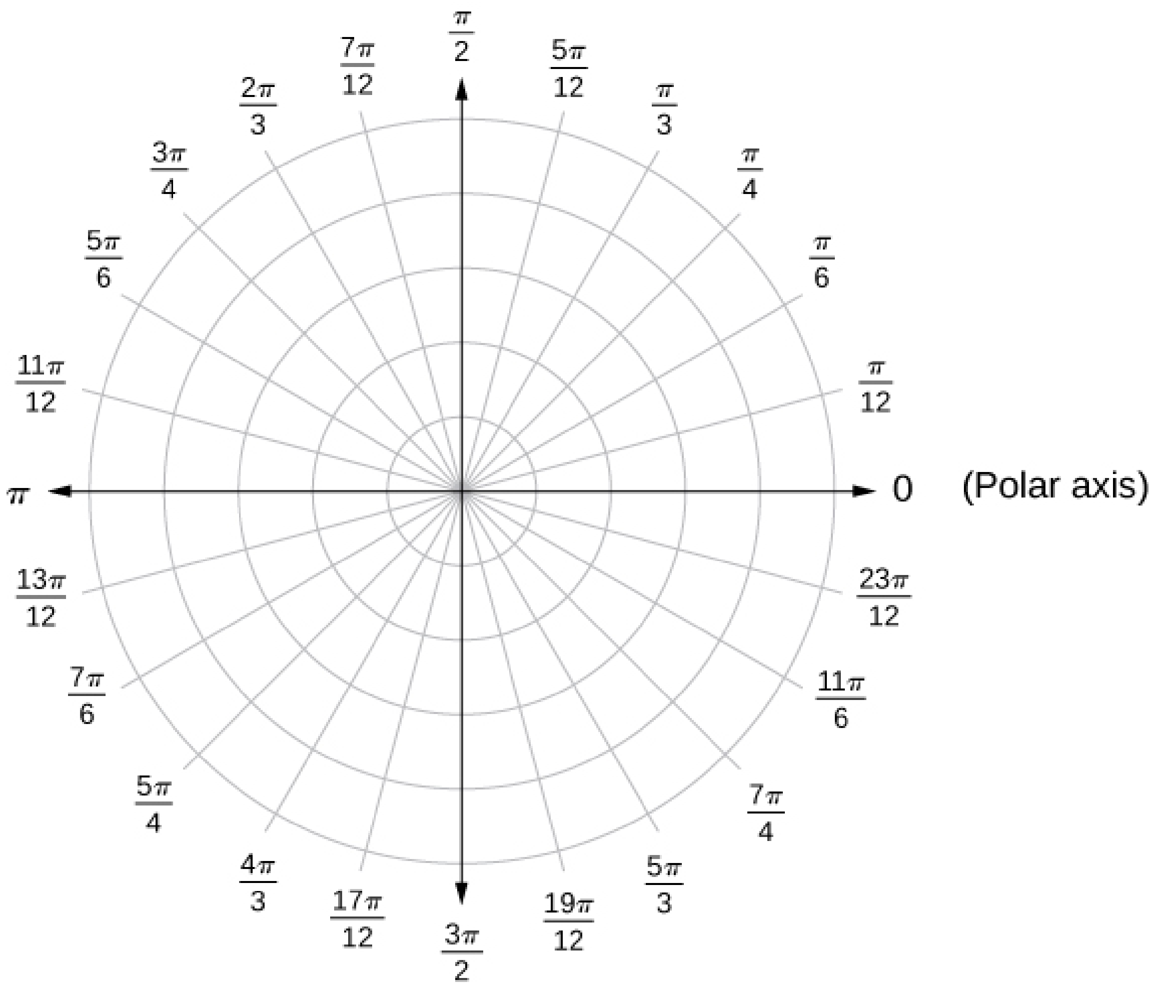
Equation:

$$\begin{aligned} x &= r\cos\theta \\ &= -2\cos\left(\frac{4\pi}{3}\right) \\ &= -2\left(-\frac{1}{2}\right) = 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &= r\sin\theta \\ &= -2\sin\left(\frac{4\pi}{3}\right) \\ &= -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}. \end{aligned}$$

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Note that the polar representation of a point in the plane also has a visual interpretation. In particular, r is the directed distance that the point lies from the origin, and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis. Positive angles are measured in a

counterclockwise direction and negative angles are measured in a clockwise direction. The polar coordinate system appears in the following figure.



The polar coordinate system.

The line segment starting from the center of the graph going to the right (called the positive *x*-axis in the Cartesian system) is the **polar axis**. The center point is the **pole**, or origin, of the coordinate system, and corresponds to $r = 0$. The innermost circle shown in [link] contains all points a distance of 1 unit from the pole, and is represented by the equation $r = 1$. Then $r = 2$ is the set of points 2 units from the pole, and so on. The line segments

emanating from the pole correspond to fixed angles. To plot a point in the polar coordinate system, start with the angle. If the angle is positive, then measure the angle from the polar axis in a counterclockwise direction. If it is negative, then measure it clockwise. If the value of r is positive, move that distance along the terminal ray of the angle. If it is negative, move along the ray that is opposite the terminal ray of the given angle.

Example:
Exercise:

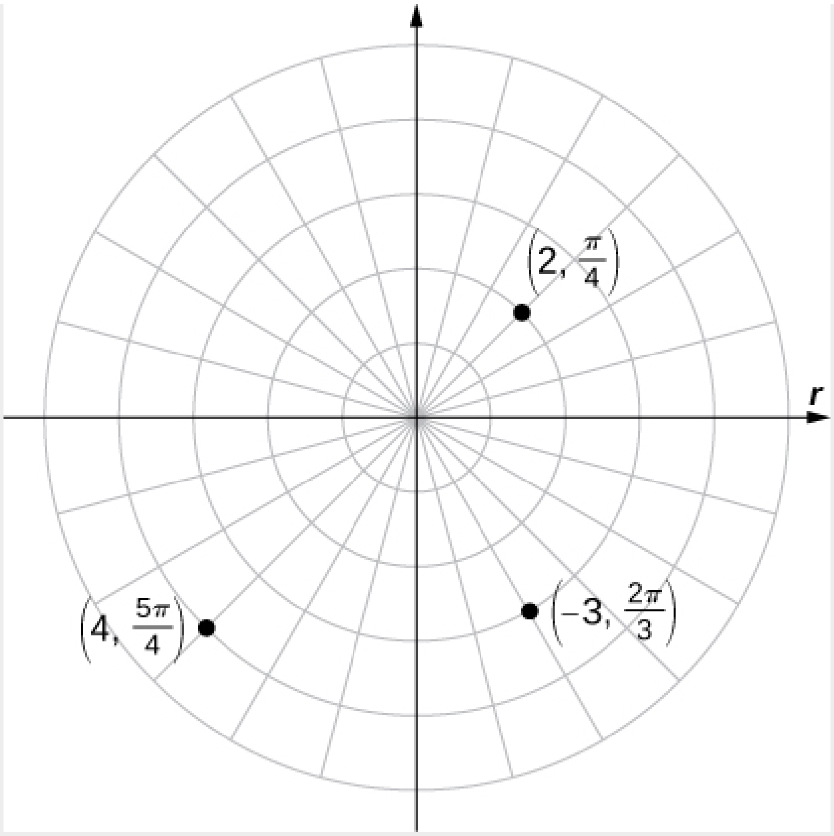
Problem:
Plotting Points in the Polar Plane

Plot each of the following points on the polar plane.

a. $\left(2, \frac{\pi}{4}\right)$
b. $\left(-3, \frac{2\pi}{3}\right)$
c. $\left(4, \frac{5\pi}{4}\right)$

Solution:

The three points are plotted in the following figure.

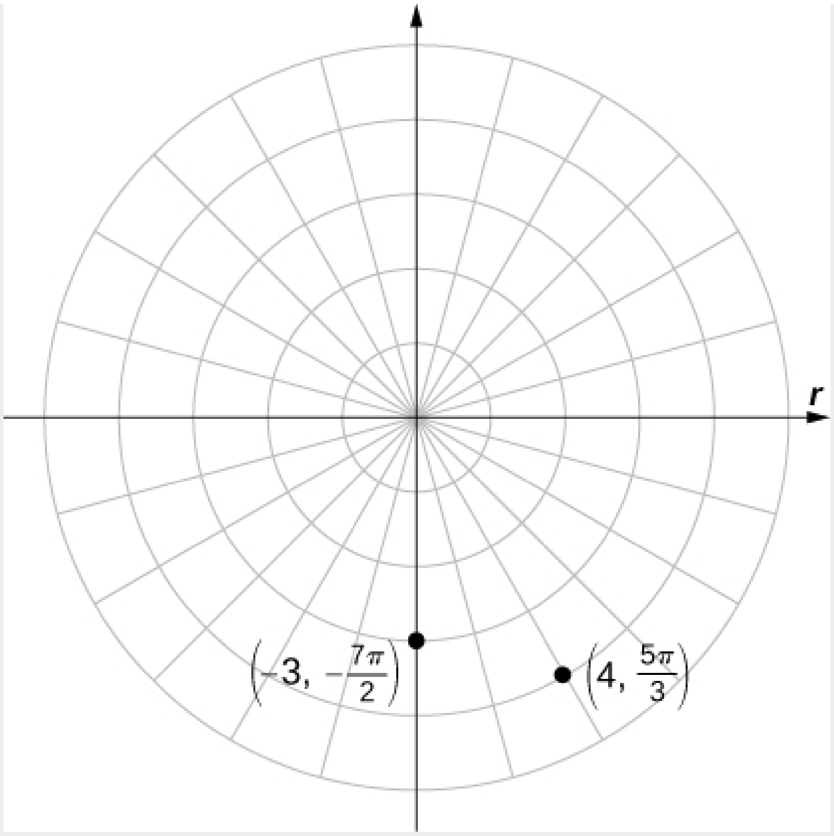


Three points plotted in the polar coordinate system.

Note:
Exercise:

Problem: Plot $\left(4, \frac{5\pi}{3}\right)$ and $\left(-3, -\frac{7\pi}{2}\right)$ on the polar plane.

Solution:



Hint

Start with θ, then use r.

Polar Curves

Now that we know how to plot points in the polar coordinate system, we can discuss how to plot curves. In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. In a similar fashion, we can graph a curve that is generated by a function $r = f(\theta)$.

The general idea behind graphing a function in polar coordinates is the same as graphing a function in rectangular coordinates. Start with a list of values for the independent variable (θ in this case) and calculate the corresponding values of the dependent variable r. This process generates a list of ordered

pairs, which can be plotted in the polar coordinate system. Finally, connect the points, and take advantage of any patterns that may appear. The function may be periodic, for example, which indicates that only a limited number of values for the independent variable are needed.

Note:
Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

1. Create a table with two columns. The first column is for θ, and the second column is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

Note:
Watch this video for more information on sketching polar curves.

Example:
Exercise:

Problem:
Graphing a Function in Polar Coordinates

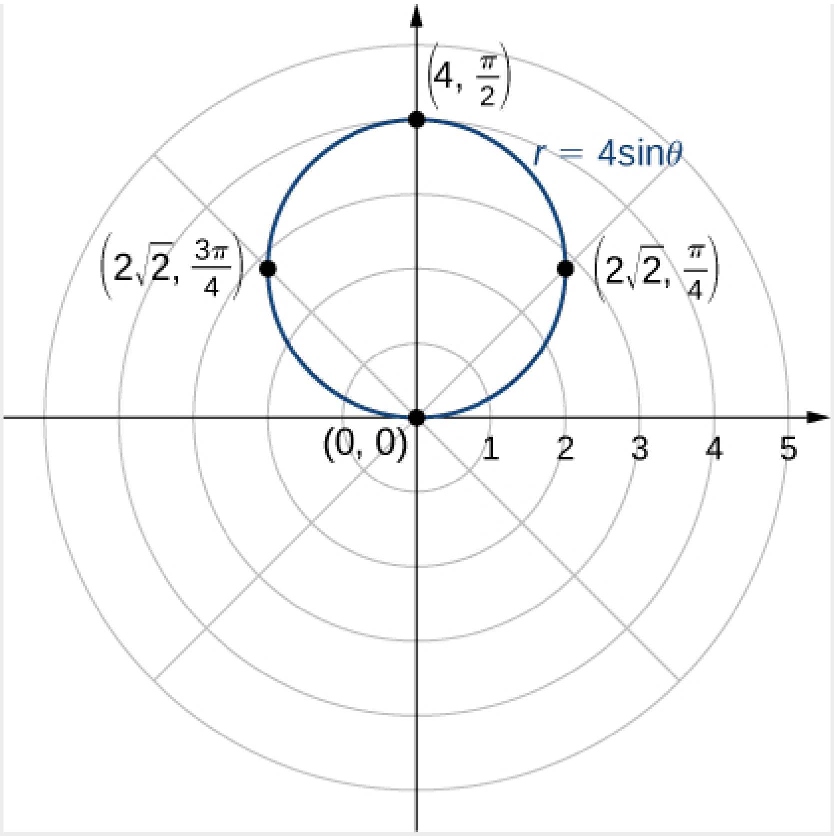
Graph the curve defined by the function $r = 4\sin\theta$. Identify the curve and rewrite the equation in rectangular coordinates.

Solution:

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π. The result of steps 1–

3 appear in the following table. [link] shows the graph based on this table.

θ	$r = 4\sin\theta$		θ	$r = 4\sin\theta$
0	0		π	0
$\frac{\pi}{6}$	2		$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$		$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$		$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4		$\frac{3\pi}{2}$	4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$		$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$		$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2		$\frac{11\pi}{6}$	-2
			2π	0



The graph of the function $r = 4\sin\theta$ is a circle.

This is the graph of a circle. The equation $r = 4\sin\theta$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2 = 4r\sin\theta$. Next use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

Equation:

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + \left(y^2 - 4y\right) &= 0 \\ x^2 + \left(y^2 - 4y + 4\right) &= 0 + 4 \\ x^2 + (y-2)^2 &= 4. \end{aligned}$$

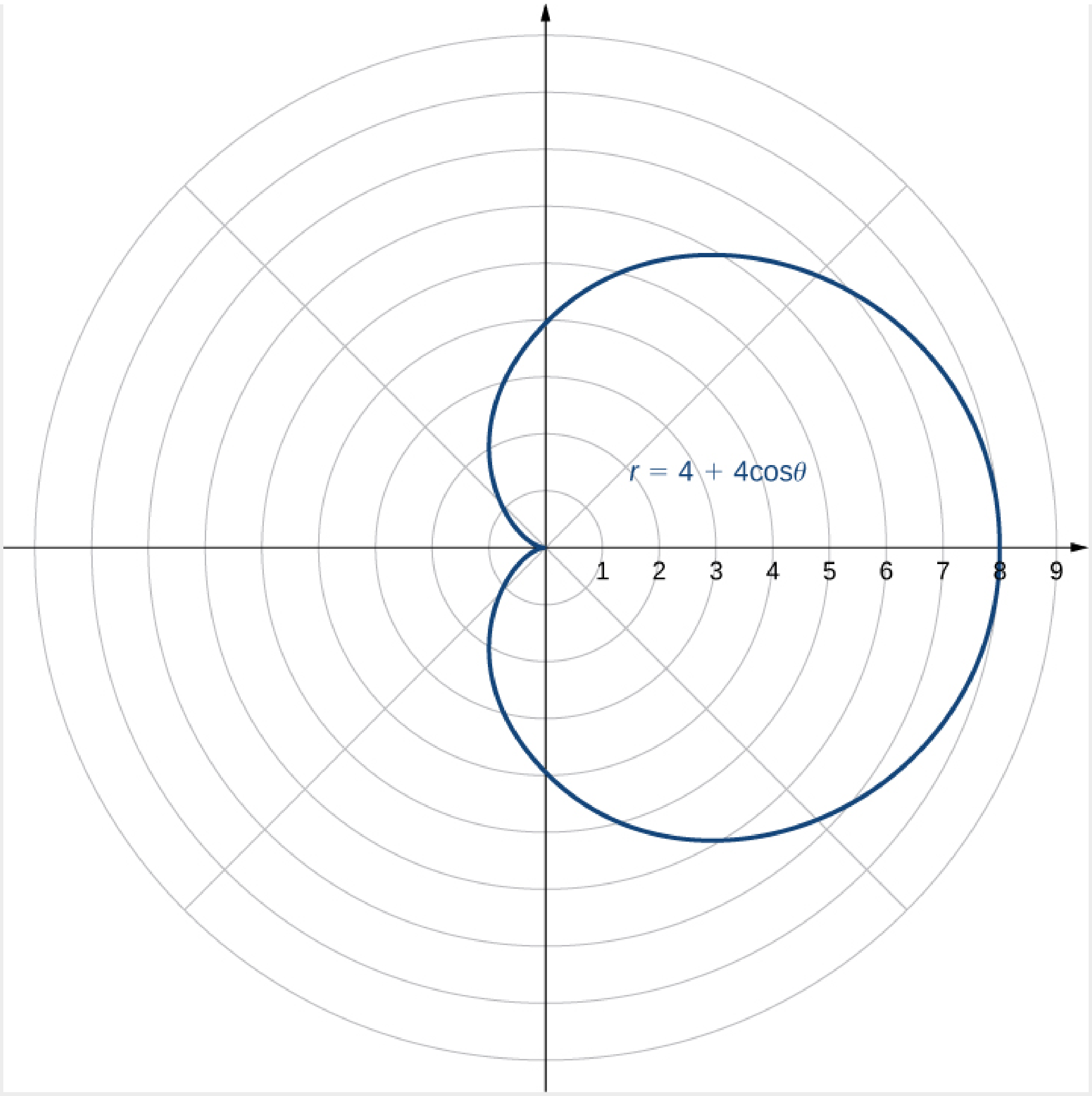
This is the equation of a circle with radius 2 and center $(0, 2)$ in the rectangular coordinate system.

Note:
Exercise:

Problem:

Create a graph of the curve defined by the function $r = 4 + 4\cos\theta$.

Solution:



The name of this shape is a cardioid, which we will study further later in this section.

Hint

Follow the problem-solving strategy for creating a graph in polar coordinates.

The graph in [link] was that of a circle. The equation of the circle can be transformed into rectangular coordinates using the coordinate transformation

formulas in [link]. [link] gives some more examples of functions for transforming from polar to rectangular coordinates.

Example:
Exercise:

Problem:
Transforming Polar Equations to Rectangular Coordinates

Rewrite each of the following equations in rectangular coordinates and identify the graph.

a. $\theta = \frac{\pi}{3}$
b. $r = 3$
c. $r = 6\cos\theta - 8\sin\theta$

Solution:

a. Take the tangent of both sides. This gives $\tan\theta = \tan(\pi/3) = \sqrt{3}$. Since $\tan\theta = y/x$ we can replace the left-hand side of this equation by y/x. This gives $y/x = \sqrt{3}$, which can be rewritten as $y = x\sqrt{3}$. This is the equation of a straight line passing through the origin with slope $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan K$.
b. First, square both sides of the equation. This gives $r^2 = 9$. Next replace r^2 with $x^2 + y^2$. This gives the equation $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3. In general, any polar equation of the form $r = k$ where k is a positive constant represents a circle of radius k centered at the origin. (*Note*: when squaring both sides of an equation it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case we do

not introduce new points. For example, $\left(-3, \frac{\pi}{3}\right)$ is the same point as $\left(3, \frac{4\pi}{3}\right)$.)

c. Multiply both sides of the equation by r. This leads to $r^2 = 6r\cos\theta - 8r\sin\theta$. Next use the formulas

Equation:

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

This gives

Equation:

$$\begin{aligned} r^2 &= 6\left(r\cos\theta\right) - 8\left(r\sin\theta\right) \\ x^2 + y^2 &= 6x - 8y. \end{aligned}$$

To put this equation into standard form, first move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

Equation:

$$\begin{aligned} x^2 + y^2 &= 6x - 8y \\ x^2 - 6x + y^2 + 8y &= 0 \\ \left(x^2 - 6x\right) + \left(y^2 + 8y\right) &= 0 \\ \left(x^2 - 6x + 9\right) + \left(y^2 + 8y + 16\right) &= 9 + 16 \\ (x-3)^2 + (y+4)^2 &= 25. \end{aligned}$$

This is the equation of a circle with center at $(3, -4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

Note:
Exercise:

Problem:

Rewrite the equation $r = \sec\theta \tan\theta$ in rectangular coordinates and identify its graph.

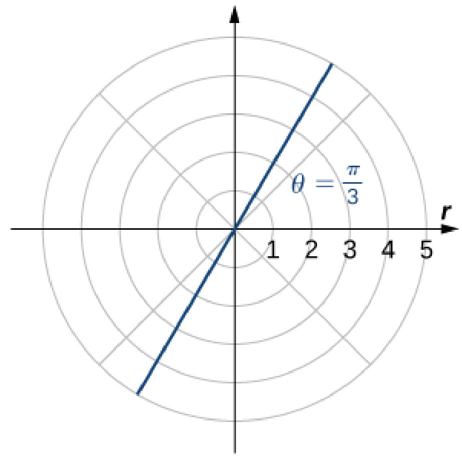
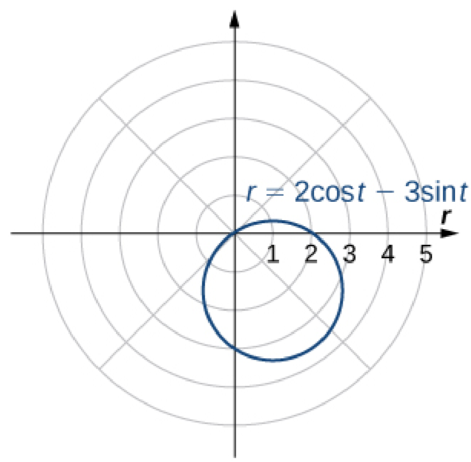
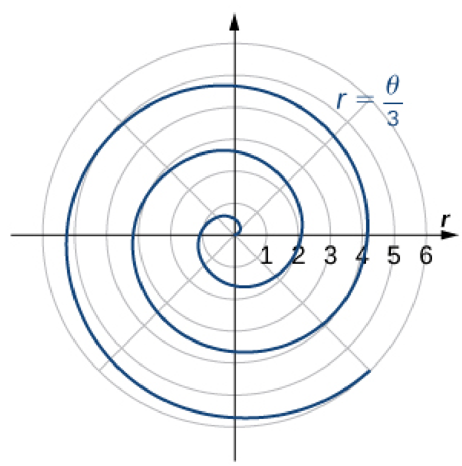
Solution:

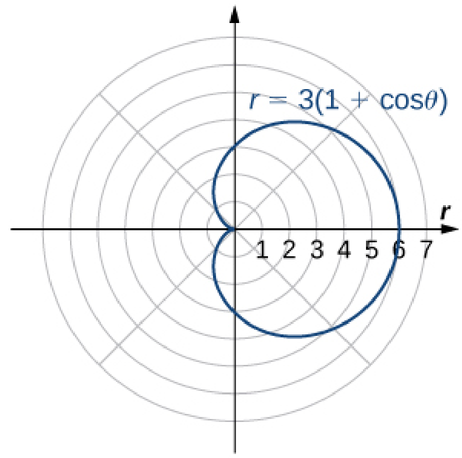
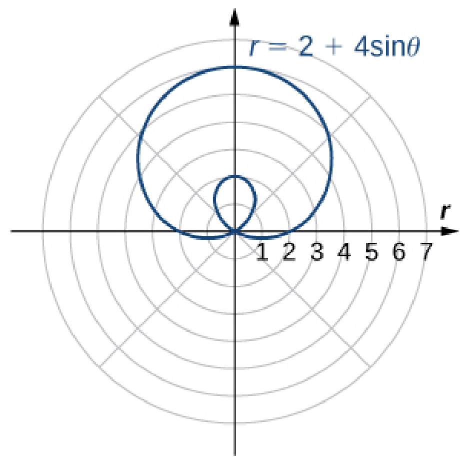
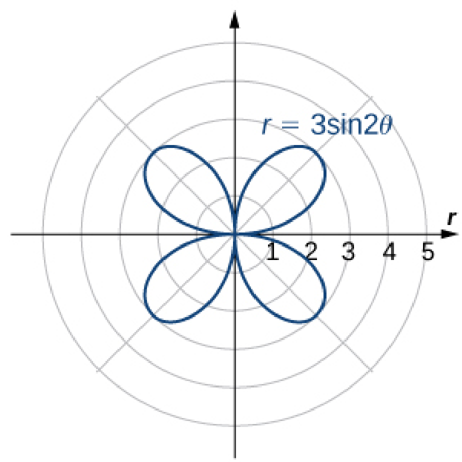
$y = x^2$, which is the equation of a parabola opening upward.

Hint

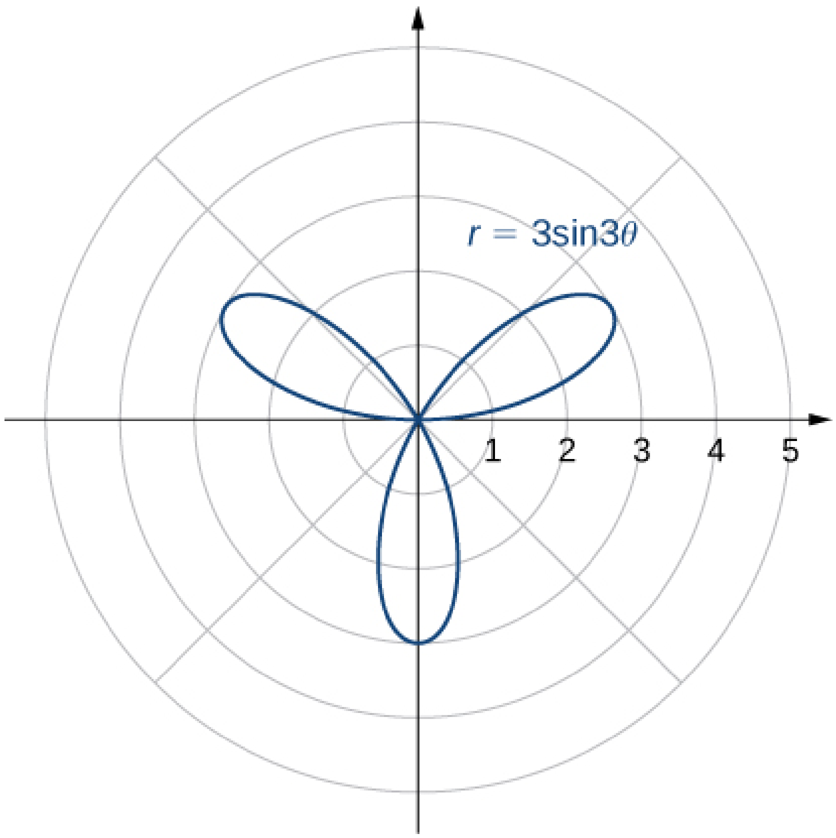
Convert to sine and cosine, then multiply both sides by cosine.

We have now seen several examples of drawing graphs of curves defined by **polar equations**. A summary of some common curves is given in the tables below. In each equation, *a* and *b* are arbitrary constants.

Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

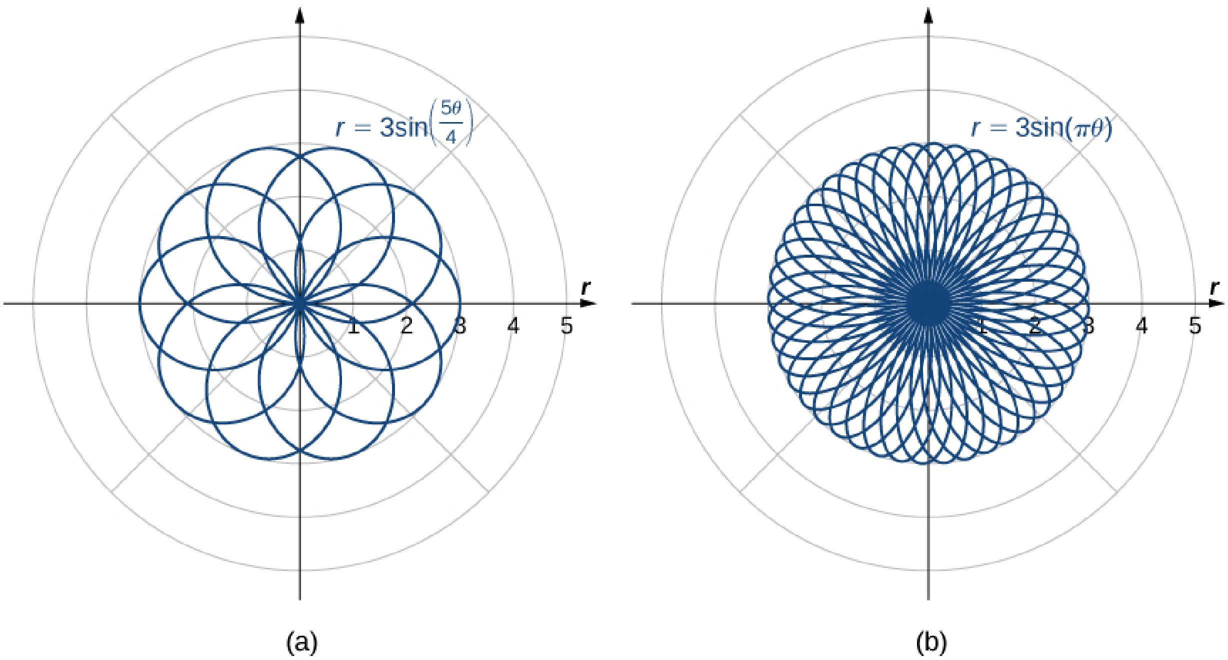
A **cardioid** is a special case of a **limaçon** (pronounced “lee-mah-son”), in which $a=b$ or $a=-b$. The **rose** is a very interesting curve. Notice that the graph of $r=3\sin 2\theta$ has four petals. However, the graph of $r=3\sin 3\theta$ has three petals as shown.



Graph of $r=3\sin 3\theta$.

If the coefficient of θ is even, the graph has twice as many petals as the coefficient. If the coefficient of θ is odd, then the number of petals equals the coefficient. You are encouraged to explore why this happens. Even more interesting graphs emerge when the coefficient of θ is not an integer. For example, if it is rational, then the curve is closed; that is, it eventually ends where it started ([link](a)). However, if the coefficient is irrational, then the curve never closes ([link](b)). Although it may appear that the curve is

closed, a closer examination reveals that the petals just above the positive *x* axis are slightly thicker. This is because the petal does not quite match up with the starting point.



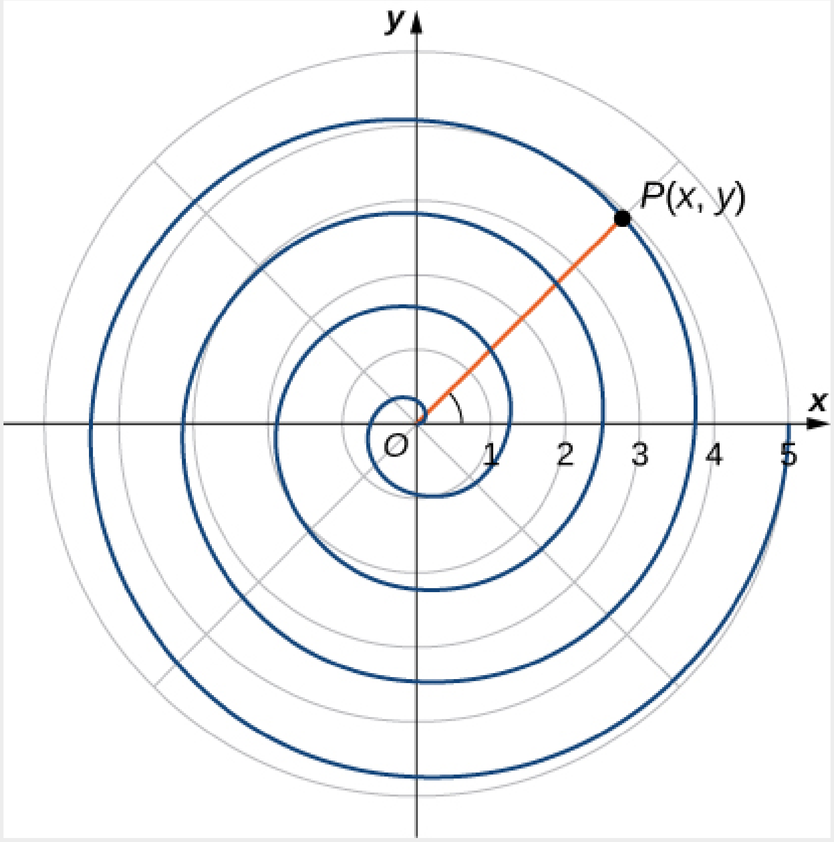
Polar rose graphs of functions with (a) rational coefficient and (b) irrational coefficient. Note that the rose in part (b) would actually fill the entire circle if plotted in full.

Since the curve defined by the graph of $r = 3\sin(\pi\theta)$ never closes, the curve depicted in [link](b) is only a partial depiction. In fact, this is an example of a **space-filling curve**. A space-filling curve is one that in fact occupies a two-dimensional subset of the real plane. In this case the curve occupies the circle of radius 3 centered at the origin.

Example:
Exercise:

Problem:
Chapter Opener: Describing a Spiral

Recall the chambered nautilus introduced in the chapter opener. This creature displays a spiral when half the outer shell is cut away. It is possible to describe a spiral using rectangular coordinates. [link] shows a spiral in rectangular coordinates. How can we describe this curve mathematically?



How can we describe a spiral graph mathematically?

Solution:

As the point P travels around the spiral in a counterclockwise direction, its distance d from the origin increases. Assume that the distance d is a constant multiple k of the angle θ that the line segment

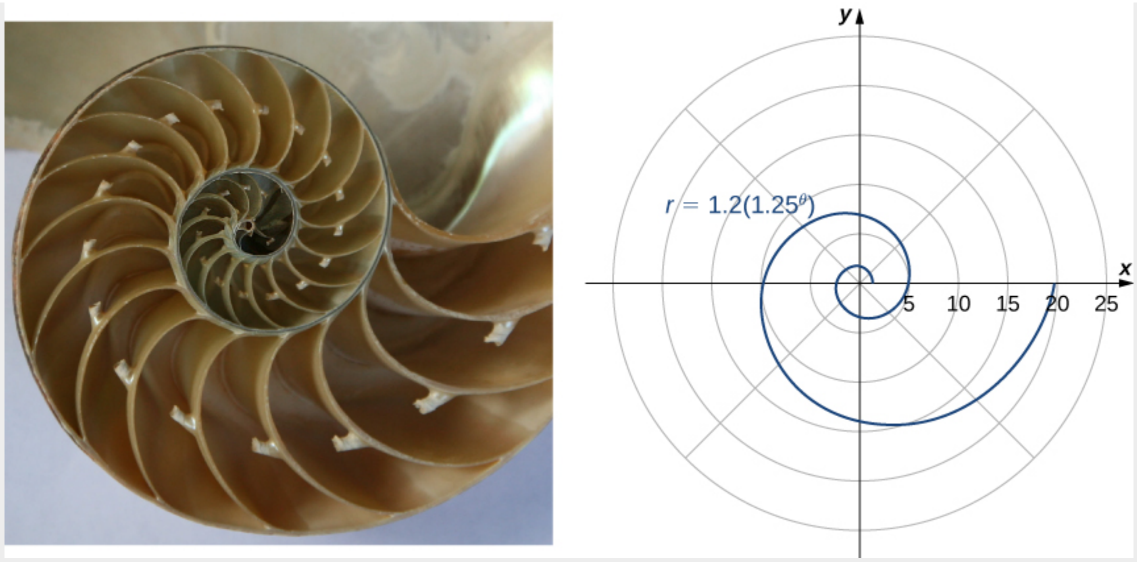
OP makes with the positive *x*-axis. Therefore $d(P,O)=k\theta$, where O is the origin. Now use the distance formula and some trigonometry:

Equation:

$$\begin{aligned} d(P,O) &= k\theta \\ \sqrt{(x-0)^2+(y-0)^2} &= k\,\text{arctan}\left(\frac{y}{x}\right) \\ \sqrt{x^2+y^2} &= k\,\text{arctan}\left(\frac{y}{x}\right) \\ \text{arctan}\left(\frac{y}{x}\right) &= \frac{\sqrt{x^2+y^2}}{k} \\ y &= x\tan\left(\frac{\sqrt{x^2+y^2}}{k}\right). \end{aligned}$$

Although this equation describes the spiral, it is not possible to solve it directly for either *x* or *y*. However, if we use polar coordinates, the equation becomes much simpler. In particular, $d(P,O)=r$, and θ is the second coordinate. Therefore the equation for the spiral becomes $r=k\theta$. Note that when $\theta=0$ we also have $r=0$, so the spiral emanates from the origin. We can remove this restriction by adding a constant to the equation. Then the equation for the spiral becomes $r=a+k\theta$ for arbitrary constants a and k. This is referred to as an Archimedean spiral, after the Greek mathematician Archimedes.

Another type of spiral is the logarithmic spiral, described by the function $r=a\cdot b^{\theta}$. A graph of the function $r=1.2\left(1.25^{\theta}\right)$ is given in [link]. This spiral describes the shell shape of the chambered nautilus.



A logarithmic spiral is similar to the shape of the chambered nautilus shell. (credit: modification of work by Jitze Couperus, Flickr)

Suppose a curve is described in the polar coordinate system via the function $r = f(\theta)$. Since we have conversion formulas from polar to rectangular coordinates given by

Equation:

$$x = r\cos\theta$$
$$y = r\sin\theta,$$

it is possible to rewrite these formulas using the function

Equation:

$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta.$$

This step gives a parameterization of the curve in rectangular coordinates using θ as the parameter. For example, the spiral formula $r = a + b\theta$ from [link] becomes

Equation:

$$\begin{aligned} x &= (a + b\theta)\cos\theta \\ y &= (a + b\theta)\sin\theta. \end{aligned}$$

Letting θ range from $-\infty$ to ∞ generates the entire spiral.

Symmetry in Polar Coordinates

When studying symmetry of functions in rectangular coordinates (i.e., in the form $y = f(x)$), we talk about symmetry with respect to the *y*-axis and symmetry with respect to the origin. In particular, if $f(-x) = f(x)$ for all x in the domain of f, then f is an even function and its graph is symmetric with respect to the *y*-axis. If $f(-x) = -f(x)$ for all x in the domain of f, then f is an odd function and its graph is symmetric with respect to the origin. By determining which types of symmetry a graph exhibits, we can learn more about the shape and appearance of the graph. Symmetry can also reveal other properties of the function that generates the graph. Symmetry in polar curves works in a similar fashion.

Note:

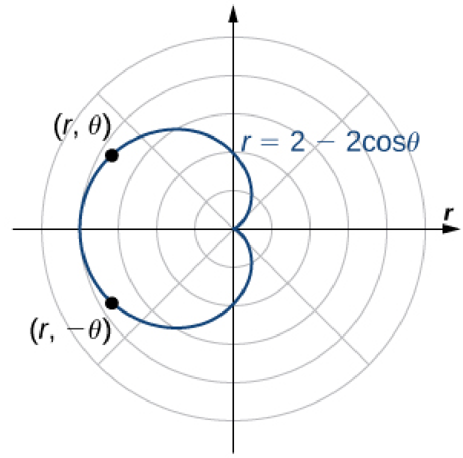
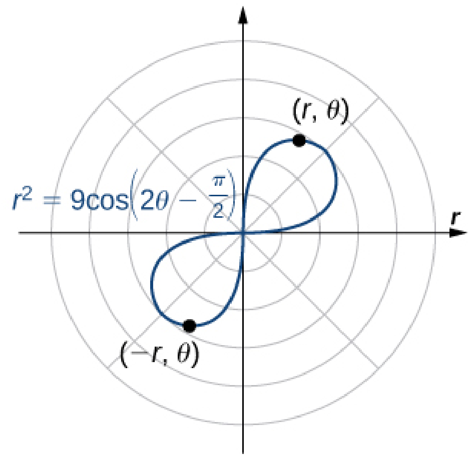
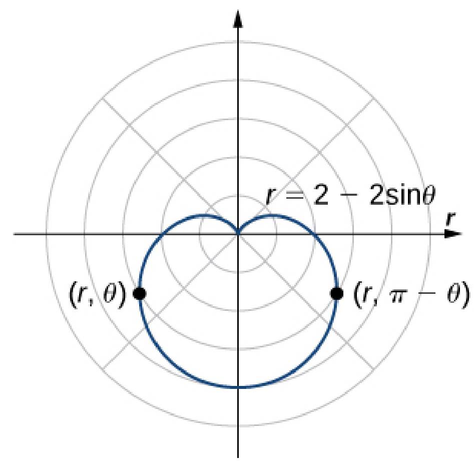
Symmetry in Polar Curves and Equations

Consider a curve generated by the function $r = f(\theta)$ in polar coordinates.

i. The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged by replacing θ with $-\theta$.
ii. The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when replacing r with $-r$, or θ with $\pi + \theta$.

iii. The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when θ is replaced by $\pi - \theta$.

The following table shows examples of each type of symmetry.

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

Example:

Exercise:

Problem:
Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

Solution:

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing θ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r with $-r$ and θ with $\pi - \theta$ yields
   Equation:

$$\begin{aligned}-r &= 3\sin(2(\pi-\theta)) \\ -r &= 3\sin(2\pi - 2\theta) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta.\end{aligned}$$

   Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace r with $-r$, which yields $-r = 3\sin(2\theta)$. Multiplying both sides by −1 gives $r = -3\sin(2\theta)$, which does not agree with the original equation. Therefore the equation does not pass the test for this symmetry. However, returning to the original equation and replacing θ with $\theta + \pi$ gives
   Equation:

$$
\begin{aligned}
r &= 3\sin\left(2\left(\theta+\pi\right)\right) \\
&= 3\sin\left(2\theta+2\pi\right) \\
&= 3\left(\sin 2\theta\cos 2\pi+\cos 2\theta\sin 2\pi\right) \\
&= 3\sin 2\theta.
\end{aligned}
$$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta=\frac{\pi}{2}$, first replace both r with $-r$ and θ with $-\theta$.
Equation:

$$
\begin{aligned}
-r &= 3\sin\left(2\left(-\theta\right)\right) \\
-r &= 3\sin\left(-2\theta\right) \\
-r &= -3\sin 2\theta.
\end{aligned}
$$

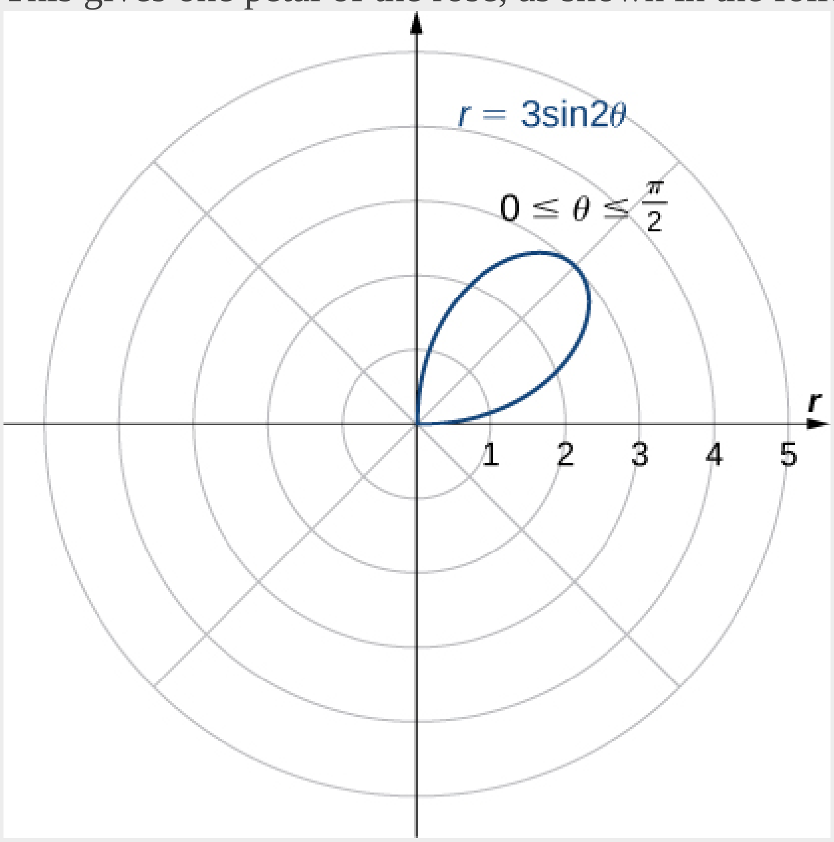
Multiplying both sides of this equation by -1 gives $r=3\sin 2\theta$, which is the original equation. Therefore the graph is symmetric about the vertical line $\theta=\frac{\pi}{2}$.

This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of θ between 0 and $\pi/2$ and then reflect the resulting graph.

θ	r
0	0
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2}\approx 2.6$

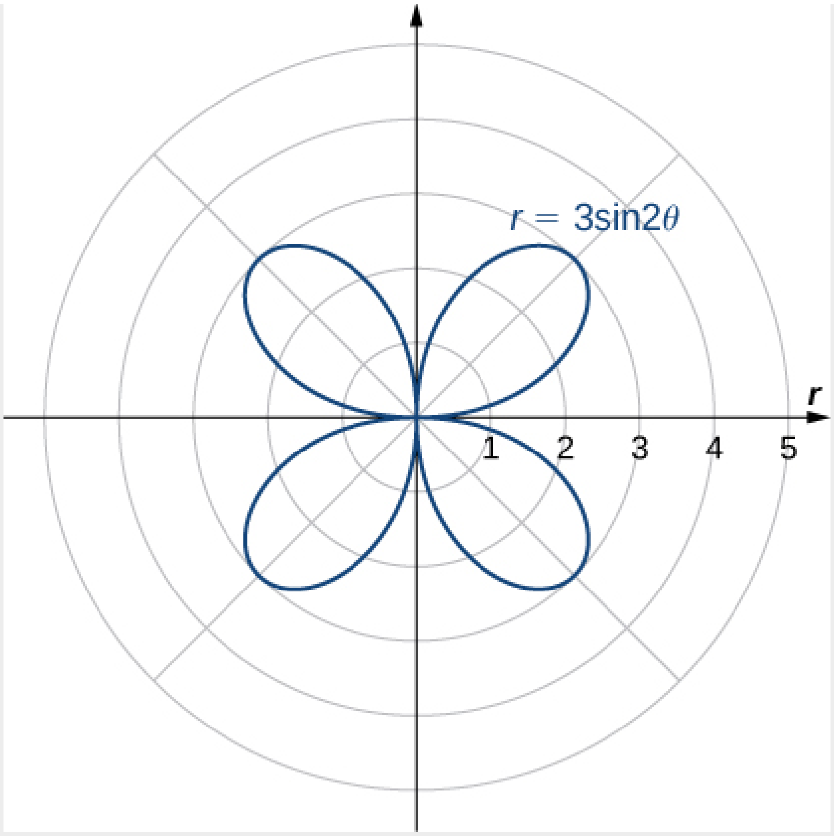
θ	r
$\frac{\pi}{4}$	3
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{2}$	0

This gives one petal of the rose, as shown in the following graph.



The graph of the equation between $\theta = 0$ and $\theta = \pi/2$.

Reflecting this image into the other three quadrants gives the entire graph as shown.



The entire graph of the equation is called a four-petaled rose.

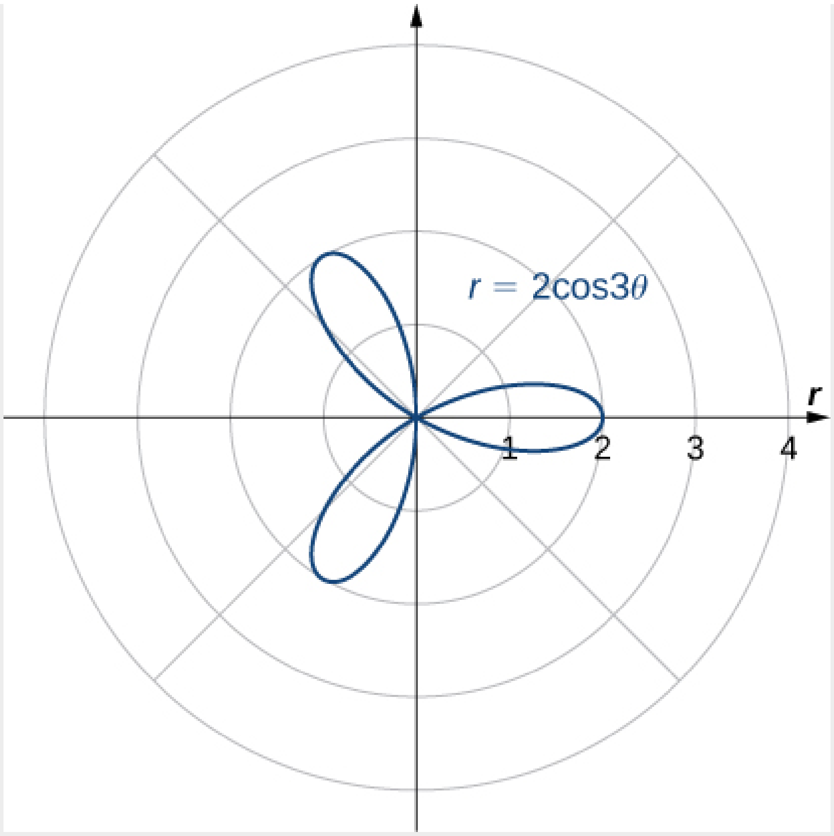
Note:
Exercise:

Problem:

Determine the symmetry of the graph determined by the equation $r = 2\cos(3\theta)$ and create a graph.

Solution:

Symmetric with respect to the polar axis.



Hint

Use [link].

Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas
  Equation:

  $$x = r\cos\theta \text{ and } y = r\sin\theta$$

  and
  Equation:

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}.$$

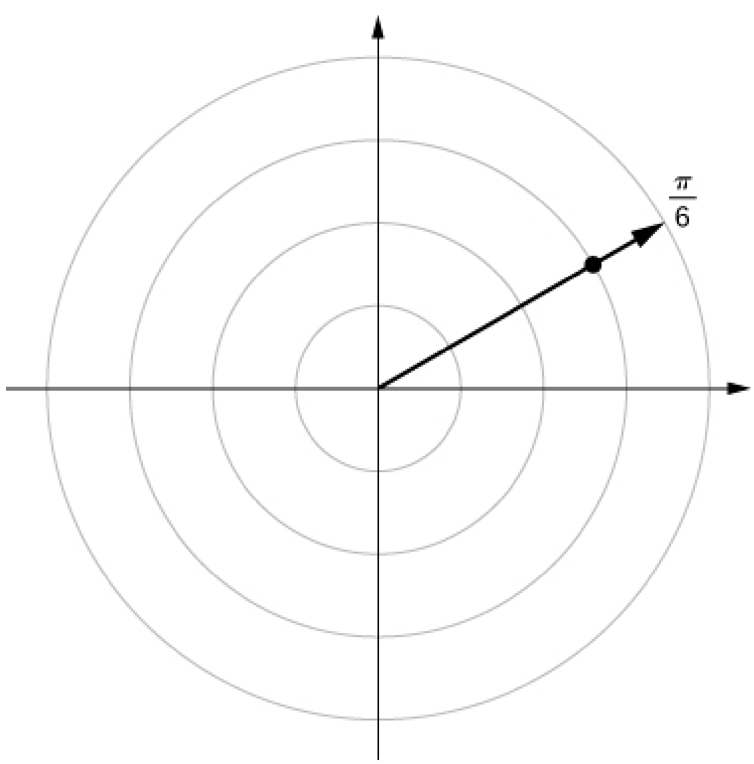
- To sketch a polar curve from a given polar function, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

In the following exercises, plot the point whose polar coordinates are given by first constructing the angle θ and then marking off the distance r along the ray.

Exercise:

Problem: $\left(3, \frac{\pi}{6}\right)$

Solution:



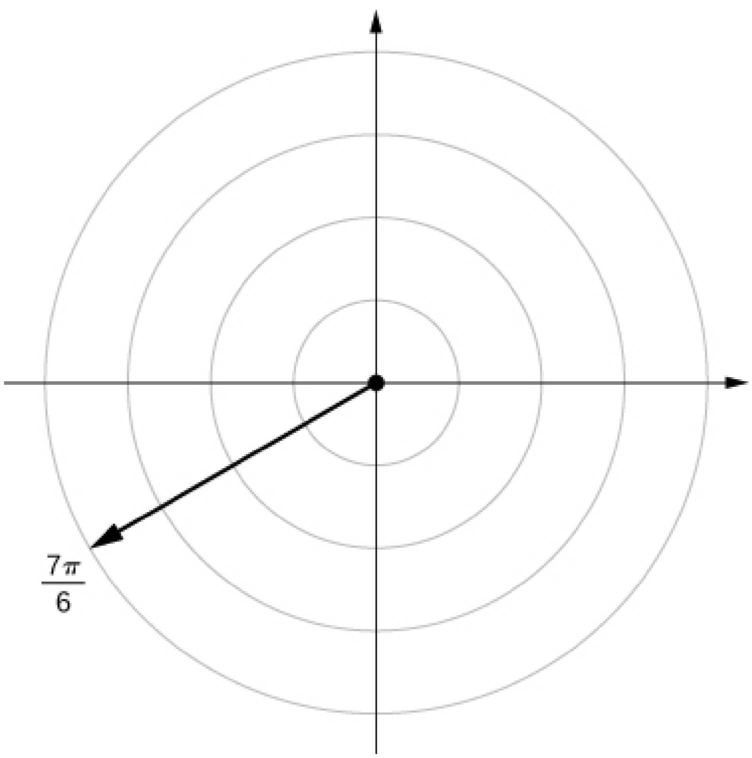
Exercise:

Problem: $\left(-2, \frac{5\pi}{3}\right)$

Exercise:

Problem: $\left(0, \frac{7\pi}{6}\right)$

Solution:



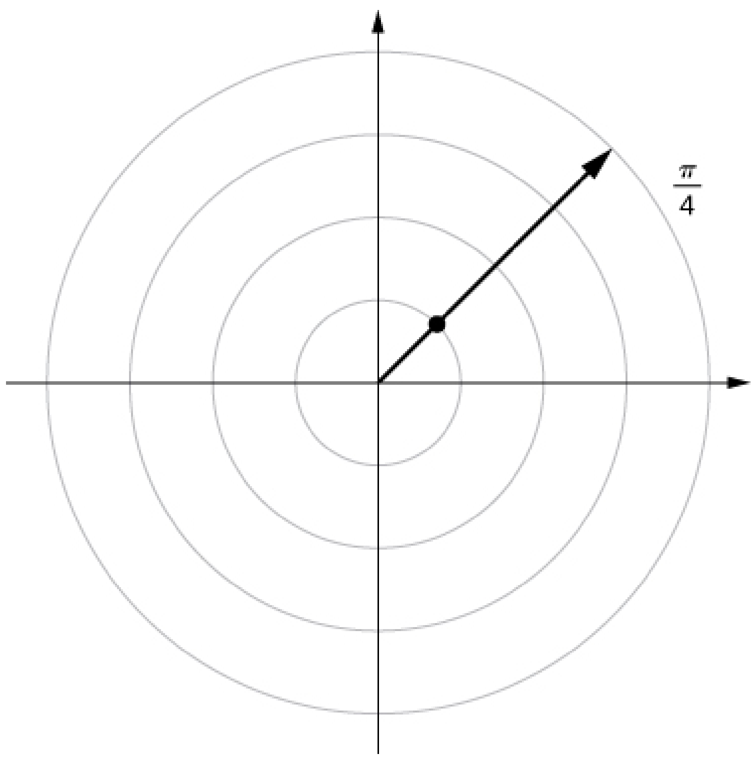
Exercise:

Problem: $\left(-4, \frac{3\pi}{4}\right)$

Exercise:

Problem: $\left(1, \frac{\pi}{4}\right)$

Solution:



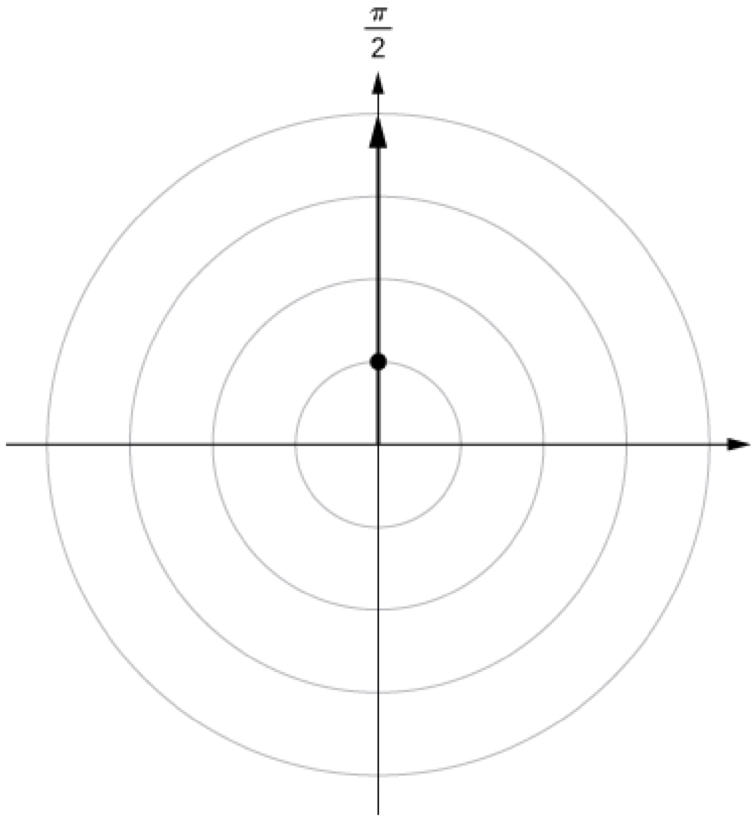
Exercise:

Problem: $\left(2, \frac{5\pi}{6}\right)$

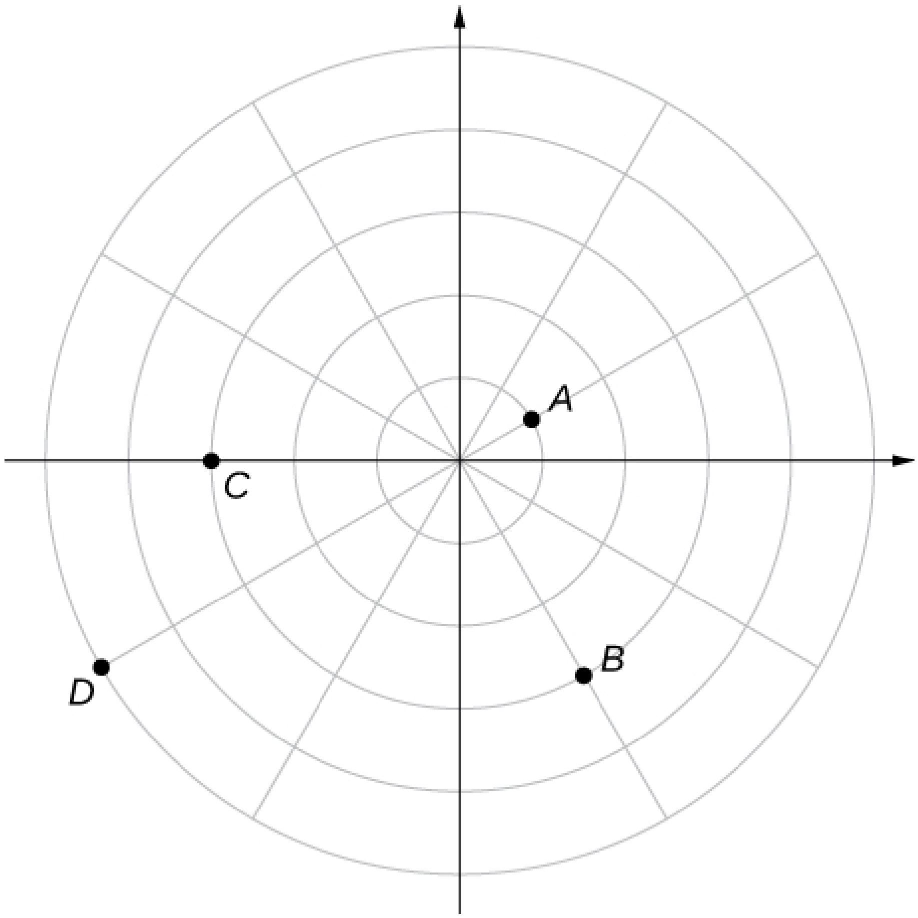
Exercise:

Problem: $\left(1, \frac{\pi}{2}\right)$

Solution:



For the following exercises, consider the polar graph below. Give two sets of polar coordinates for each point.



Exercise:

Problem: Coordinates of point A.

Exercise:

Problem: Coordinates of point B.

Solution:

$B\left(3, \frac{-\pi}{3}\right) \quad B\left(-3, \frac{2\pi}{3}\right)$

Exercise:

Problem: Coordinates of point C.

Exercise:

Problem: Coordinates of point D.

Solution:

$D\left(5, \frac{7\pi}{6}\right) D\left(-5, \frac{\pi}{6}\right)$

For the following exercises, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point in $(0, 2\pi]$. Round to three decimal places.

Exercise:

Problem: $(2,\ 2)$

Exercise:

Problem: $(3, -4)$ (3, −4)

Solution:

$(5, -0.927) \quad (-5, -0.927 + \pi)$

Exercise:

Problem: $(8,\ 15)$

Exercise:

Problem: $(-6,\ 8)$

Solution:

$(10, -0.927) \, (-10, -0.927 + \pi)$

Exercise:

Problem: $(4,\ 3)$

Exercise:

Problem: $\left(3,-\sqrt{3}\right)$

Solution:

$\left(2\sqrt{3},-0.524\right)\left(-2\sqrt{3},-0.524+\pi\right)$

For the following exercises, find rectangular coordinates for the given point in polar coordinates.

Exercise:

Problem: $\left(2,\frac{5\pi}{4}\right)$

Exercise:

Problem: $\left(-2,\frac{\pi}{6}\right)$

Solution:

$\left(-\sqrt{3},\quad -1\right)$

Exercise:

Problem: $\left(5,\frac{\pi}{3}\right)$

Exercise:

Problem: $\left(1,\frac{7\pi}{6}\right)$

Solution:

$\left(-\frac{\sqrt{3}}{2},\quad \frac{-1}{2}\right)$

Exercise:

Problem: $\left(-3, \frac{3\pi}{4}\right)$

Exercise:

Problem: $\left(0, \frac{\pi}{2}\right)$

Solution:

$(0, \quad 0)$

Exercise:

Problem: $(-4.5, 6.5)$

For the following exercises, determine whether the graphs of the polar equation are symmetric with respect to the x-axis, the y-axis, or the origin.

Exercise:

Problem: $r = 3\sin(2\theta)$

Solution:

Symmetry with respect to the *x*-axis, *y*-axis, and origin.

Exercise:

Problem: $r^2 = 9\cos\theta$

Exercise:

Problem: $r = \cos\left(\frac{\theta}{5}\right)$

Solution:

Symmetric with respect to *x*-axis only.

Exercise:

Problem: $r = 2\sec\theta$

Exercise:

Problem: $r = 1 + \cos\theta$

Solution:

Symmetry with respect to *x*-axis only.

For the following exercises, describe the graph of each polar equation. Confirm each description by converting into a rectangular equation.

Exercise:

Problem: $r = 3$

Exercise:

Problem: $\theta = \frac{\pi}{4}$

Solution:

Line $y = x$

Exercise:

Problem: $r = \sec\theta$

Exercise:

Problem: $r = \csc\theta$

Solution:

$y = 1$

For the following exercises, convert the rectangular equation to polar form and sketch its graph.

Exercise:

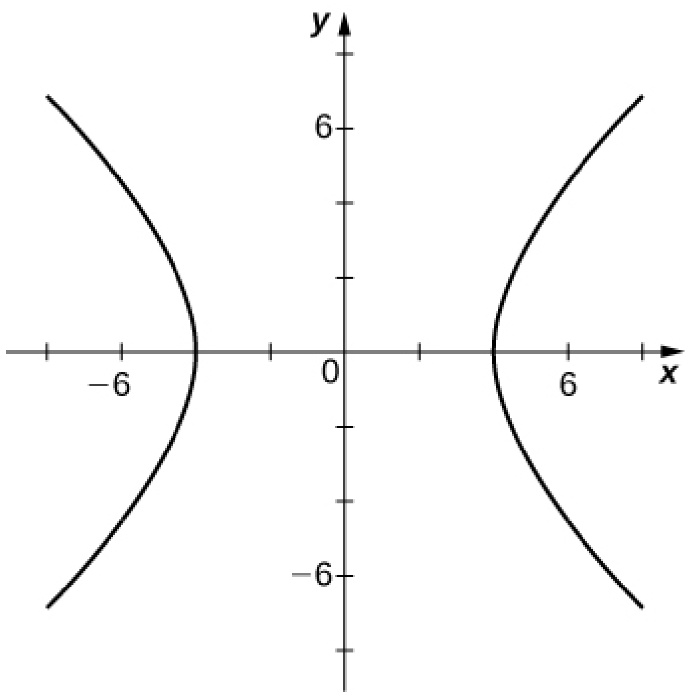
Problem: $x^2 + y^2 = 16$

Exercise:

Problem: $x^2 - y^2 = 16$

Solution:

Hyperbola; polar form $r^2\cos(2\theta) = 16$ or $r^2 = 16\sec\theta$.



Exercise:

Problem: $x = 8$

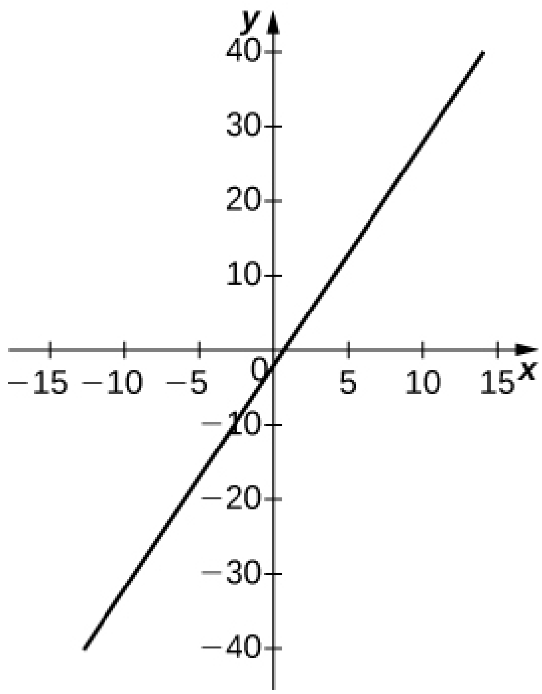
For the following exercises, convert the rectangular equation to polar form and sketch its graph.

Exercise:

Problem: $3x - y = 2$

Solution:

$r = \frac{2}{3\cos\theta - \sin\theta}$



Exercise:

Problem: $y^2 = 4x$

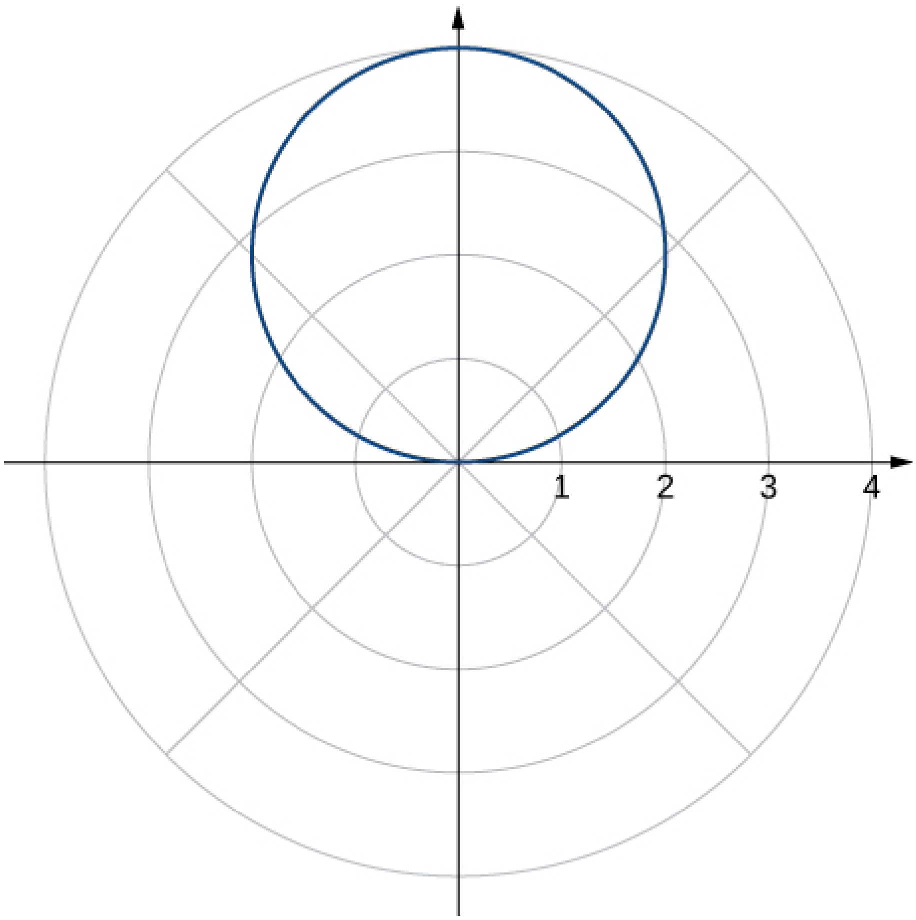
For the following exercises, convert the polar equation to rectangular form and sketch its graph.

Exercise:

Problem: $r = 4\sin\theta$

Solution:

$x^2 + y^2 = 4y$



Exercise:

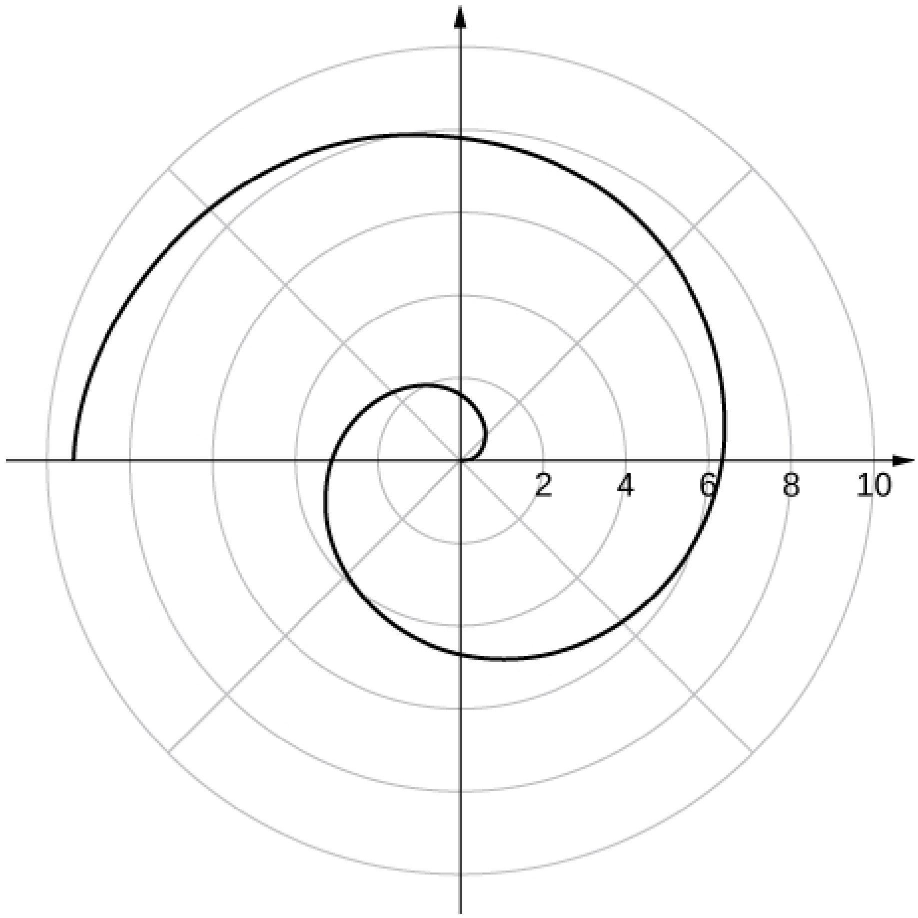
Problem: $r = 6 \cos \theta$

Exercise:

Problem: $r = \theta$

Solution:

$x \tan \sqrt{x^2 + y^2} = y$



Exercise:

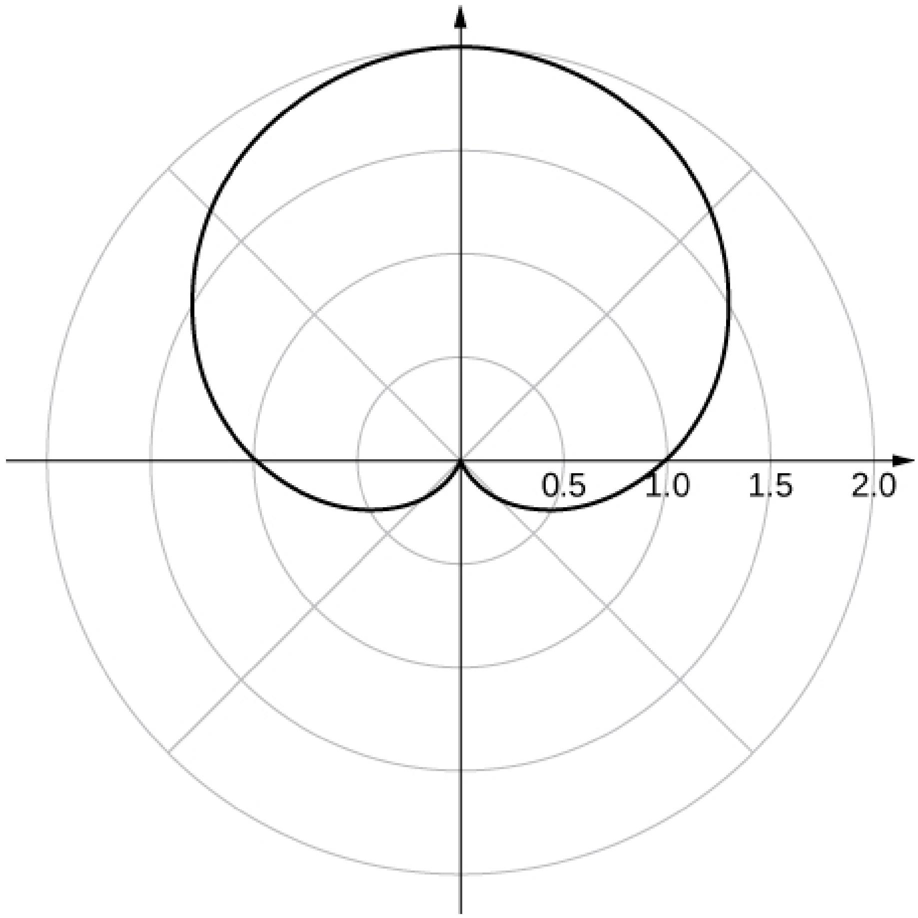
Problem: $r = \cot\theta \csc\theta$

For the following exercises, sketch a graph of the polar equation and identify any symmetry.

Exercise:

Problem: $r = 1 + \sin\theta$

Solution:



y-axis symmetry

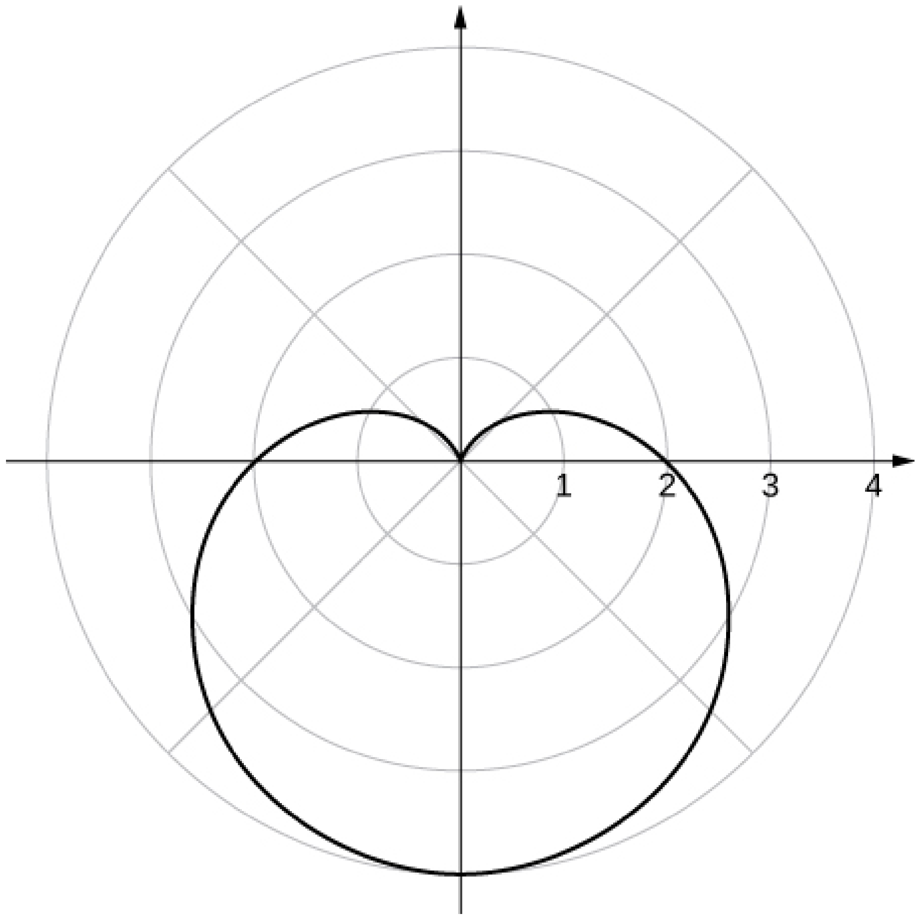
Exercise:

Problem: $r = 3 - 2\cos\theta$

Exercise:

Problem: $r = 2 - 2\sin\theta$

Solution:



y-axis symmetry

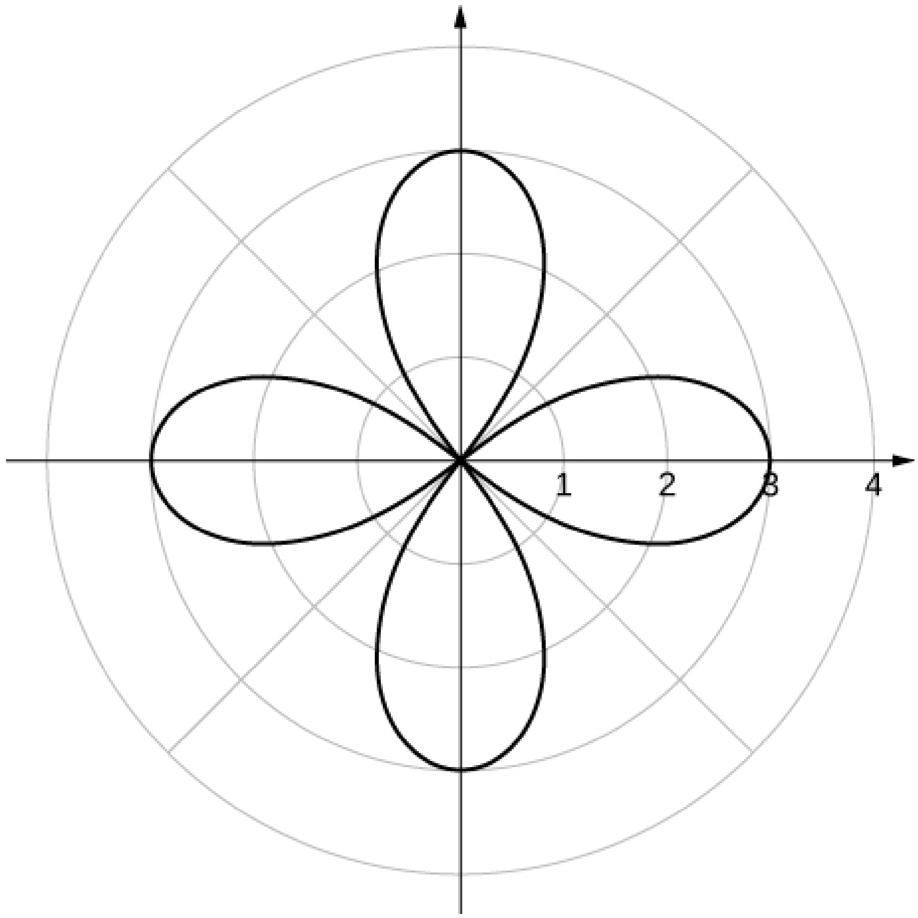
Exercise:

Problem: $r = 5 - 4\sin\theta$

Exercise:

Problem: $r = 3\cos(2\theta)$

Solution:



x- and y-axis symmetry and symmetry about the pole

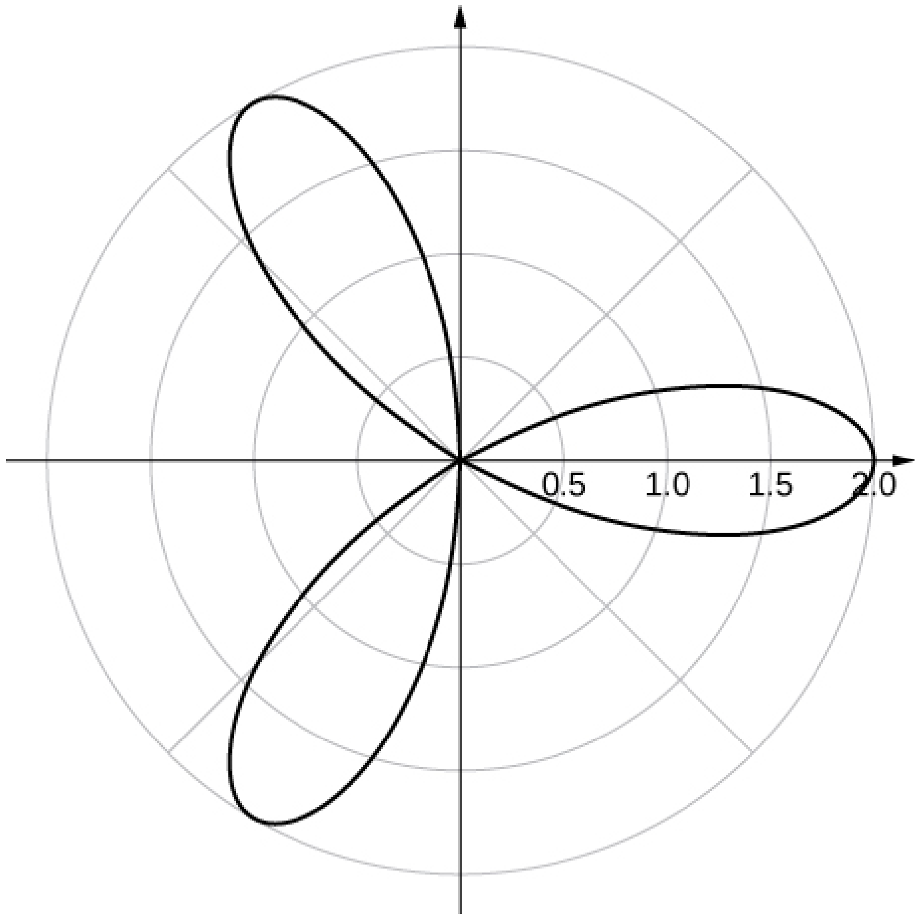
Exercise:

Problem: $r = 3\sin(2\theta)$

Exercise:

Problem: $r = 2\cos(3\theta)$

Solution:



x-axis symmetry

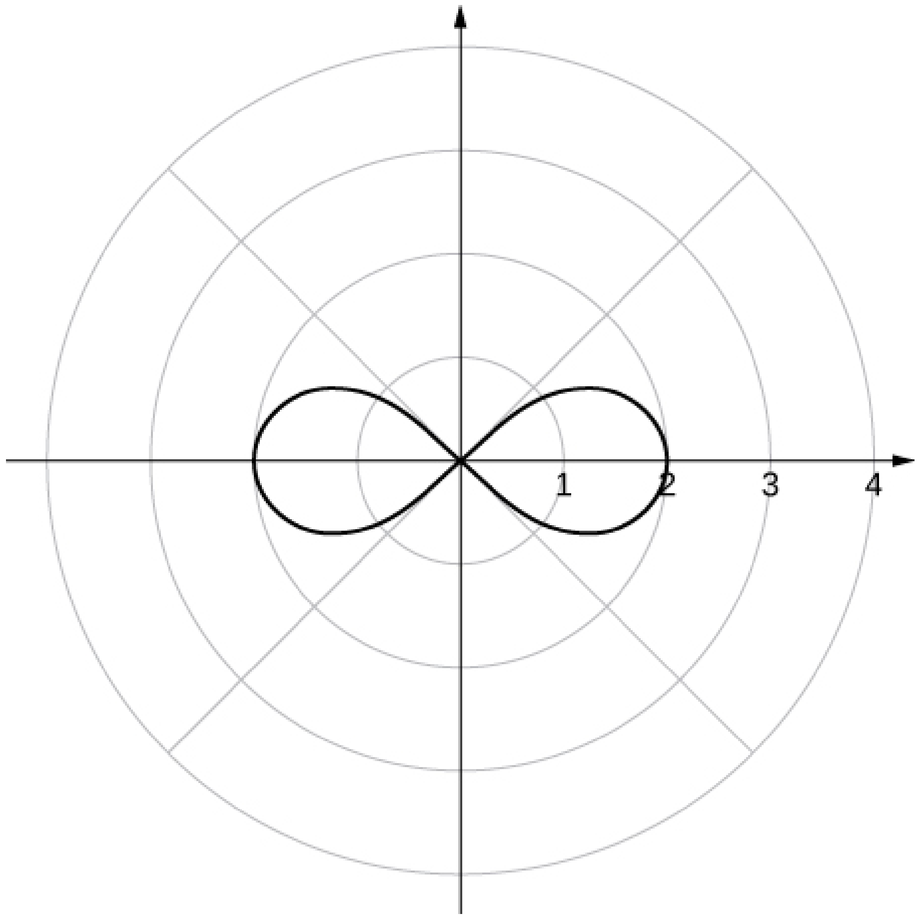
Exercise:

Problem: $r = 3\cos\left(\frac{\theta}{2}\right)$

Exercise:

Problem: $r^2 = 4\cos(2\theta)$

Solution:



x- and *y*-axis symmetry and symmetry about the pole

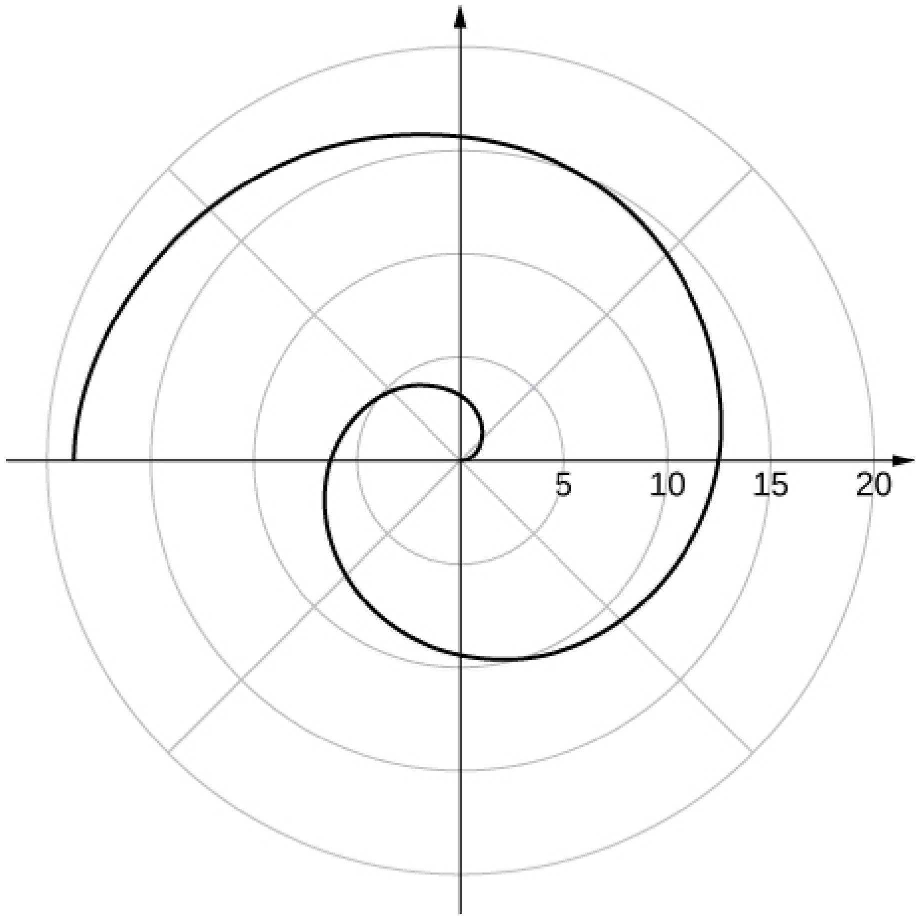
Exercise:

Problem: $r^2 = 4\sin\theta$

Exercise:

Problem: $r = 2\theta$

Solution:



no symmetry

Exercise:

Problem:

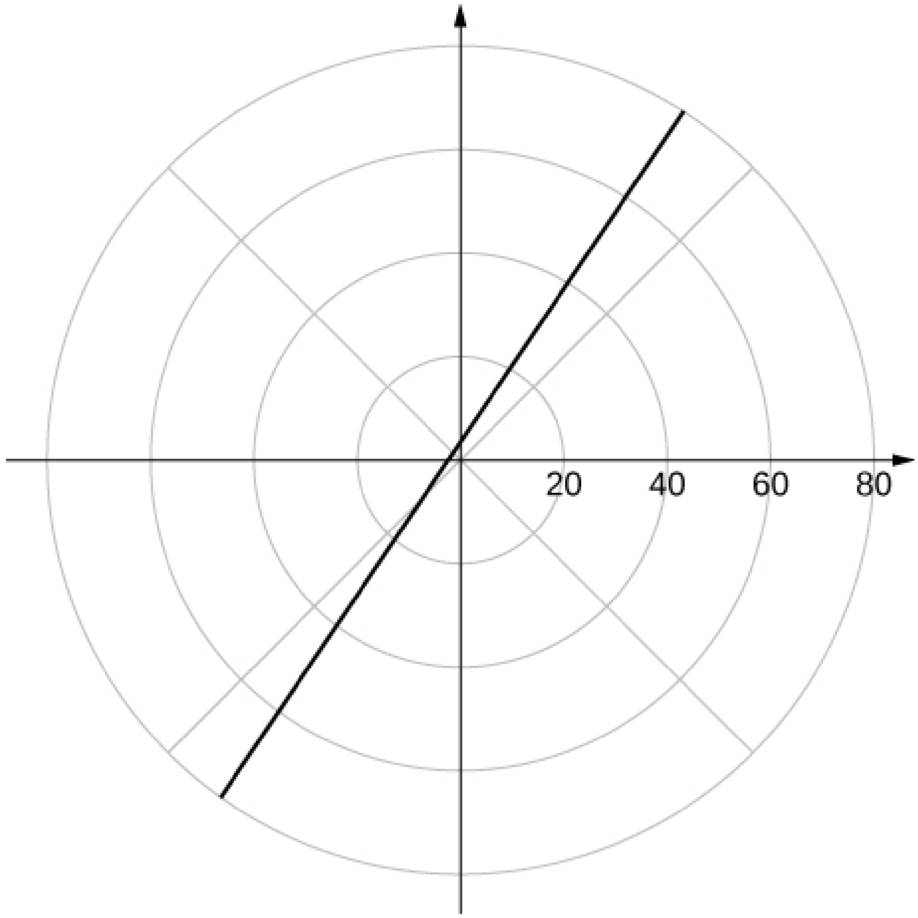
[T] The graph of $r = 2\cos(2\theta)\sec(\theta)$. is called a *strophoid*. Use a graphing utility to sketch the graph, and, from the graph, determine the asymptote.

Exercise:

Problem:

[T] Use a graphing utility and sketch the graph of $r = \frac{6}{2\sin\theta - 3\cos\theta}$.

Solution:



a line

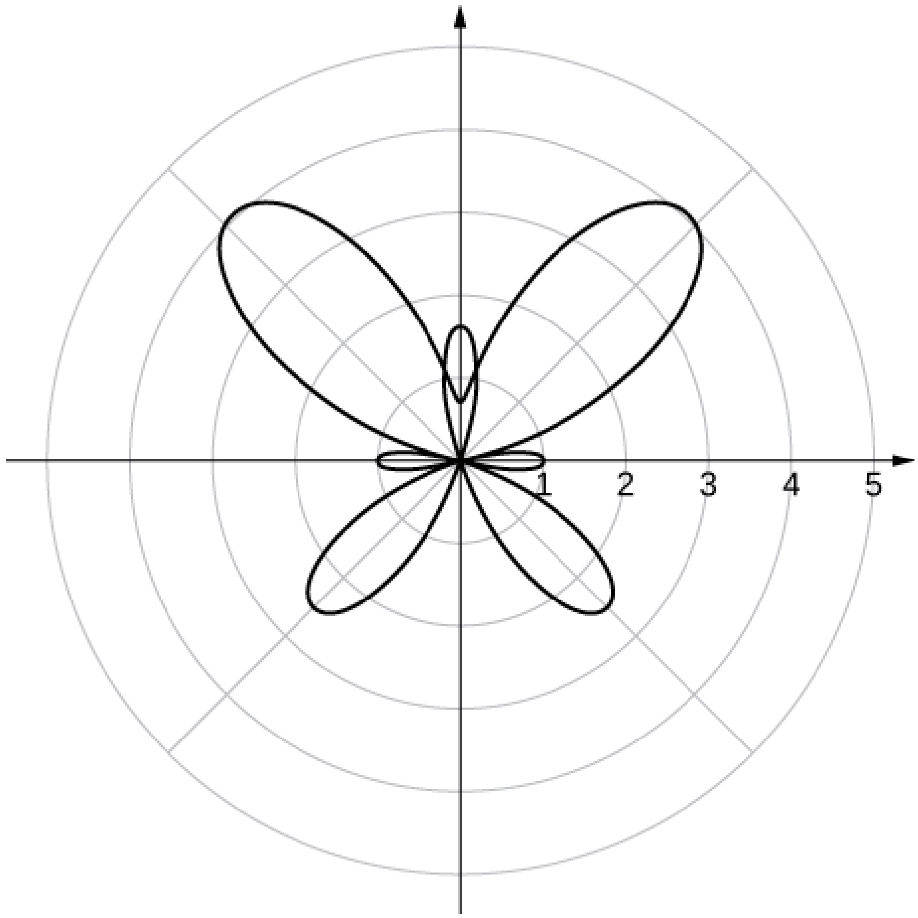
Exercise:

Problem: **[T]** Use a graphing utility to graph $r = \frac{1}{1 - \cos \theta}$.

Exercise:

Problem: **[T]** Use technology to graph $r = e^{\sin(\theta)} - 2 \cos(4\theta)$.

Solution:



Exercise:

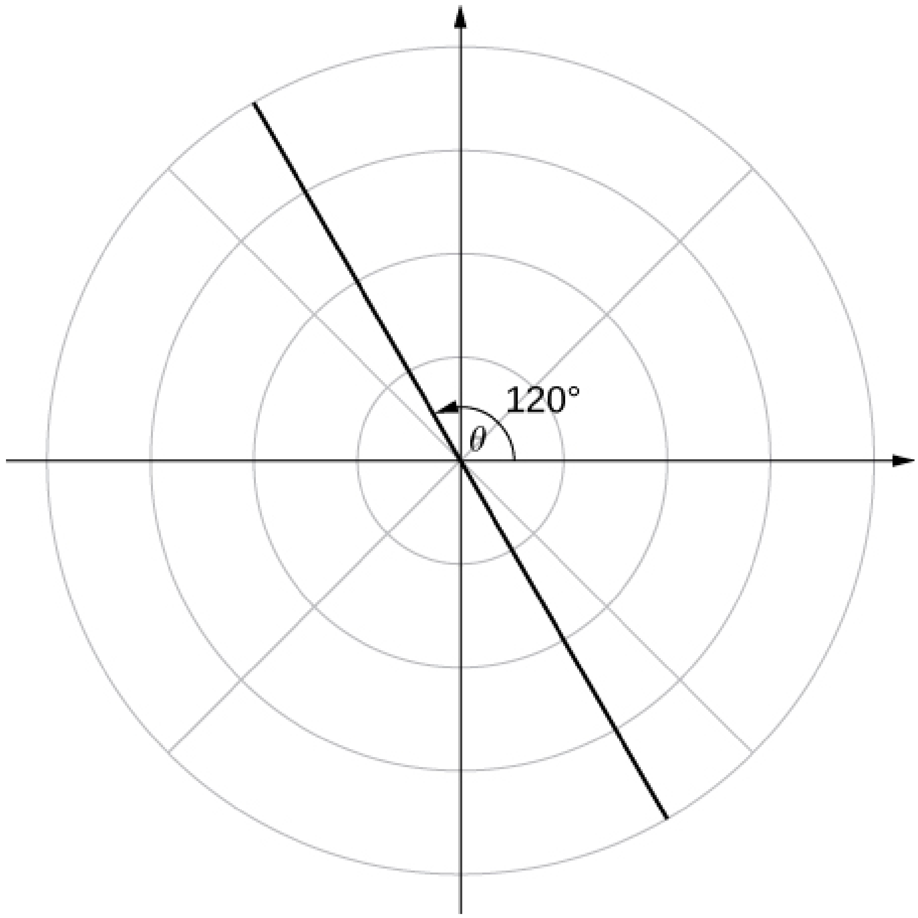
Problem:

[T] Use technology to plot $r = \sin\left(\frac{3\theta}{7}\right)$ (use the interval $0 \le \theta \le 14\pi$).

Exercise:

Problem: Without using technology, sketch the polar curve $\theta = \frac{2\pi}{3}$.

Solution:



Exercise:

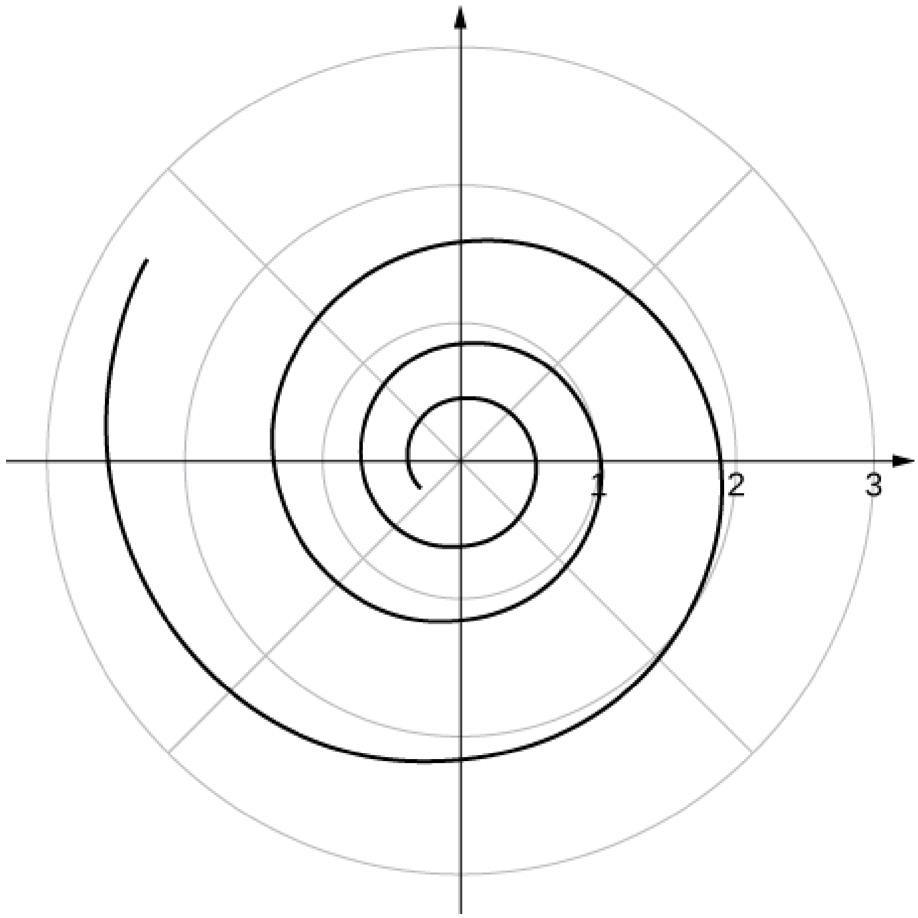
Problem:

[T] Use a graphing utility to plot $r = \theta \sin \theta$ for $-\pi \leq \theta \leq \pi$.

Exercise:

Problem: **[T]** Use technology to plot $r = e^{-0.1\theta}$ for $-10 \leq \theta \leq 10$.

Solution:



Exercise:

Problem:

[T] There is a curve known as the "*Black Hole*." Use technology to plot $r = e^{-0.01\theta}$ for $-100 \leq \theta \leq 100$.

Exercise:

Problem:

[T] Use the results of the preceding two problems to explore the graphs of $r = e^{-0.001\theta}$ and $r = e^{-0.0001\theta}$ for $|\theta| > 100$.

Solution:

Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

Glossary

angular coordinate
: θ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial (x) axis, measured counterclockwise

cardioid
: a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

limaçon
: the graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid

polar axis
: the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

polar coordinate system
: a system for locating points in the plane. The coordinates are r, the radial coordinate, and θ, the angular coordinate

polar equation
: an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

pole
: the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

radial coordinate
: r the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

rose

graph of the polar equation $r = a \cos 2\theta$ or $r = a \sin 2\theta$ for a positive constant a

space-filling curve

a curve that completely occupies a two-dimensional subset of the real plane

Area and Arc Length in Polar Coordinates

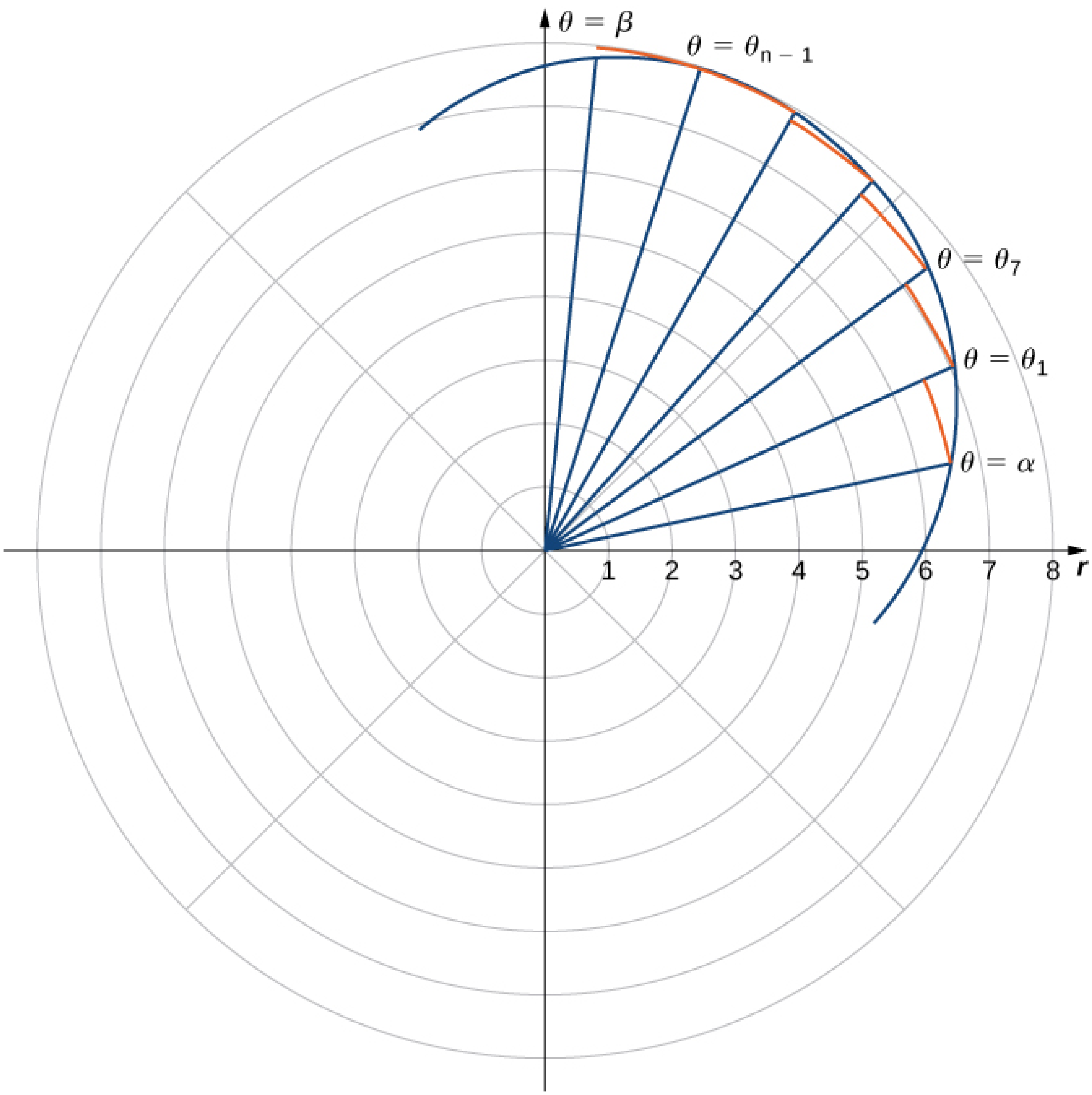
- Apply the formula for area of a region in polar coordinates.
- Determine the arc length of a polar curve.

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval, the area between the curve and the *x*-axis is given by $A = \int_a^b f(x)\,dx$. This fact, along with the formula for evaluating this integral, is summarized in the Fundamental Theorem of Calculus. Similarly, the arc length of this curve is given by $L = \int_a^b \sqrt{1 + (f'(x))^2}dx$. In this section, we study analogous formulas for area and arc length in the polar coordinate system.

Areas of Regions Bounded by Polar Curves

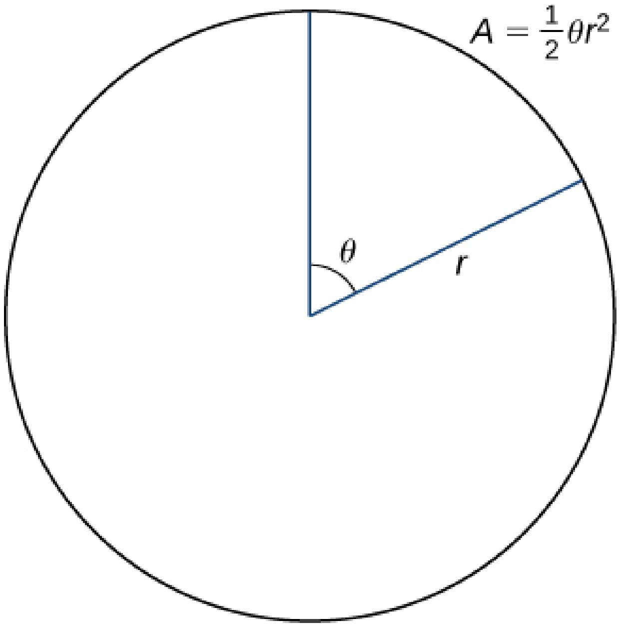
We have studied the formulas for area under a curve defined in rectangular coordinates and parametrically defined curves. Now we turn our attention to deriving a formula for the area of a region bounded by a polar curve. Recall that the proof of the Fundamental Theorem of Calculus used the concept of a Riemann sum to approximate the area under a curve by using rectangles. For polar curves we use the Riemann sum again, but the rectangles are replaced by sectors of a circle.

Consider a curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into *n* equal-width subintervals. The width of each subinterval is given by the formula $\Delta\theta = (\beta - \alpha)/n$, and the *i*th partition point θ_i is given by the formula $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan\theta_i$ passing through the pole as shown in the following graph.



A partition of a typical curve in polar coordinates.

The line segments are connected by arcs of constant radius. This defines sectors whose areas can be calculated by using a geometric formula. The area of each sector is then used to approximate the area between successive line segments. We then sum the areas of the sectors to approximate the total area. This approach gives a Riemann sum approximation for the total area. The formula for the area of a sector of a circle is illustrated in the following figure.



The area of a sector of a circle is given by $A = \frac{1}{2}\theta r^2$.

Recall that the area of a circle is $A = \pi r^2$. When measuring angles in radians, 360 degrees is equal to 2π radians. Therefore a fraction of a circle can be measured by the central angle θ. The fraction of the circle is given by $\frac{\theta}{2\pi}$, so the area of the sector is this fraction multiplied by the total area:

Equation:

$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}\theta r^2.$$

Since the radius of a typical sector in [link] is given by $r_i = f(\theta_i)$, the area of the *i*th sector is given by

Equation:

$$A_i = \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

Therefore a Riemann sum that approximates the area is given by

Equation:

$$A_n = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

We take the limit as $n \to \infty$ to get the exact area:
Equation:

$$A = \lim_{n\to\infty} A_n = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

This gives the following theorem.

Note:
Area of a Region Bounded by a Polar Curve
Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is
Equation:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta.$$

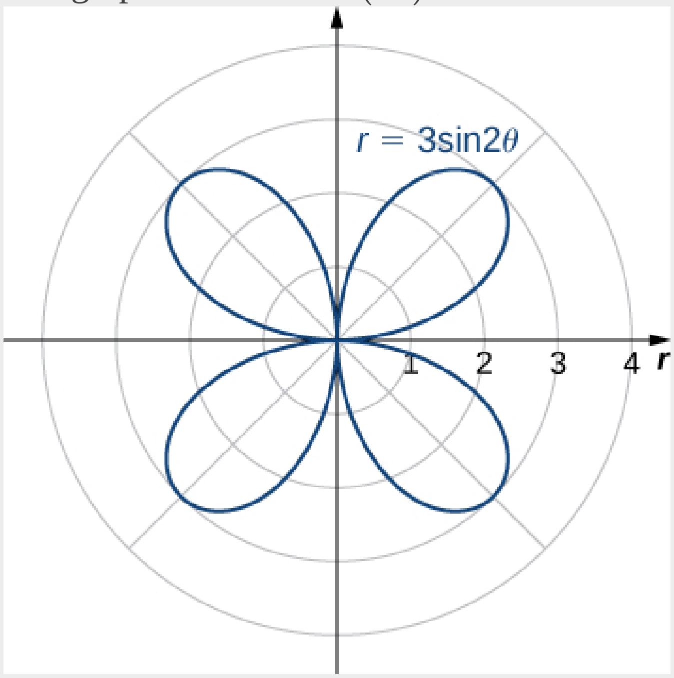
Example:
Exercise:

Problem:
Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Solution:

The graph of $r = 3\sin(2\theta)$ follows.



The graph of $r = 3\sin(2\theta)$.

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for θ. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use [link] with $f(\theta) = 3\sin(2\theta)$, $\alpha = 0$, and $\beta = \pi/2$:

Equation:

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} [3\sin(2\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta)\, d\theta. \end{aligned}$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$:

Equation:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)\,d\theta \\ &= \frac{9}{2}\int_0^{\pi/2} \frac{(1-\cos(4\theta))}{2}\,d\theta \\ &= \frac{9}{4}\left(\int_0^{\pi/2} 1-\cos(4\theta)\,d\theta\right) \\ &= \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Big|_0^{\pi/2} \\ &= \frac{9}{4}\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \frac{9}{4}\left(0 - \frac{\sin 4(0)}{4}\right) \\ &= \frac{9\pi}{8}. \end{aligned}$$

Note:

Exercise:

Problem:

Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

Solution:

$A = 3\pi/2$

Hint

Use [link]. Be sure to determine the correct limits of integration before evaluating.

[link] involved finding the area inside one curve. We can also use [link] to find the area between two polar curves. However, we often need to find the points of intersection of the curves and determine which function defines the outer curve or the inner curve between these two points.

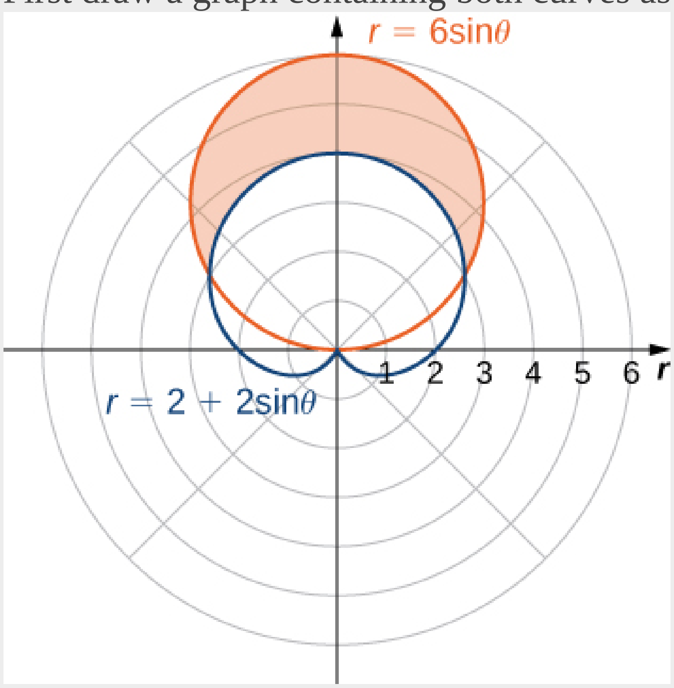
Example:
Exercise:

Problem:
Finding the Area between Two Polar Curves

Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.

Solution:

First draw a graph containing both curves as shown.



The region between the curves $r = 2 + 2\sin\theta$ and $r = 6\sin\theta$.

To determine the limits of integration, first find the points of intersection by setting the two functions equal to each other and solving for θ:

Equation:

$$\begin{aligned} 6\sin\theta &= 2+2\sin\theta \\ 4\sin\theta &= 2 \\ \sin\theta &= \tfrac{1}{2}. \end{aligned}$$

This gives the solutions $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, which are the limits of integration. The circle $r = 3\sin\theta$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin\theta$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

Equation:

$$\begin{aligned} A &= \text{circle} - \text{cardioid} \\ &= \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} [6\sin\theta]^2 d\theta - \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} [2+2\sin\theta]^2 d\theta \\ &= \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} 36\sin^2\theta\, d\theta - \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} 4+8\sin\theta+4\sin^2\theta\, d\theta \\ &= 18\int_{\pi/6}^{5\pi/6} \frac{1-\cos(2\theta)}{2} d\theta - 2\int_{\pi/6}^{5\pi/6} 1+2\sin\theta+\frac{1-\cos(2\theta)}{2} d\theta \\ &= 9\left[\theta - \tfrac{\sin(2\theta)}{2}\right]_{\pi/6}^{5\pi/6} - 2\left[\tfrac{3\theta}{2} - 2\cos\theta - \tfrac{\sin(2\theta)}{4}\right]_{\pi/6}^{5\pi/6} \\ &= 9\left(\tfrac{5\pi}{6} - \tfrac{\sin 2(5\pi/6)}{2}\right) - 9\left(\tfrac{\pi}{6} - \tfrac{\sin 2(\pi/6)}{2}\right) \\ &\quad -\left(3\left(\tfrac{5\pi}{6}\right) - 4\cos\tfrac{5\pi}{6} - \tfrac{\sin 2(5\pi/6)}{2}\right) + \left(3\left(\tfrac{\pi}{6}\right) - 4\cos\tfrac{\pi}{6} - \tfrac{\sin 2(\pi/6)}{2}\right) \\ &= 4\pi. \end{aligned}$$

Note:
Exercise:

Problem:

Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

Solution:

$A = \frac{4\pi}{3} + 4\sqrt{3}$

Hint

Use [link] and take advantage of symmetry.

In [link] we found the area inside the circle and outside the cardioid by first finding their intersection points. Notice that solving the equation directly for θ yielded two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, in the graph there are three intersection points. The third intersection point is the origin. The reason why this point did not show up as a solution is because the origin is on both graphs but for different values of θ. For example, for the cardioid we get

Equation:

$$\begin{array}{rcl} 2 + 2\sin\theta & = & 0 \\ \sin\theta & = & -1, \end{array}$$

so the values for θ that solve this equation are $\theta = \frac{3\pi}{2} + 2n\pi$, where *n* is any integer. For the circle we get

Equation:

$$6\sin\theta = 0.$$

The solutions to this equation are of the form $\theta = n\pi$ for any integer value of *n*. These two solution sets have no points in common. Regardless of this fact, the curves intersect at the origin. This case must always be taken into consideration.

Arc Length in Polar Curves

Here we derive a formula for the arc length of a curve defined in polar coordinates.

In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \leq t \leq b$ is given by

Equation:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

Equation:

$$x = r\cos\theta = f(\theta)\cos\theta \text{ and } y = r\sin\theta = f(\theta)\sin\theta,$$

and we replace the parameter *t* by θ. Then

Equation:

$$\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta.\end{aligned}$$

We replace dt by $d\theta$, and the lower and upper limits of integration are α and β, respectively. Then the arc length formula becomes

Equation:

$$
\begin{aligned}
L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\
&= \int_\alpha^\beta \sqrt{\left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2} d\theta \\
&= \int_\alpha^\beta \sqrt{\left(f'(\theta)\right)^2\left(\cos^2\theta + \sin^2\theta\right) + \left(f(\theta)\right)^2\left(\cos^2\theta + \sin^2\theta\right)} d\theta \\
&= \int_\alpha^\beta \sqrt{\left(f'(\theta)\right)^2 + \left(f(\theta)\right)^2} d\theta \\
&= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.
\end{aligned}
$$

This gives us the following theorem.

Note:
Arc Length of a Curve Defined by a Polar Function
Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is
Equation:

$$L = \int_\alpha^\beta \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2} d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Example:
Exercise:

Problem:
Finding the Arc Length of a Polar Curve

Find the arc length of the cardioid $r = 2 + 2\cos\theta$.

Solution:

When $\theta = 0, r = 2 + 2\cos 0 = 4$. Furthermore, as θ goes from 0 to 2π, the cardioid is traced out exactly once. Therefore these are the limits of integration. Using $f(\theta) = 2 + 2\cos\theta$, $\alpha = 0$, and $\beta = 2\pi$, [link] becomes

Equation:

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{[2 + 2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{4 + 8\cos\theta + 4\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\ &= 2\int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta. \end{aligned}$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1$, add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2\alpha$. Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2)$, so the integral becomes

Equation:

$$\begin{aligned} L &= 2\int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= 2\int_{0}^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta \\ &= 2\int_{0}^{2\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right| d\theta. \end{aligned}$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and $\frac{\pi}{2}$. Thus,

Equation:

$$\begin{aligned} L &= 4\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta \\ &= 8\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta \\ &= 8\left(2\sin\left(\tfrac{\theta}{2}\right)\right)_0^{\pi} \\ &= 16. \end{aligned}$$

Note:
Exercise:

Problem: Find the total arc length of $r = 3\sin\theta$.

Solution:

$s = 3\pi$

Hint

Use [link]. To determine the correct limits, make a table of values.

Key Concepts

- The area of a region in polar coordinates defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta$.

- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral
$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Key Equations

- **Area of a region bounded by a polar curve**
$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$
- **Arc length of a polar curve**
$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

For the following exercises, determine a definite integral that represents the area.
Exercise:

Problem: Region enclosed by $r = 4$

Exercise:

Problem: Region enclosed by $r = 3\sin\theta$

Solution:

$$\frac{9}{2} \int_{0}^{\pi} \sin^2\theta \, d\theta$$

Exercise:

Problem: Region in the first quadrant within the cardioid $r = 1 + \sin\theta$

Exercise:

Problem: Region enclosed by one petal of $r = 8\sin(2\theta)$

Solution:

$$32\int_0^{\pi/2} \sin^2(2\theta)d\theta$$

Exercise:

Problem: Region enclosed by one petal of $r = \cos(3\theta)$

Exercise:

Problem: Region below the polar axis and enclosed by $r = 1 - \sin\theta$

Solution:

$$\tfrac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2 d\theta$$

Exercise:

Problem: Region in the first quadrant enclosed by $r = 2 - \cos\theta$

Exercise:

Problem: Region enclosed by the inner loop of $r = 2 - 3\sin\theta$

Solution:

$$\int_{\sin^{-1}(2/3)}^{\pi/2} (2-3\sin\theta)^2 d\theta$$

Exercise:

Problem: Region enclosed by the inner loop of $r = 3 - 4\cos\theta$

Exercise:

Problem: Region enclosed by $r = 1 - 2\cos\theta$ and outside the inner loop

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Solution:

$$\int_0^{\pi} (1 - 2\cos\theta)^2 d\theta - \int_0^{\pi/3} (1 - 2\cos\theta)^2 d\theta$$

Exercise:

Problem: Region common to $r = 3\sin\theta$ and $r = 2 - \sin\theta$

Exercise:

Problem: Region common to $r = 2$ and $r = 4\cos\theta$

Solution:

$$4\int_0^{\pi/3} d\theta + 16\int_{\pi/3}^{\pi/2} \left(\cos^2\theta\right) d\theta$$

Exercise:

Problem: Region common to $r = 3\cos\theta$ and $r = 3\sin\theta$

For the following exercises, find the area of the described region.

Exercise:

Problem: Enclosed by $r = 6\sin\theta$

Solution:

9π

Exercise:

Problem: Above the polar axis enclosed by $r = 2 + \sin\theta$

Exercise:

Problem: Below the polar axis and enclosed by $r = 2 - \cos\theta$

Solution:

$\frac{9\pi}{4}$

Exercise:

Problem: Enclosed by one petal of $r = 4\cos(3\theta)$

Exercise:

Problem: Enclosed by one petal of $r = 3\cos(2\theta)$

Solution:

$\frac{9\pi}{8}$

Exercise:

Problem: Enclosed by $r = 1 + \sin\theta$

Exercise:

Problem: Enclosed by the inner loop of $r = 3 + 6\cos\theta$

Solution:

$\frac{18\pi - 27\sqrt{3}}{2}$

Exercise:

Problem: Enclosed by $r = 2 + 4\cos\theta$ and outside the inner loop

Exercise:

Problem: Common interior of $r = 4\sin(2\theta)$ and $r = 2$

Solution:

$\frac{4}{3}\left(4\pi - 3\sqrt{3}\right)$

Exercise:

Problem: Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$

Exercise:

Problem: Common interior of $r = 6\sin\theta$ and $r = 3$

Solution:

$\frac{3}{2}\left(4\pi - 3\sqrt{3}\right)$

Exercise:

Problem: Inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$

Exercise:

Problem: Common interior of $r = 2 + 2\cos\theta$ and $r = 2\sin\theta$

Solution:

$2\pi - 4$

For the following exercises, find a definite integral that represents the arc length.

Exercise:

Problem: $r = 4\cos\theta$ on the interval $0 \leq \theta \leq \frac{\pi}{2}$

Exercise:

Problem: $r = 1 + \sin\theta$ on the interval $0 \leq \theta \leq 2\pi$

Solution:

$$\int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta d\theta}$$

Exercise:

Problem: $r = 2\sec\theta$ on the interval $0 \le \theta \le \frac{\pi}{3}$

Exercise:

Problem: $r = e^{\theta}$on the interval $0 \le \theta \le 1$

Solution:

$$\sqrt{2}\int_0^1 e^{\theta} d\theta$$

For the following exercises, find the length of the curve over the given interval.

Exercise:

Problem: $r = 6$ on the interval $0 \le \theta \le \frac{\pi}{2}$

Exercise:

Problem: $r = e^{3\theta}$on the interval $0 \le \theta \le 2$

Solution:

$$\frac{\sqrt{10}}{3}\left(e^6 - 1\right)$$

Exercise:

Problem: $r = 6\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

Exercise:

Problem: $r = 8 + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Solution:

32

Exercise:

Problem: $r = 1 - \sin\theta$ on the interval $0 \leq \theta \leq 2\pi$

For the following exercises, use the integration capabilities of a calculator to approximate the length of the curve.

Exercise:

Problem: **[T]** $r = 3\theta$ on the interval $0 \leq \theta \leq \frac{\pi}{2}$

Solution:

6.238

Exercise:

Problem: **[T]** $r = \frac{2}{\theta}$ on the interval $\pi \leq \theta \leq 2\pi$

Exercise:

Problem: **[T]** $r = \sin^2\left(\frac{\theta}{2}\right)$ on the interval $0 \leq \theta \leq \pi$

Solution:

2

Exercise:

Problem: **[T]** $r = 2\theta^2$ on the interval $0 \leq \theta \leq \pi$

Exercise:

Problem: **[T]** $r = \sin(3\cos\theta)$ on the interval $0 \leq \theta \leq \pi$

Solution:

4.39

For the following exercises, use the familiar formula from geometry to find the area of the region described and then confirm by using the definite integral.

Exercise:

Problem: $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

Exercise:

Problem: $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

Solution:

$$A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2} \text{ and } \frac{1}{2}\int_0^{\pi}(1 + 2\sin\theta\cos\theta)d\theta = \frac{\pi}{2}$$

Exercise:

Problem: $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

For the following exercises, use the familiar formula from geometry to find the length of the curve and then confirm using the definite integral.

Exercise:

Problem: $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

Solution:

$$C = 2\pi\left(\frac{3}{2}\right) = 3\pi \text{ and } \int_0^{\pi} 3d\theta = 3\pi$$

Exercise:

Problem: $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

Exercise:

Problem: $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Solution:

$C = 2\pi\,(5) = 10\pi$ and $\int_0^{\pi} 10\,d\theta = 10\pi$

Exercise:

Problem:

Verify that if $y = r\sin\theta = f(\theta)\sin\theta$ then $\frac{dy}{d\theta} = f\prime(\theta)\sin\theta + f(\theta)\cos\theta$.

For the following exercises, find the slope of a tangent line to a polar curve $r = f\,(\theta)$. Let $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so the polar equation $r = f\,(\theta)$ is now written in parametric form.

Exercise:

Problem:

Use the definition of the derivative $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ and the product rule to derive the derivative of a polar equation.

Solution:

$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

Exercise:

Problem: $r = 1 - \sin\theta;\ \left(\frac{1}{2}, \frac{\pi}{6}\right)$

Exercise:

Problem: $r = 4\cos\theta;\ \left(2, \frac{\pi}{3}\right)$

Solution:

The slope is $\frac{1}{\sqrt{3}}$.

Exercise:

Problem: $r = 8\sin\theta;\ \left(4, \frac{5\pi}{6}\right)$

Exercise:

Problem: $r = 4 + \sin\theta$; $\left(3, \frac{3\pi}{2}\right)$

Solution:

The slope is 0.

Exercise:

Problem: $r = 6 + 3\cos\theta$; $(3, \pi)$

Exercise:

Problem: $r = 4\cos(2\theta)$; tips of the leaves

Solution:

At $(4, 0)$, the slope is undefined. At $\left(-4, \frac{\pi}{2}\right)$, the slope is 0.

Exercise:

Problem: $r = 2\sin(3\theta)$; tips of the leaves

Exercise:

Problem: $r = 2\theta$; $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

Solution:

The slope is undefined at $\theta = \frac{\pi}{4}$.

Exercise:

Problem:

Find the points on the interval $-\pi \leq \theta \leq \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.

Exercise:

Problem:

For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

Solution:

Slope = −1.

For the following exercises, find the slope of the tangent line to the given polar curve at the point given by the value of θ.

Exercise:

Problem: $r = 3\cos\theta, \theta = \frac{\pi}{3}$

Exercise:

Problem: $r = \theta, \theta = \frac{\pi}{2}$

Solution:

Slope is $\frac{-2}{\pi}$.

Exercise:

Problem: $r = \ln\theta, \theta = e$

Exercise:

Problem: **[T]** Use technology: $r = 2 + 4\cos\theta$ at $\theta = \frac{\pi}{6}$

Solution:

Calculator answer: −0.836.

For the following exercises, find the points at which the following polar curves have a horizontal or vertical tangent line.

Exercise:

Problem: $r = 4\cos\theta$

Exercise:

Problem: $r^2 = 4\cos(2\theta)$

Solution:

Horizontal tangent at $\left(\pm\sqrt{2}, \frac{\pi}{6}\right)$, $\left(\pm\sqrt{2}, -\frac{\pi}{6}\right)$.

Exercise:

Problem: $r = 2\sin(2\theta)$

Exercise:

Problem: The cardioid $r = 1 + \sin\theta$

Solution:

Horizontal tangents at $\frac{\pi}{2}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$. Vertical tangents at $\frac{\pi}{6}$, $\frac{5\pi}{6}$ and also at the pole $(0,0)$.

Exercise:

Problem:

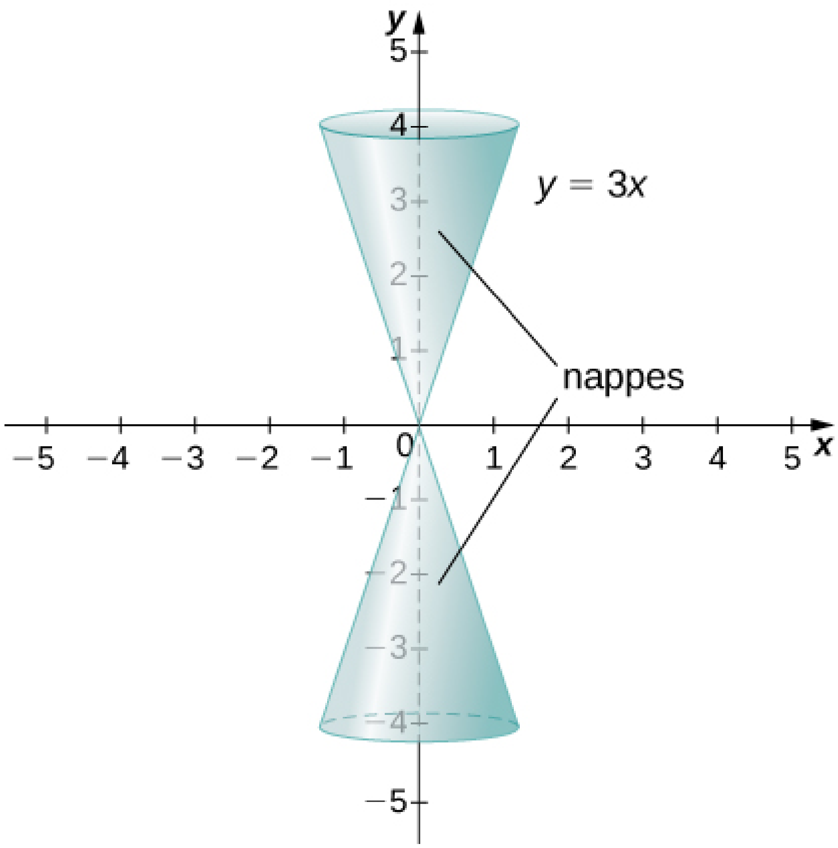
Show that the curve $r = \sin\theta\tan\theta$ (called a *cissoid of Diocles*) has the line $x = 1$ as a vertical asymptote.

Conic Sections

- Identify the equation of a parabola in standard form with given focus and directrix.
- Identify the equation of an ellipse in standard form with given foci.
- Identify the equation of a hyperbola in standard form with given foci.
- Recognize a parabola, ellipse, or hyperbola from its eccentricity value.
- Write the polar equation of a conic section with eccentricity e.
- Identify when a general equation of degree two is a parabola, ellipse, or hyperbola.

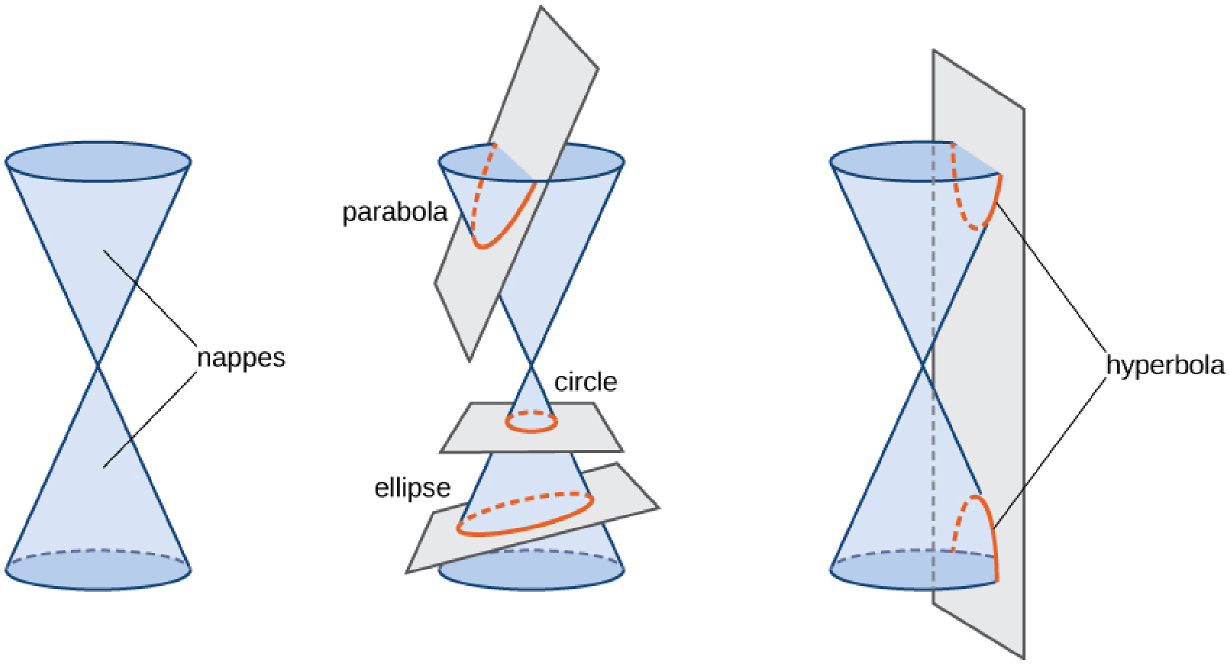
Conic sections have been studied since the time of the ancient Greeks, and were considered to be an important mathematical concept. As early as 320 BCE, such Greek mathematicians as Menaechmus, Appollonius, and Archimedes were fascinated by these curves. Appollonius wrote an entire eight-volume treatise on conic sections in which he was, for example, able to derive a specific method for identifying a conic section through the use of geometry. Since then, important applications of conic sections have arisen (for example, in astronomy), and the properties of conic sections are used in radio telescopes, satellite dish receivers, and even architecture. In this section we discuss the three basic conic sections, some of their properties, and their equations.

Conic sections get their name because they can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called **nappes**. One nappe is what most people mean by "cone," having the shape of a party hat. A right circular cone can be generated by revolving a line passing through the origin around the y-axis as shown.



A cone generated by revolving the line $y = 3x$ around the y-axis.

Conic sections are generated by the intersection of a plane with a cone ([link]). If the plane is parallel to the axis of revolution (the y-axis), then the **conic section** is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than 90°), then the conic section is an ellipse.



The four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone.

Parabolas

A parabola is generated when a plane intersects a cone parallel to the generating line. In this case, the plane intersects only one of the nappes. A parabola can also be defined in terms of distances.

> **Note:**
> Definition
> A parabola is the set of all points whose distance from a fixed point, called the **focus**, is equal to the distance from a fixed line, called the **directrix**. The point halfway between the focus and the directrix is called the **vertex** of the parabola.

A graph of a typical parabola appears in [link]. Using this diagram in conjunction with the distance formula, we can derive an equation for a parabola. Recall the distance formula: Given point P with coordinates (x_1, y_1) and point Q with coordinates $(x_2, \text{ y}_2)$, the distance between them is given by the formula

Equation:

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Then from the definition of a parabola and [link], we get

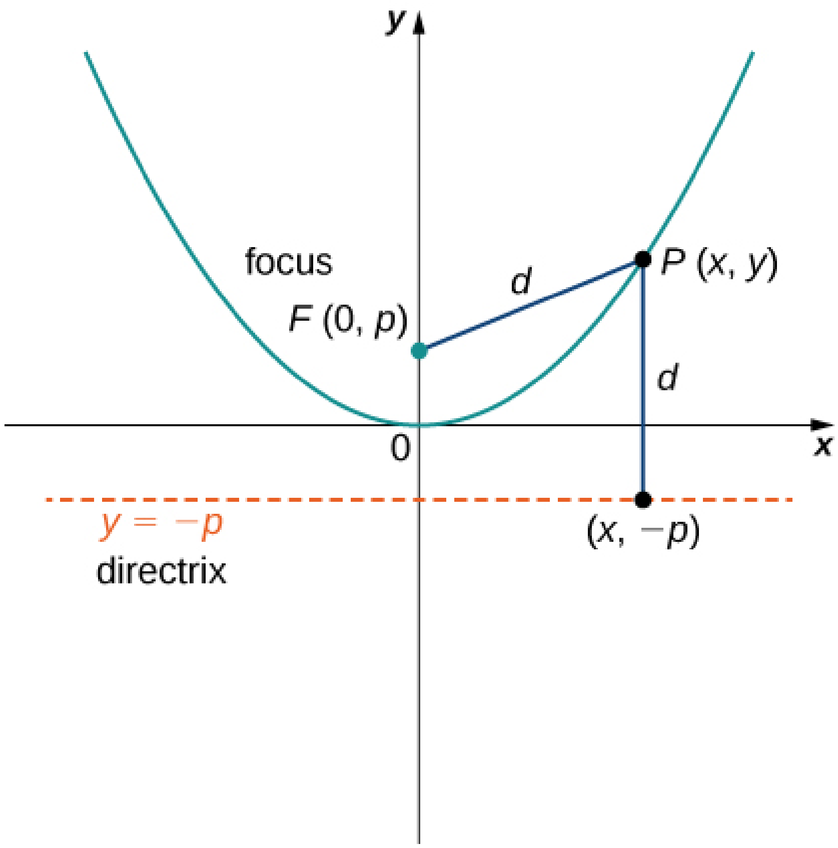
Equation:

$$\begin{array}{rcl} d(F, P) & = & d(P, Q) \\ \sqrt{(0 - x)^2 + (p - y)^2} & = & \sqrt{(x - x)^2 + (-p - y)^2}. \end{array}$$

Squaring both sides and simplifying yields

Equation:

$$\begin{array}{rcl} x^2 + (p - y)^2 & = & 0^2 + (-p - y)^2 \\ x^2 + p^2 - 2py + y^2 & = & p^2 + 2py + y^2 \\ x^2 - 2py & = & 2py \\ x^2 & = & 4py. \end{array}$$



A typical parabola in which the distance from the focus to the vertex is represented by the variable p.

Now suppose we want to relocate the vertex. We use the variables (h, k) to denote the coordinates of the vertex. Then if the focus is directly above the vertex, it has coordinates $(h, k+p)$ and the directrix has the equation $y = k - p$. Going through the same derivation yields the formula $(x-h)^2 = 4p\,(y-k)$. Solving this equation for y leads to the following theorem.

Note:
Equations for Parabolas

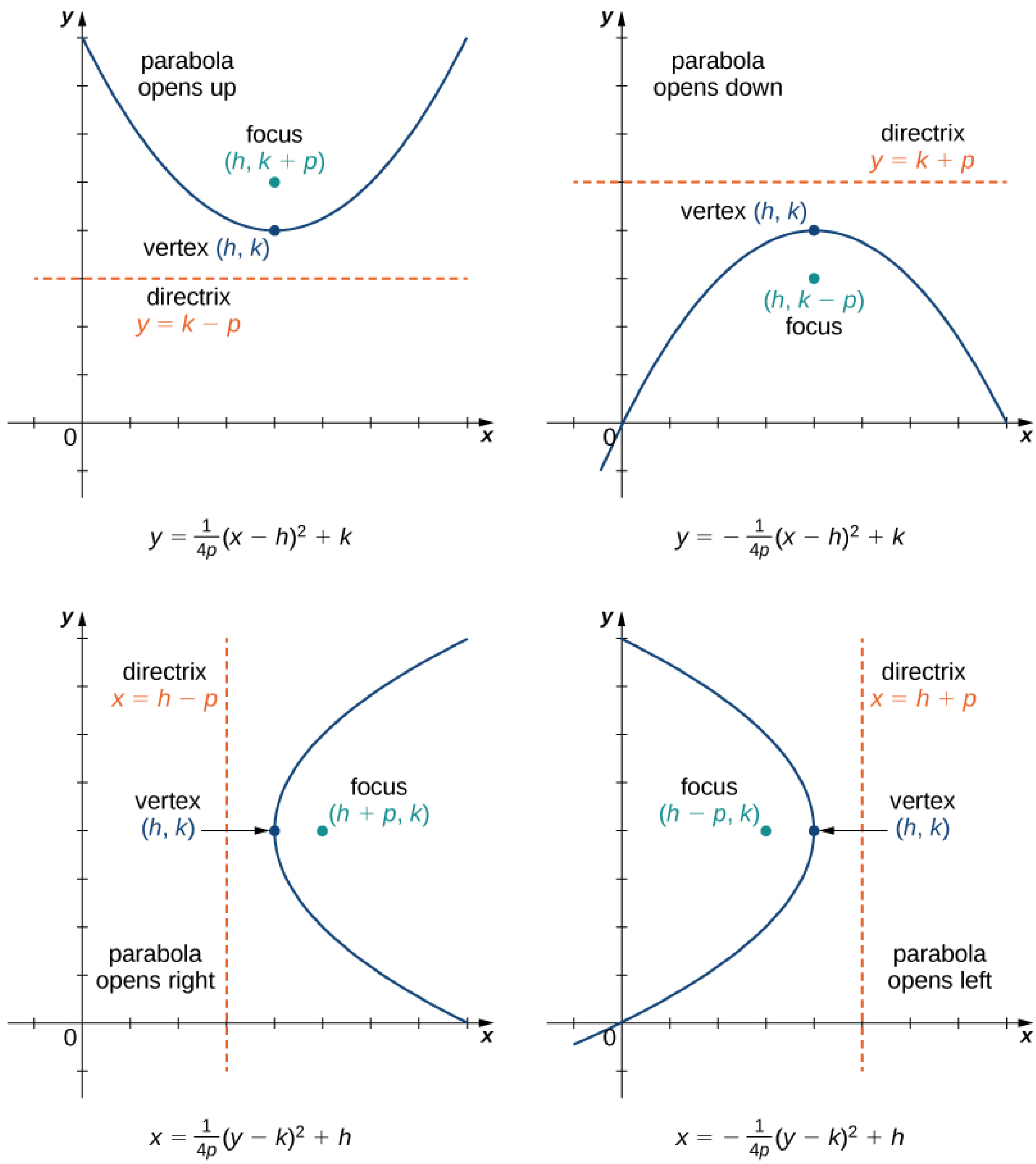
Given a parabola opening upward with vertex located at (h, k) and focus located at $(h, k+p)$, where p is a constant, the equation for the parabola is given by

Equation:

$$y = \frac{1}{4p}(x-h)^2 + k.$$

This is the **standard form** of a parabola.

We can also study the cases when the parabola opens down or to the left or the right. The equation for each of these cases can also be written in standard form as shown in the following graphs.



Four parabolas, opening in various directions, along with their equations in standard form.

In addition, the equation of a parabola can be written in the **general form**, though in this form the values of *h*, *k*, and *p* are not immediately recognizable. The general form of a parabola is written as
Equation:

$$ax^2 + bx + cy + d = 0 \quad \text{or} \quad ay^2 + bx + cy + d = 0.$$

The first equation represents a parabola that opens either up or down. The second equation represents a parabola that opens either to the left or to the right. To put the equation into standard form, use the method of completing the square.

Example:
Exercise:

Problem:
Converting the Equation of a Parabola from General into Standard Form

Put the equation $x^2 - 4x - 8y + 12 = 0$ into standard form and graph the resulting parabola.

Solution:

Since *y* is not squared in this equation, we know that the parabola opens either upward or downward. Therefore we need to solve this equation for *y*, which will put the equation into standard form. To do that, first add $8y$ to both sides of the equation:
Equation:

$$8y = x^2 - 4x + 12.$$

The next step is to complete the square on the right-hand side. Start by grouping the first two terms on the right-hand side using parentheses:
Equation:

$$8y = \left(x^2 - 4x\right) + 12.$$

Next determine the constant that, when added inside the parentheses, makes the quantity inside the parentheses a perfect square trinomial. To do this, take half the coefficient of *x* and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4.$ Add 4 inside the parentheses and subtract 4 outside the parentheses, so the value of the equation is not changed:

Equation:

$$8y = \left(x^2 - 4x + 4\right) + 12 - 4.$$

Now combine like terms and factor the quantity inside the parentheses:

Equation:

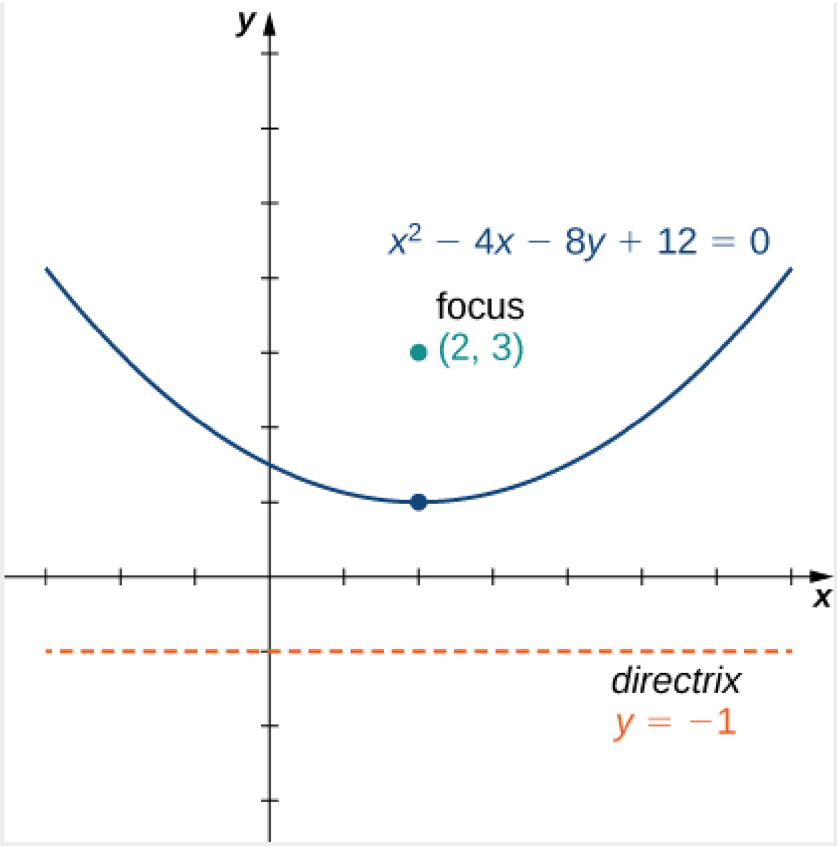
$$8y = (x - 2)^2 + 8.$$

Finally, divide by 8:

Equation:

$$y = \frac{1}{8}(x - 2)^2 + 1.$$

This equation is now in standard form. Comparing this to [link] gives $h = 2$, $k = 1$, and $p = 2$. The parabola opens up, with vertex at $(2, 1)$, focus at $(2, 3)$, and directrix $y = -1$. The graph of this parabola appears as follows.



The parabola in [link].

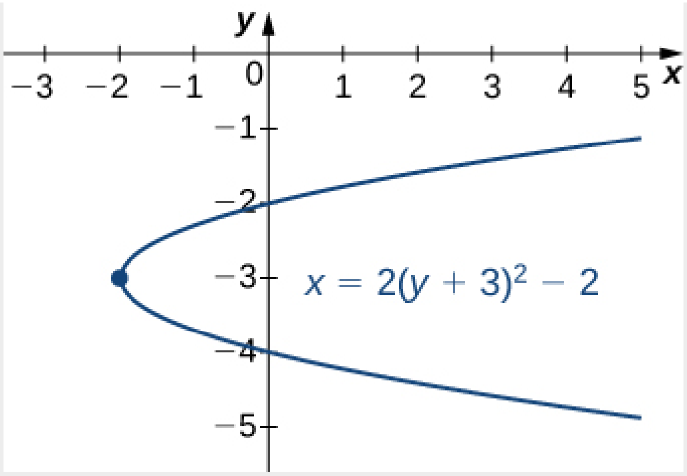
Note:
Exercise:

Problem:

Put the equation $2y^2 - x + 12y + 16 = 0$ into standard form and graph the resulting parabola.

Solution:

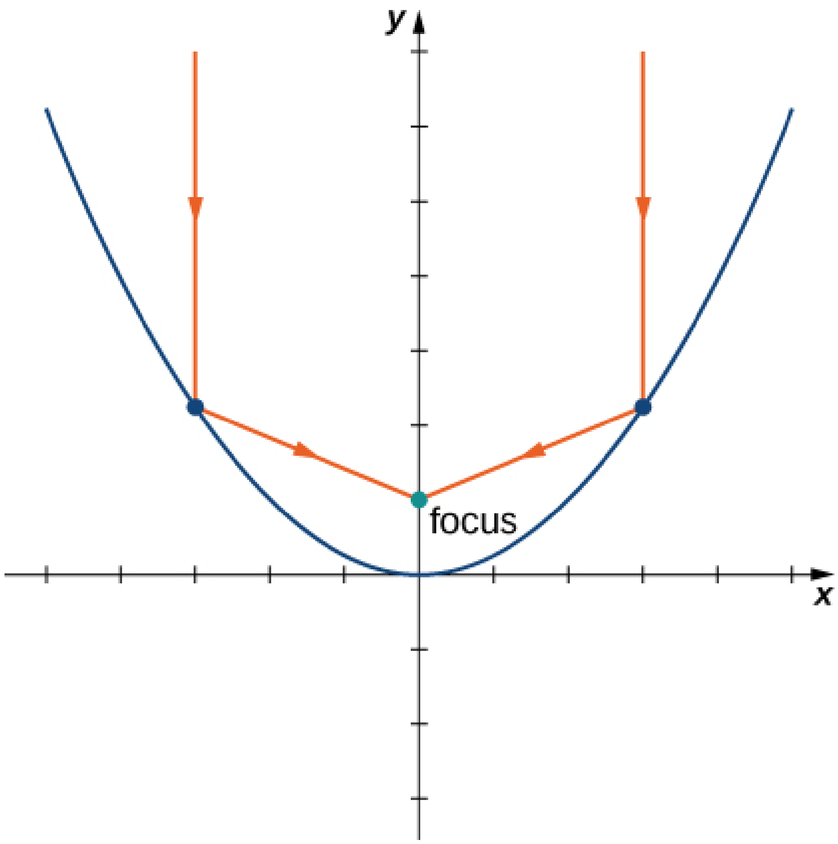
$x = 2(y+3)^2 - 2$



Hint

Solve for *x*. Check which direction the parabola opens.

The axis of symmetry of a vertical (opening up or down) parabola is a vertical line passing through the vertex. The parabola has an interesting reflective property. Suppose we have a satellite dish with a parabolic cross section. If a beam of electromagnetic waves, such as light or radio waves, comes into the dish in a straight line from a satellite (parallel to the axis of symmetry), then the waves reflect off the dish and collect at the focus of the parabola as shown.

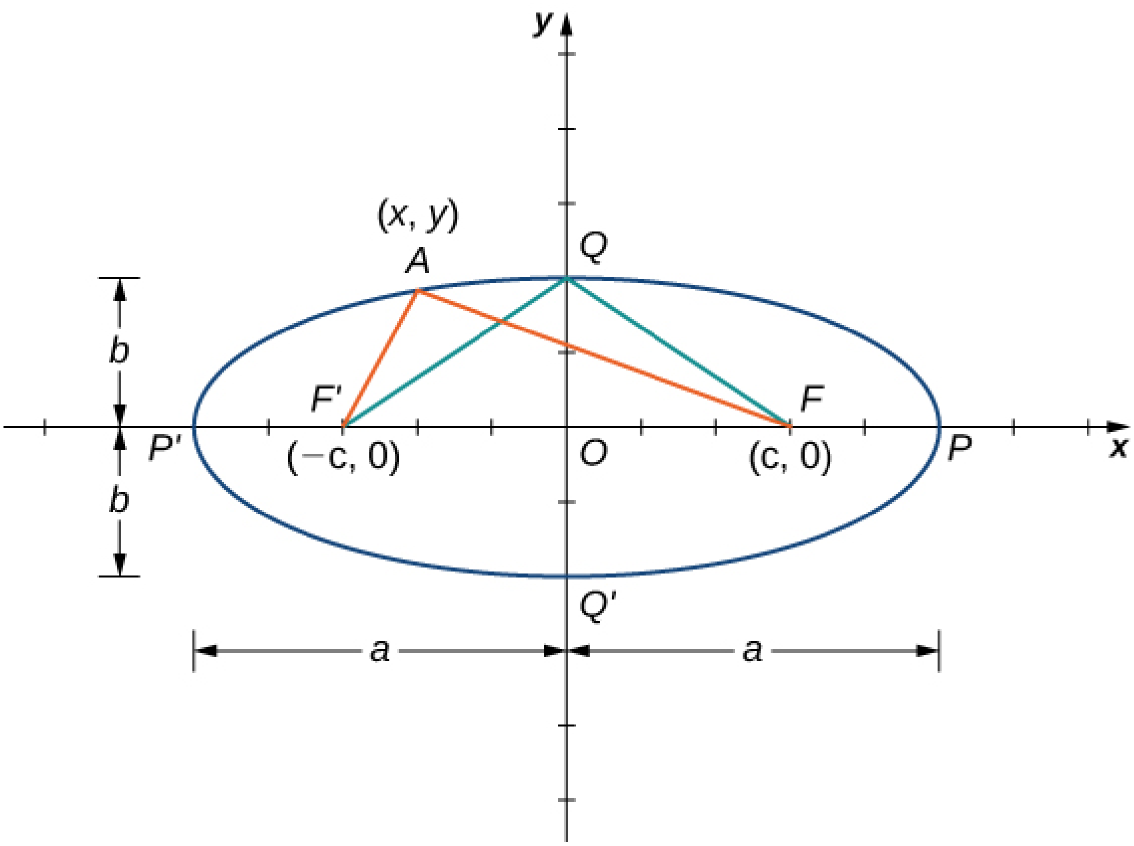


Consider a parabolic dish designed to collect signals from a satellite in space. The dish is aimed directly at the satellite, and a receiver is located at the focus of the parabola. Radio waves coming in from the satellite are reflected off the surface of the parabola to the receiver, which collects and decodes the digital signals. This allows a small receiver to gather signals from a wide angle of sky. Flashlights and headlights in a car work on the same principle, but in reverse: the source of the light (that is, the light bulb) is located at the focus and the reflecting surface on the parabolic mirror focuses the beam straight ahead. This allows a small light bulb to illuminate a wide angle of space in front of the flashlight or car.

Ellipses

An ellipse can also be defined in terms of distances. In the case of an ellipse, there are two foci (plural of focus), and two directrices (plural of directrix). We look at the directrices in more detail later in this section.

Note:
Definition
An *ellipse* is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.



A typical ellipse in which the sum of the distances from any point on the ellipse to the foci is constant.

A graph of a typical ellipse is shown in [link]. In this figure the foci are labeled as F and F'. Both are the same fixed distance from the origin, and this distance is represented by the variable c. Therefore the coordinates of F are $(c, 0)$ and the coordinates of F' are $(-c, 0)$. The points P and P' are located at the ends of the **major axis** of the ellipse, and have coordinates $(a, 0)$ and $(-a, 0)$, respectively. The major axis is always the longest

distance across the ellipse, and can be horizontal or vertical. Thus, the length of the major axis in this ellipse is $2a$. Furthermore, P and P' are called the vertices of the ellipse. The points Q and Q' are located at the ends of the **minor axis** of the ellipse, and have coordinates $(0, b)$ and $(0, -b)$, respectively. The minor axis is the shortest distance across the ellipse. The minor axis is perpendicular to the major axis.

According to the definition of the ellipse, we can choose any point on the ellipse and the sum of the distances from this point to the two foci is constant. Suppose we choose the point *P*. Since the coordinates of point *P* are $(a, 0)$, the sum of the distances is
Equation:

$$d(P, F) + d\left(P, F'\right) = (a - c) + (a + c) = 2a.$$

Therefore the sum of the distances from an arbitrary point *A* with coordinates (x, y) is also equal to $2a$. Using the distance formula, we get
Equation:

$$\begin{aligned} d(A, F) + d(A, F') &= 2a \\ \sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} &= 2a. \end{aligned}$$

Subtract the second radical from both sides and square both sides:
Equation:

$$\begin{aligned} \sqrt{(x - c)^2 + y^2} &= 2a - \sqrt{(x + c)^2 + y^2} \\ (x - c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2 \\ x^2 - 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ -2cx &= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx. \end{aligned}$$

Now isolate the radical on the right-hand side and square again:

Equation:

$$\begin{aligned} -2cx &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx \\ 4a\sqrt{(x+c)^2 + y^2} &= 4a^2 + 4cx \\ \sqrt{(x+c)^2 + y^2} &= a + \tfrac{cx}{a} \\ (x+c)^2 + y^2 &= a^2 + 2cx + \tfrac{c^2x^2}{a^2} \\ x^2 + 2cx + c^2 + y^2 &= a^2 + 2cx + \tfrac{c^2x^2}{a^2} \\ x^2 + c^2 + y^2 &= a^2 + \tfrac{c^2x^2}{a^2}. \end{aligned}$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

Equation:

$$\begin{aligned} x^2 - \tfrac{c^2x^2}{a^2} + y^2 &= a^2 - c^2 \\ \tfrac{(a^2-c^2)x^2}{a^2} + y^2 &= a^2 - c^2. \end{aligned}$$

Divide both sides by $a^2 - c^2$. This gives the equation

Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we refer back to [link], then the length of each of the two green line segments is equal to a. This is true because the sum of the distances from the point Q to the foci F and F' is equal to $2a$, and the lengths of these two line segments are equal. This line segment forms a right triangle with hypotenuse length a and leg lengths b and c. From the Pythagorean theorem, $a^2 + b^2 = c^2$ and $b^2 = a^2 - c^2$. Therefore the equation of the ellipse becomes

Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if the center of the ellipse is moved from the origin to a point (h, k), we have the following standard form of an ellipse.

> **Note:**
> Equation of an Ellipse in Standard Form
> Consider the ellipse with center (h, k), a horizontal major axis with length $2a$, and a vertical minor axis with length $2b$. Then the equation of this ellipse in standard form is
> **Equation:**
>
> $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
>
> and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 - b^2$. The equations of the directrices are $x = h \pm \frac{a^2}{c}$.
> If the major axis is vertical, then the equation of the ellipse becomes
> **Equation:**
>
> $$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$
>
> and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 - b^2$. The equations of the directrices in this case are $y = k \pm \frac{a^2}{c}$.

If the major axis is horizontal, then the ellipse is called horizontal, and if the major axis is vertical, then the ellipse is called vertical. The equation of an ellipse is in general form if it is in the form

$Ax^2 + By^2 + Cx + Dy + E = 0$, where A and B are either both positive or both negative. To convert the equation from general to standard form, use the method of completing the square.

Example:
Exercise:

Problem:
Finding the Standard Form of an Ellipse

Put the equation $9x^2 + 4y^2 - 36x + 24y + 36 = 0$ into standard form and graph the resulting ellipse.

Solution:

First subtract 36 from both sides of the equation:
Equation:

$$9x^2 + 4y^2 - 36x + 24y = -36.$$

Next group the *x* terms together and the *y* terms together, and factor out the common factor:
Equation:

$$\begin{aligned} \left(9x^2 - 36x\right) + \left(4y^2 + 24y\right) &= -36 \\ 9\left(x^2 - 4x\right) + 4\left(y^2 + 6y\right) &= -36. \end{aligned}$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of *x* and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4$. In the second set of parentheses, take half the coefficient of *y* and square it. This gives $\left(\frac{6}{2}\right)^2 = 9$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually

adding 36 to the left-hand side. Similarly, we are adding 36 to the second set as well. Therefore the equation becomes

Equation:

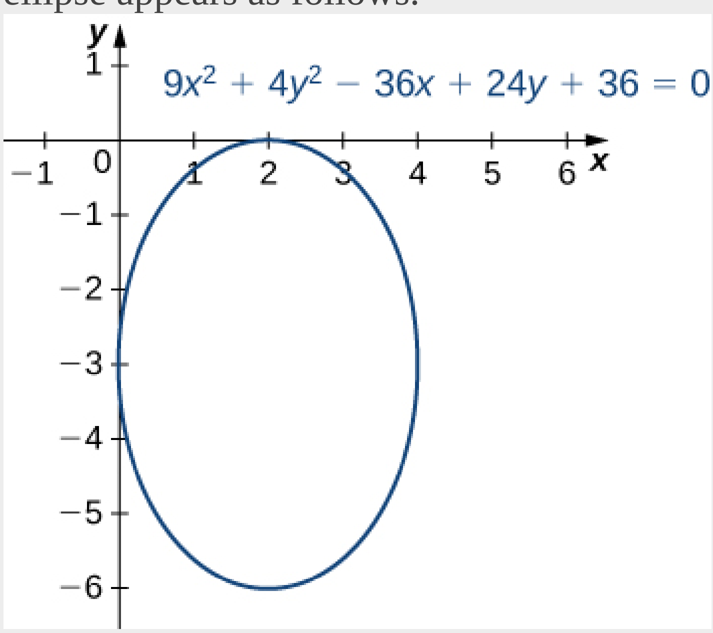
$$\begin{aligned} 9\left(x^2-4x+4\right)+4\left(y^2+6y+9\right) &= -36+36+36 \\ 9\left(x^2-4x+4\right)+4\left(y^2+6y+9\right) &= 36. \end{aligned}$$

Now factor both sets of parentheses and divide by 36:

Equation:

$$\begin{aligned} 9(x-2)^2+4(y+3)^2 &= 36 \\ \frac{9(x-2)^2}{36}+\frac{4(y+3)^2}{36} &= 1 \\ \frac{(x-2)^2}{4}+\frac{(y+3)^2}{9} &= 1. \end{aligned}$$

The equation is now in standard form. Comparing this to [link] gives $h=2, k=-3, a=3$, and $b=2$. This is a vertical ellipse with center at $(2,-3)$, major axis 6, and minor axis 4. The graph of this ellipse appears as follows.



The ellipse in [link].

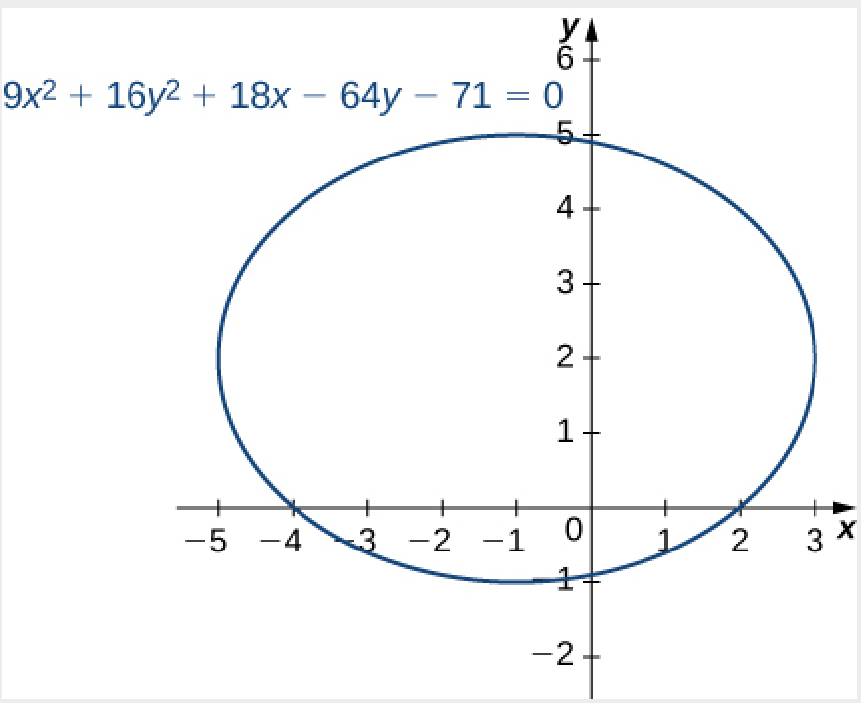
Note:
Exercise:

Problem:

Put the equation $9x^2 + 16y^2 + 18x - 64y - 71 = 0$ into standard form and graph the resulting ellipse.

Solution:

$\frac{(x+1)^2}{16} + \frac{(y-2)^2}{9} = 1$

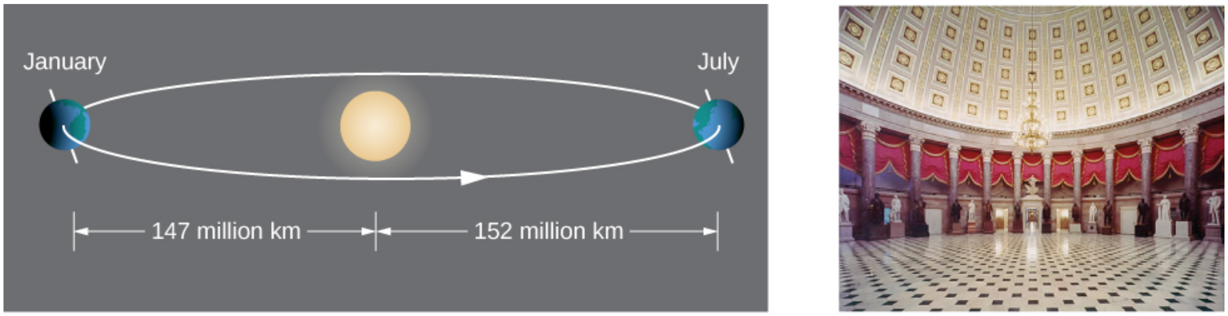


Hint

Move the constant over and complete the square.

According to Kepler’s first law of planetary motion, the orbit of a planet around the Sun is an ellipse with the Sun at one of the foci as shown in [link](a). Because Earth’s orbit is an ellipse, the distance from the Sun varies throughout the year. A commonly held misconception is that Earth is closer to the Sun in the summer. In fact, in summer for the northern hemisphere, Earth is farther from the Sun than during winter. The difference in season is caused by the tilt of Earth’s axis in the orbital plane. Comets that orbit the Sun, such as Halley’s Comet, also have elliptical orbits, as do moons orbiting the planets and satellites orbiting Earth.

Ellipses also have interesting reflective properties: A light ray emanating from one focus passes through the other focus after mirror reflection in the ellipse. The same thing occurs with a sound wave as well. The National Statuary Hall in the U.S. Capitol in Washington, DC, is a famous room in an elliptical shape as shown in [link](b). This hall served as the meeting place for the U.S. House of Representatives for almost fifty years. The location of the two foci of this semi-elliptical room are clearly identified by marks on the floor, and even if the room is full of visitors, when two people stand on these spots and speak to each other, they can hear each other much more clearly than they can hear someone standing close by. Legend has it that John Quincy Adams had his desk located on one of the foci and was able to eavesdrop on everyone else in the House without ever needing to stand. Although this makes a good story, it is unlikely to be true, because the original ceiling produced so many echoes that the entire room had to be hung with carpets to dampen the noise. The ceiling was rebuilt in 1902 and only then did the now-famous whispering effect emerge. Another famous whispering gallery—the site of many marriage proposals—is in Grand Central Station in New York City.



(a) (b)

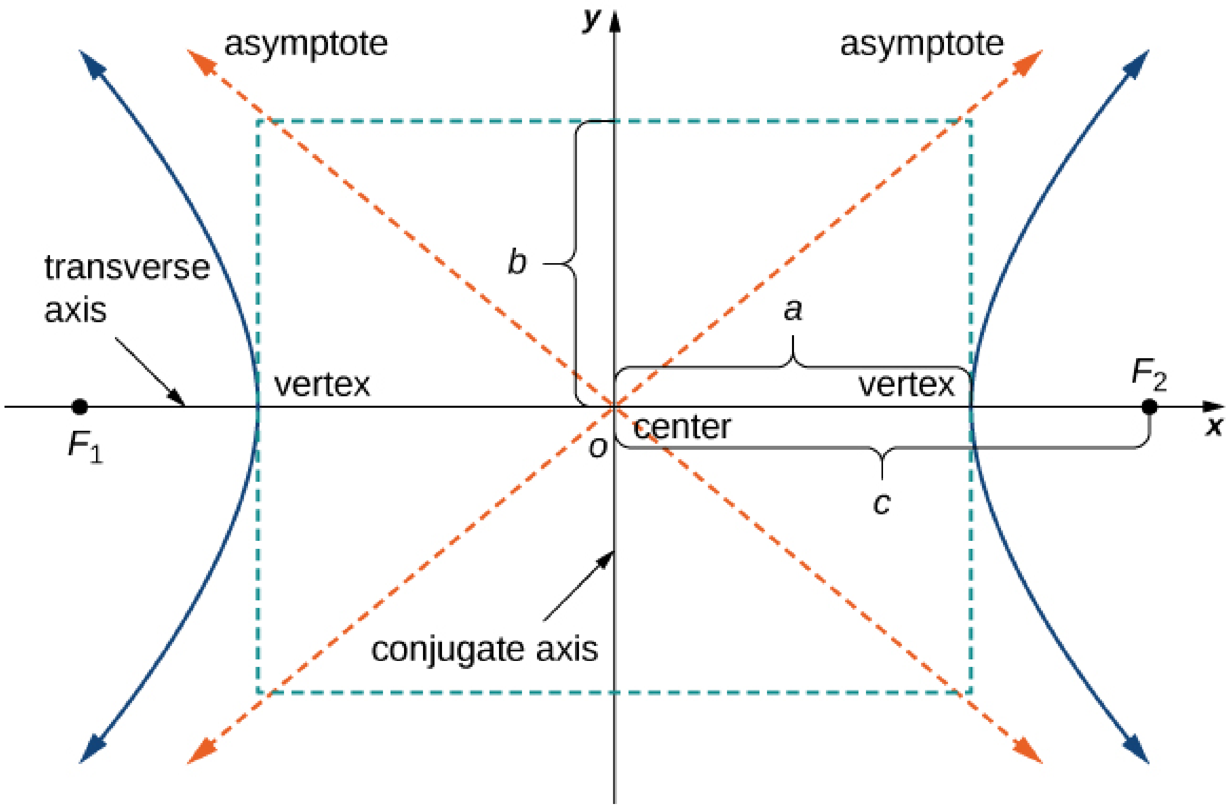
(a) Earth’s orbit around the Sun is an ellipse with the Sun at one focus. (b) Statuary Hall in the U.S. Capitol is a whispering gallery with an elliptical cross section.

Hyperbolas

A hyperbola can also be defined in terms of distances. In the case of a hyperbola, there are two foci and two directrices. Hyperbolas also have two asymptotes.

> **Note:**
> Definition
> A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

A graph of a typical hyperbola appears as follows.



A typical hyperbola in which the difference of the distances from any point on the ellipse to the foci is constant. The transverse axis is also called the major axis, and the conjugate axis is also called the minor axis.

The derivation of the equation of a hyperbola in standard form is virtually identical to that of an ellipse. One slight hitch lies in the definition: The difference between two numbers is always positive. Let P be a point on the hyperbola with coordinates (x, y). Then the definition of the hyperbola gives $|d(P, F_1) - d(P, F_2)| = \text{constant}$. To simplify the derivation, assume that P is on the right branch of the hyperbola, so the absolute value bars drop. If it is on the left branch, then the subtraction is reversed. The vertex of the right branch has coordinates $(a, 0)$, so

Equation:

$$d(P, F_1) - d(P, F_2) = (c + a) - (c - a) = 2a.$$

This equation is therefore true for any point on the hyperbola. Returning to the coordinates (x, y) for P:

Equation:

$$\begin{aligned} d\left(P, F_1\right) - d\left(P, F_2\right) &= 2a \\ \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a. \end{aligned}$$

Add the second radical from both sides and square both sides:

Equation:

$$\begin{aligned} \sqrt{(x-c)^2 + y^2} &= 2a + \sqrt{(x+c)^2 + y^2} \\ (x-c)^2 + y^2 &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\ x^2 - 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ -2cx &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + 2cx. \end{aligned}$$

Now isolate the radical on the right-hand side and square again:

Equation:

$$\begin{aligned} -2cx &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + 2cx \\ 4a\sqrt{(x+c)^2 + y^2} &= -4a^2 - 4cx \\ \sqrt{(x+c)^2 + y^2} &= -a - \tfrac{cx}{a} \\ (x+c)^2 + y^2 &= a^2 + 2cx + \tfrac{c^2x^2}{a^2} \\ x^2 + 2cx + c^2 + y^2 &= a^2 + 2cx + \tfrac{c^2x^2}{a^2} \\ x^2 + c^2 + y^2 &= a^2 + \tfrac{c^2x^2}{a^2}. \end{aligned}$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:
Equation:

$$\begin{aligned} x^2 - \frac{c^2x^2}{a^2} + y^2 &= a^2 - c^2 \\ \frac{(a^2-c^2)x^2}{a^2} + y^2 &= a^2 - c^2. \end{aligned}$$

Finally, divide both sides by $a^2 - c^2$. This gives the equation
Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

We now define b so that $b^2 = c^2 - a^2$. This is possible because $c > a$. Therefore the equation of the ellipse becomes
Equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Finally, if the center of the hyperbola is moved from the origin to the point (h, k), we have the following standard form of a hyperbola.

Note:
Equation of a Hyperbola in Standard Form
Consider the hyperbola with center (h, k), a horizontal major axis, and a vertical minor axis. Then the equation of this ellipse is
Equation:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{b}{a}(x - h)$. The equations of the directrices are

Equation:

$$x = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = h \pm \frac{a^2}{c}.$$

If the major axis is vertical, then the equation of the hyperbola becomes

Equation:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{a}{b}(x - h)$. The equations of the directrices are

Equation:

$$y = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = k \pm \frac{a^2}{c}.$$

If the major axis (transverse axis) is horizontal, then the hyperbola is called horizontal, and if the major axis is vertical then the hyperbola is called vertical. The equation of a hyperbola is in general form if it is in the form $Ax^2 + By^2 + Cx + Dy + E = 0$, where *A* and *B* have opposite signs. In order to convert the equation from general to standard form, use the method of completing the square.

Example:

Exercise:

Problem:

Finding the Standard Form of a Hyperbola

Put the equation $9x^2 - 16y^2 + 36x + 32y - 124 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution:

First add 124 to both sides of the equation:
Equation:

$$9x^2 - 16y^2 + 36x + 32y = 124.$$

Next group the *x* terms together and the *y* terms together, then factor out the common factors:
Equation:

$$\begin{aligned} \left(9x^2 + 36x\right) - \left(16y^2 - 32y\right) &= 124 \\ 9\left(x^2 + 4x\right) - 16\left(y^2 - 2y\right) &= 124. \end{aligned}$$

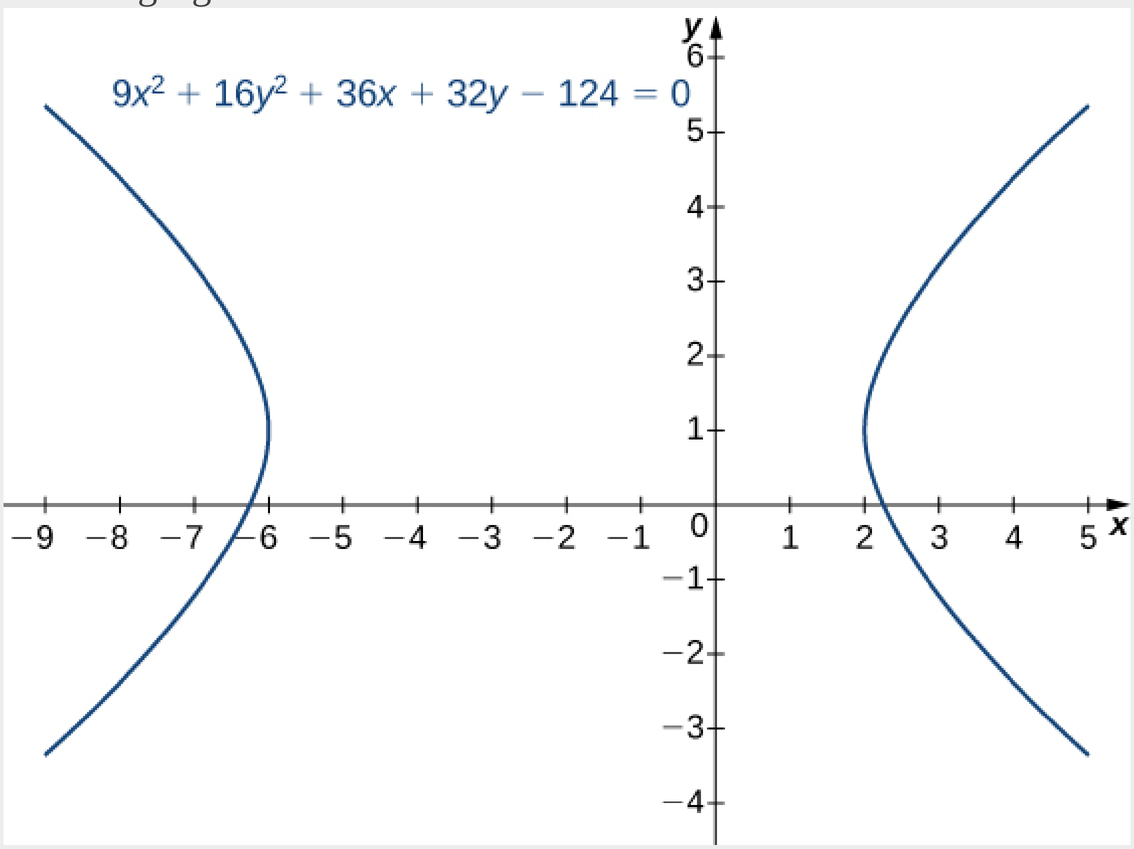
We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of *x* and square it. This gives $\left(\frac{4}{2}\right)^2 = 4$. In the second set of parentheses, take half the coefficient of *y* and square it. This gives $\left(\frac{-2}{2}\right)^2 = 1$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are subtracting 16 from the second set of parentheses. Therefore the equation becomes
Equation:

$$\begin{aligned} 9\left(x^2 + 4x + 4\right) - 16\left(y^2 - 2y + 1\right) &= 124 + 36 - 16 \\ 9\left(x^2 + 4x + 4\right) - 16\left(y^2 - 2y + 1\right) &= 144. \end{aligned}$$

Next factor both sets of parentheses and divide by 144:
Equation:

$$\begin{array}{rcl} 9(x+2)^2 - 16(y-1)^2 & = & 144 \\ \frac{9(x+2)^2}{144} - \frac{16(y-1)^2}{144} & = & 1 \\ \frac{(x+2)^2}{16} - \frac{(y-1)^2}{9} & = & 1. \end{array}$$

The equation is now in standard form. Comparing this to [link] gives $h = -2$, $k = 1$, $a = 4$, and $b = 3$. This is a horizontal hyperbola with center at $(-2, 1)$ and asymptotes given by the equations $y = 1 \pm \frac{3}{4}(x+2)$. The graph of this hyperbola appears in the following figure.



Graph of the hyperbola in [link].

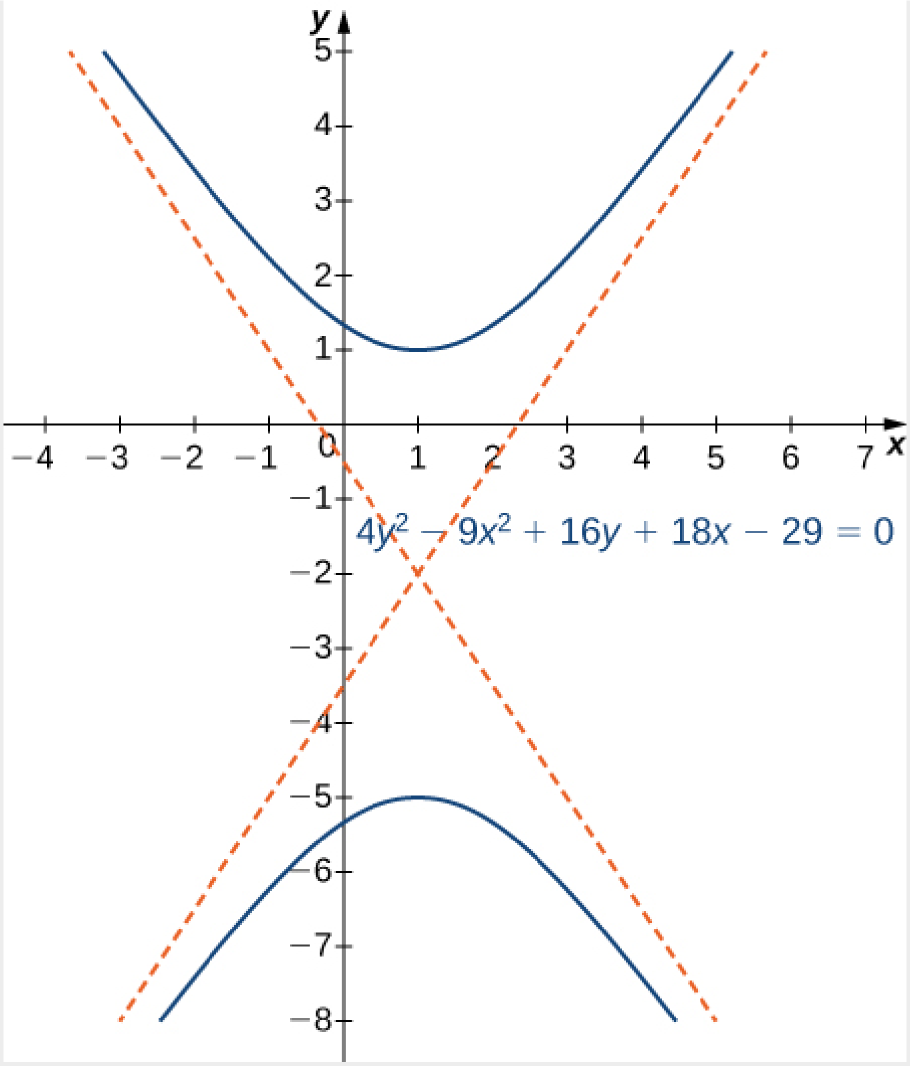
Note:
Exercise:

Problem:

Put the equation $4y^2 - 9x^2 + 16y + 18x - 29 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution:

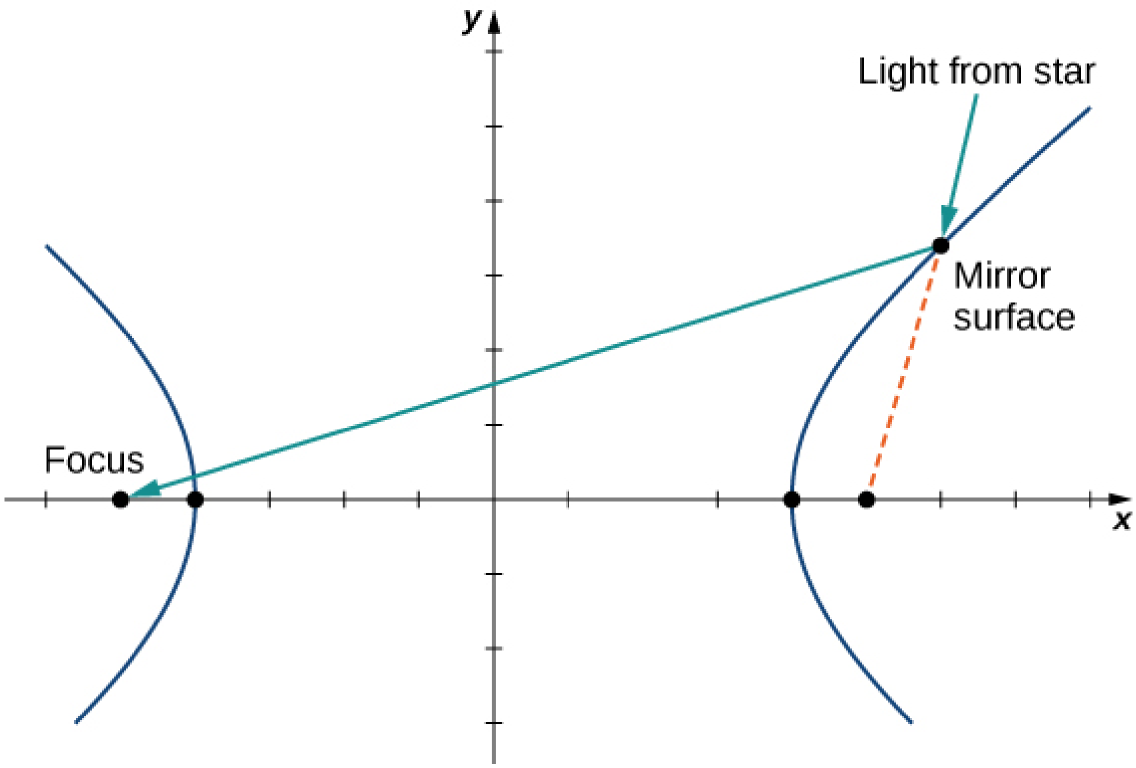
$\frac{(y+2)^2}{9} - \frac{(x-1)^2}{4} = 1$. This is a vertical hyperbola. Asymptotes $y = -2 \pm \frac{3}{2}(x-1)$.



Hint

Move the constant over and complete the square. Check which direction the hyperbola opens.

Hyperbolas also have interesting reflective properties. A ray directed toward one focus of a hyperbola is reflected by a hyperbolic mirror toward the other focus. This concept is illustrated in the following figure.



A hyperbolic mirror used to collect light from distant stars.

This property of the hyperbola has important applications. It is used in radio direction finding (since the difference in signals from two towers is constant along hyperbolas), and in the construction of mirrors inside telescopes (to reflect light coming from the parabolic mirror to the eyepiece). Another interesting fact about hyperbolas is that for a comet entering the solar system, if the speed is great enough to escape the Sun’s gravitational pull, then the path that the comet takes as it passes through the solar system is hyperbolic.

Eccentricity and Directrix

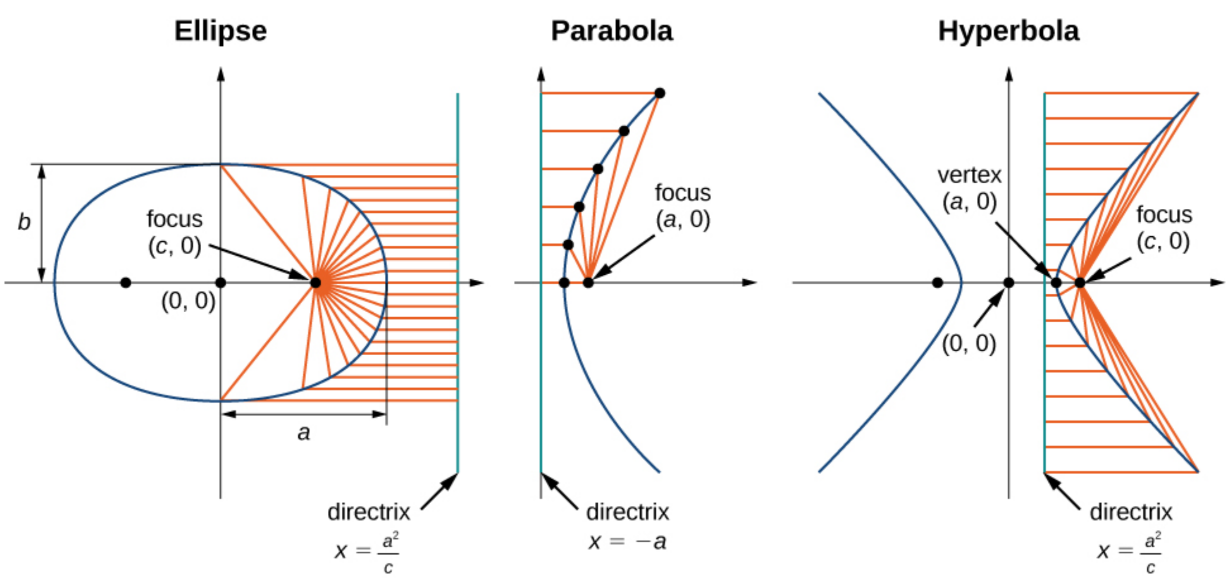
An alternative way to describe a conic section involves the directrices, the foci, and a new property called eccentricity. We will see that the value of the eccentricity of a conic section can uniquely define that conic.

Note:
Definition
The **eccentricity** e of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define the conic section as well:

1. If $e = 1$, the conic is a parabola.
2. If $e < 1$, it is an ellipse.
3. If $e > 1$, it is a hyperbola.

The eccentricity of a circle is zero. The directrix of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.

The three conic sections with their directrices appear in the following figure.



The three conic sections with their foci and directrices.

Recall from the definition of a parabola that the distance from any point on the parabola to the focus is equal to the distance from that same point to the directrix. Therefore, by definition, the eccentricity of a parabola must be 1. The equations of the directrices of a horizontal ellipse are $x = \pm\frac{a^2}{c}$. The right vertex of the ellipse is located at $(a, 0)$ and the right focus is $(c, 0)$. Therefore the distance from the vertex to the focus is $a - c$ and the distance from the vertex to the right directrix is $\frac{a^2}{c} - c$. This gives the eccentricity as

Equation:

$$e = \frac{a-c}{\frac{a^2}{c} - a} = \frac{c\left(a-c\right)}{a^2 - ac} = \frac{c\left(a-c\right)}{a\left(a-c\right)} = \frac{c}{a}.$$

Since $c < a$, this step proves that the eccentricity of an ellipse is less than 1. The directrices of a horizontal hyperbola are also located at $x = \pm\frac{a^2}{c}$, and a similar calculation shows that the eccentricity of a hyperbola is also $e = \frac{c}{a}$. However in this case we have $c > a$, so the eccentricity of a hyperbola is greater than 1.

Example:
Exercise:

Problem:
Determining Eccentricity of a Conic Section

Determine the eccentricity of the ellipse described by the equation

Equation:

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1.$$

Solution:

From the equation we see that $a = 5$ and $b = 4$. The value of c can be calculated using the equation $a^2 = b^2 + c^2$ for an ellipse. Substituting the values of a and b and solving for c gives $c = 3$. Therefore the eccentricity of the ellipse is $e = \frac{c}{a} = \frac{3}{5} = 0.6$.

Note:
Exercise:

Problem:

Determine the eccentricity of the hyperbola described by the equation
Equation:

$$\frac{(y-3)^2}{49} - \frac{(x+2)^2}{25} = 1.$$

Solution:

$e = \frac{c}{a} = \frac{\sqrt{74}}{7} \approx 1.229$

Hint

First find the values of a and b, then determine c using the equation $c^2 = a^2 + b^2$.

Polar Equations of Conic Sections

Sometimes it is useful to write or identify the equation of a conic section in polar form. To do this, we need the concept of the focal parameter. The **focal parameter** of a conic section p is defined as the distance from a focus to the nearest directrix. The following table gives the focal parameters for the different types of conics, where a is the length of the semi-major axis (i.e., half the length of the major axis), c is the distance from the origin to the

focus, and *e* is the eccentricity. In the case of a parabola, *a* represents the distance from the vertex to the focus.

Conic	***e***	***p***
Ellipse	$0 < e < 1$	$\frac{a^2 - c^2}{c} = \frac{a\left(1 - e^2\right)}{c}$
Parabola	$e = 1$	$2a$
Hyperbola	$e > 1$	$\frac{c^2 - a^2}{c} = \frac{a\left(e^2 - 1\right)}{e}$

Eccentricities and Focal Parameters of the Conic Sections

Using the definitions of the focal parameter and eccentricity of the conic section, we can derive an equation for any conic section in polar coordinates. In particular, we assume that one of the foci of a given conic section lies at the pole. Then using the definition of the various conic sections in terms of distances, it is possible to prove the following theorem.

Note:
Polar Equation of Conic Sections
The polar equation of a conic section with focal parameter *p* is given by
Equation:

$$r = \frac{ep}{1 \pm e\cos\theta} \text{ or } r = \frac{ep}{1 \pm e\sin\theta}.$$

In the equation on the left, the major axis of the conic section is horizontal, and in the equation on the right, the major axis is vertical. To work with a conic section written in polar form, first make the constant term in the denominator equal to 1. This can be done by dividing both the numerator and the denominator of the fraction by the constant that appears in front of the plus or minus in the denominator. Then the coefficient of the sine or cosine in the denominator is the eccentricity. This value identifies the conic. If cosine appears in the denominator, then the conic is horizontal. If sine appears, then the conic is vertical. If both appear then the axes are rotated. The center of the conic is not necessarily at the origin. The center is at the origin only if the conic is a circle (i.e., $e = 0$).

Example:
Exercise:

Problem:
Graphing a Conic Section in Polar Coordinates

Identify and create a graph of the conic section described by the equation

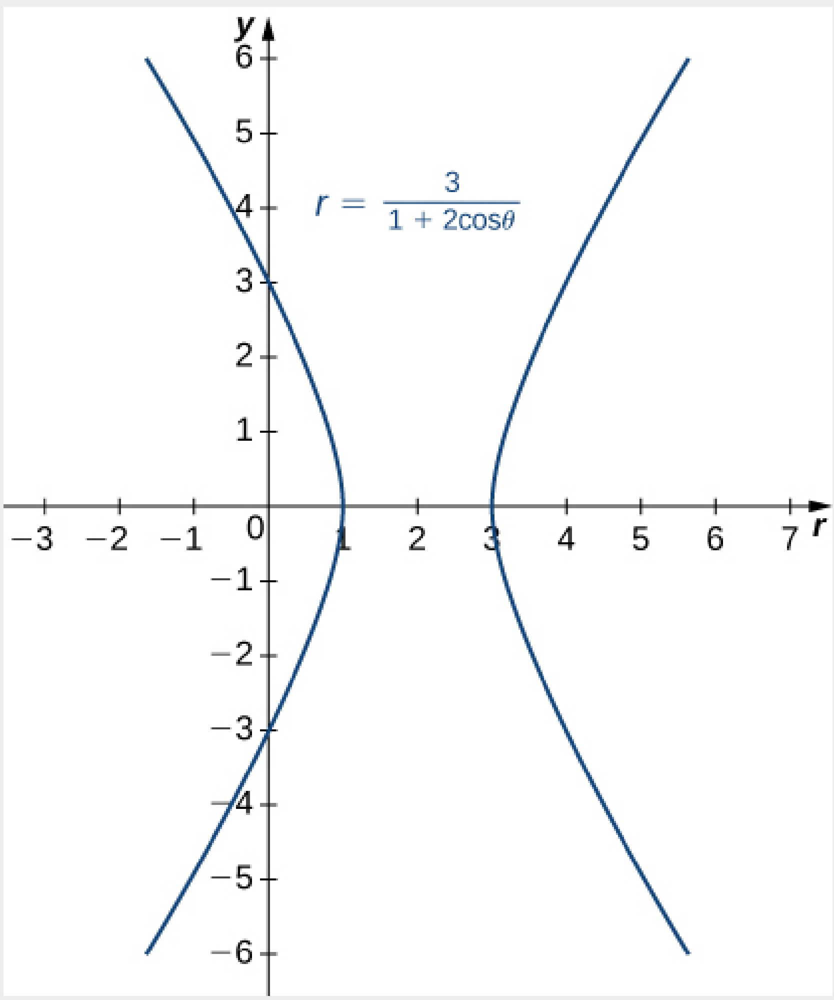
Equation:

$$r = \frac{3}{1 + 2\cos\theta}.$$

Solution:

The constant term in the denominator is 1, so the eccentricity of the conic is 2. This is a hyperbola. The focal parameter p can be calculated by using the equation $ep = 3$. Since $e = 2$, this gives $p = \frac{3}{2}$. The cosine function appears in the denominator, so the hyperbola is horizontal. Pick a few values for θ and create a table of values. Then we can graph the hyperbola ([link]).

θ	r	θ	r
0	1	π	-3
$\frac{\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$	$\frac{5\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$
$\frac{\pi}{2}$	3	$\frac{3\pi}{2}$	3
$\frac{3\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$	$\frac{7\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$



Graph of the hyperbola described in [link].

Note:
Exercise:

Problem:

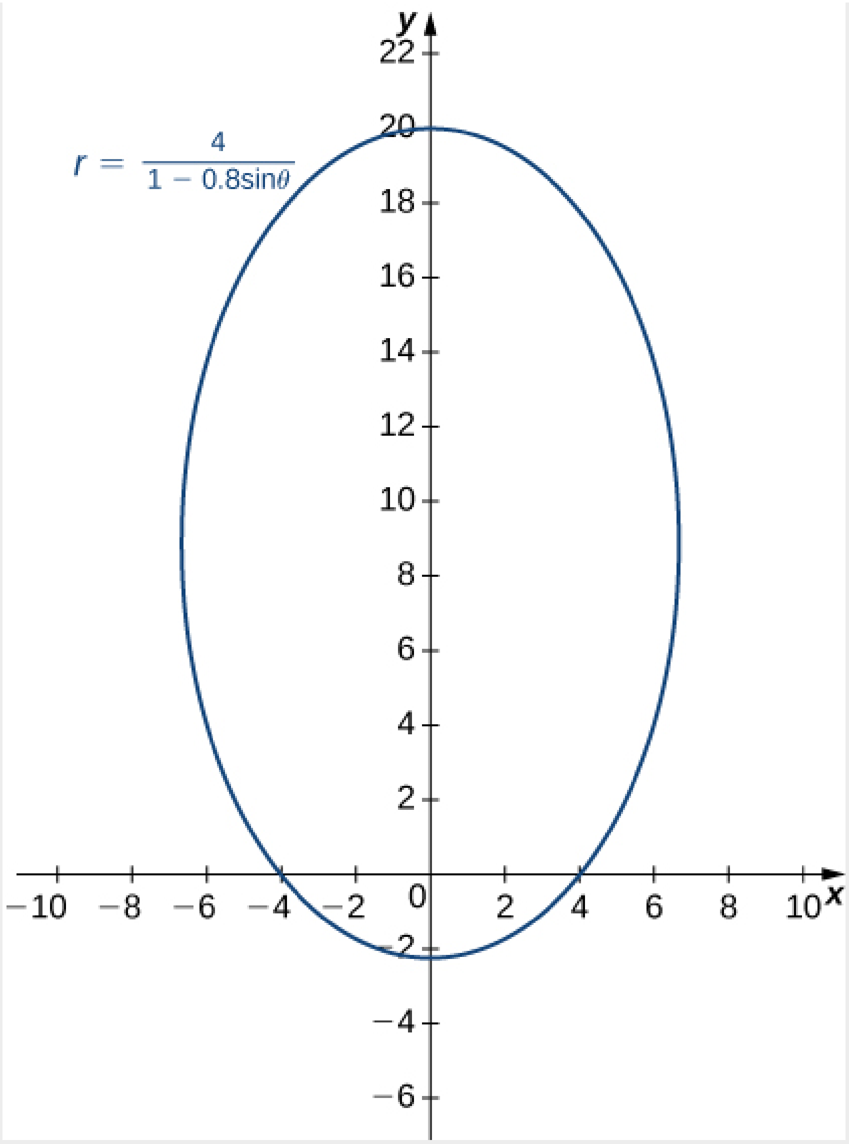
Identify and create a graph of the conic section described by the equation

Equation:

$$r = \frac{4}{1 - 0.8\sin\theta}.$$

Solution:

Here $e = 0.8$ and $p = 5$. This conic section is an ellipse.



Hint

First find the values of *e* and *p*, and then create a table of values.

General Equations of Degree Two

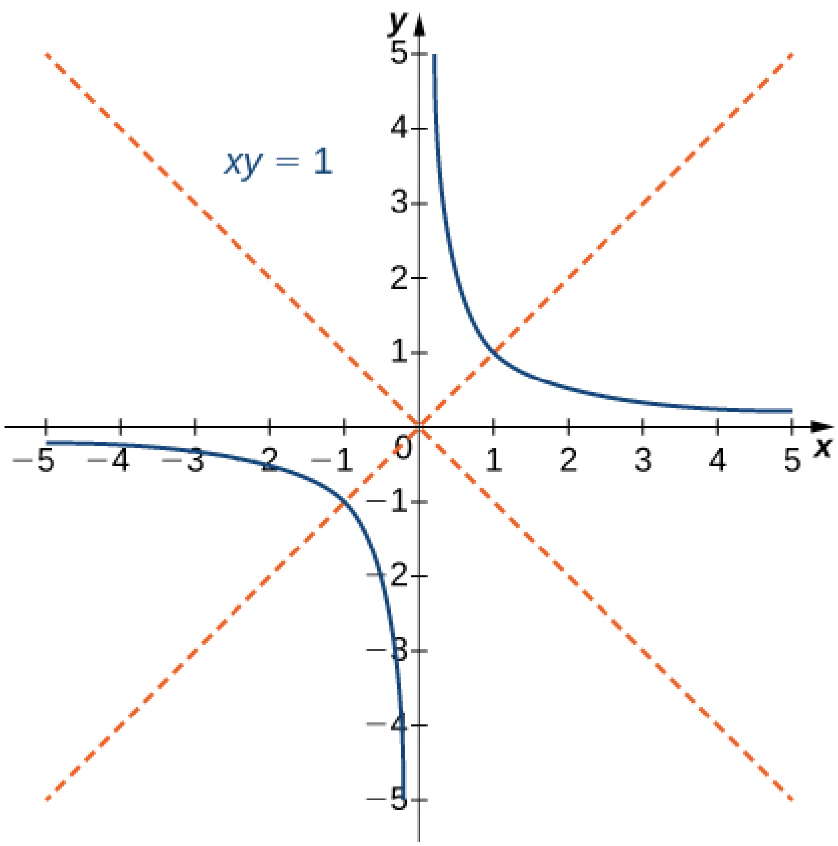
A general equation of degree two can be written in the form
Equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The graph of an equation of this form is a conic section. If $B \neq 0$ then the coordinate axes are rotated. To identify the conic section, we use the **discriminant** of the conic section $4AC - B^2$. One of the following cases must be true:

1. $4AC - B^2 > 0$. If so, the graph is an ellipse.
2. $4AC - B^2 = 0$. If so, the graph is a parabola.
3. $4AC - B^2 < 0$. If so, the graph is a hyperbola.

The simplest example of a second-degree equation involving a cross term is $xy = 1$. This equation can be solved for y to obtain $y = \frac{1}{x}$. The graph of this function is called a *rectangular hyperbola* as shown.



Graph of the equation $xy = 1$; The red lines indicate the rotated axes.

The asymptotes of this hyperbola are the *x* and *y* coordinate axes. To determine the angle θ of rotation of the conic section, we use the formula $\cot 2\theta = \frac{A-C}{B}$. In this case $A = C = 0$ and $B = 1$, so $\cot 2\theta = (0-0)/1 = 0$ and $\theta = 45^\circ$. The method for graphing a conic section with rotated axes involves determining the coefficients of the conic in the rotated coordinate system. The new coefficients are labeled $A', B', C', D', E',$ and F', and are given by the formulas
Equation:

$$\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\
B' &= 0 \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
D' &= D\cos\theta + E\sin\theta \\
E' &= -D\sin\theta + E\cos\theta \\
F' &= F.
\end{aligned}$$

The procedure for graphing a rotated conic is the following:

1. Identify the conic section using the discriminant $4AC - B^2$.
2. Determine θ using the formula $\cot 2\theta = \frac{A-C}{B}$.
3. Calculate $A', B', C', D', E',$ and F'.
4. Rewrite the original equation using $A', B', C', D', E',$ and F'.
5. Draw a graph using the rotated equation.

Example:
Exercise:

Problem:
Identifying a Rotated Conic

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

Equation:

$$13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0.$$

Solution:

In this equation, $A = 13, B = -6\sqrt{3}, C = 7, D = 0, E = 0$, and $F = -256$. The discriminant of this equation is $4AC - B^2 = 4\left(13\right)\left(7\right) - \left(-6\sqrt{3}\right)^2 = 364 - 108 = 256.$ Therefore this conic is an ellipse. To calculate the angle of rotation of the axes, use $\cot 2\theta = \frac{A-C}{B}$. This gives

Equation:

$$\begin{aligned} \cot 2\theta &= \frac{A-C}{B} \\ &= \frac{13-7}{-6\sqrt{3}} \\ &= -\frac{\sqrt{3}}{3}. \end{aligned}$$

Therefore $2\theta = 120^{\text{o}}$ and $\theta = 60^{\text{o}}$, which is the angle of the rotation of the axes.

To determine the rotated coefficients, use the formulas given above:

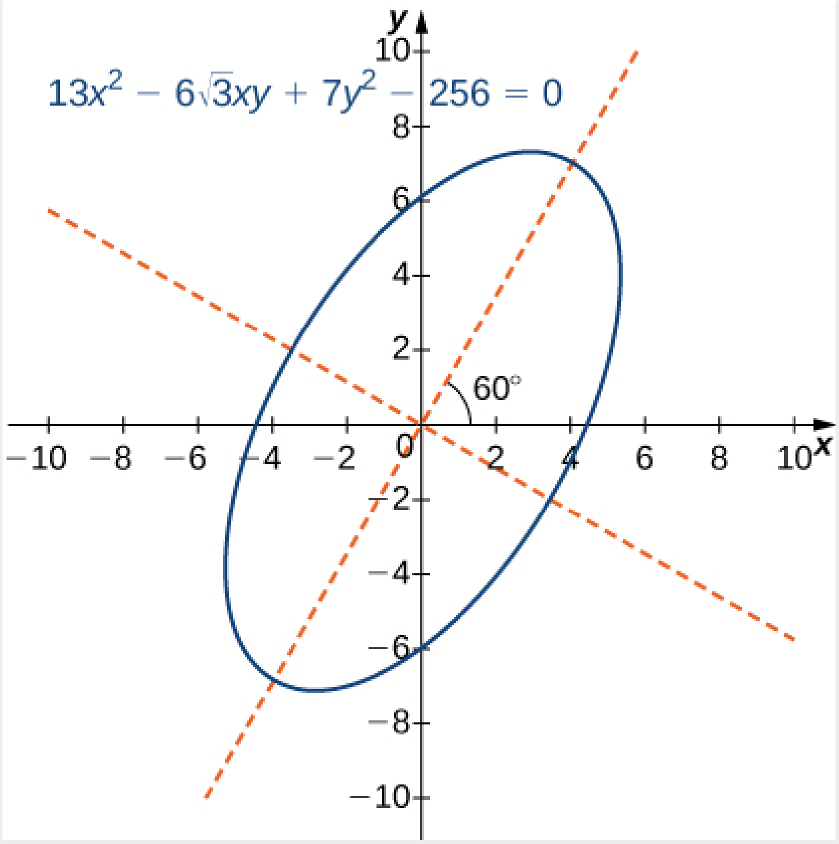
Equation:

$$\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\
&= 13\cos^2 60 + \left(-6\sqrt{3}\right)\cos 60\sin 60 + 7\sin^2 60 \\
&= 13\left(\tfrac{1}{2}\right)^2 - 6\sqrt{3}\left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right) + 7\left(\tfrac{\sqrt{3}}{2}\right)^2 \\
&= 4, \\
B' &= 0, \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
&= 13\sin^2 60 + \left(-6\sqrt{3}\right)\sin 60\cos 60 = 7\cos^2 60 \\
&= \left(\tfrac{\sqrt{3}}{2}\right)^2 + 6\sqrt{3}\left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{1}{2}\right) + 7\left(\tfrac{1}{2}\right)^2 \\
&= 16, \\
D' &= D\cos\theta + E\sin\theta \\
&= (0)\cos 60 + (0)\sin 60 \\
&= 0, \\
E' &= -D\sin\theta + E\cos\theta \\
&= -(0)\sin 60 + (0)\cos 60 \\
&= 0, \\
F' &= F \\
&= -256.
\end{aligned}$$

The equation of the conic in the rotated coordinate system becomes
Equation:

$$\begin{aligned}
4(x')^2 + 16(y')^2 &= 256 \\
\frac{(x')^2}{64} + \frac{(y')^2}{16} &= 1.
\end{aligned}$$

A graph of this conic section appears as follows.



Graph of the ellipse described by the equation $13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0$. The axes are rotated 60°. The red dashed lines indicate the rotated axes.

Note:
Exercise:

Problem:

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

Equation:

$$3x^2 + 5xy - 2y^2 - 125 = 0.$$

Solution:

The conic is a hyperbola and the angle of rotation of the axes is $\theta = 22.5^\circ$.

Hint

Follow steps 1 and 2 of the five-step method outlined above.

Key Concepts

- The equation of a vertical parabola in standard form with given focus and directrix is $y = \frac{1}{4p}(x-h)^2 + k$ where p is the distance from the vertex to the focus and (h, k) are the coordinates of the vertex.
- The equation of a horizontal ellipse in standard form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the major axis has length $2a$, the minor axis has length $2b$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$.
- The equation of a horizontal hyperbola in standard form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the vertices are located at $(h \pm a, k)$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 + b^2$.
- The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is equal to 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle is 0.
- The polar equation of a conic section with eccentricity e is $r = \frac{ep}{1 \pm e\cos\theta}$ or $r = \frac{ep}{1 \pm e\sin\theta}$, where p represents the focal parameter.
- To identify a conic generated by the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, first calculate the discriminant $D = 4AC - B^2$. If $D > 0$ then the conic is an ellipse, if $D = 0$ then the conic is a parabola, and if $D < 0$ then the conic is a hyperbola.

For the following exercises, determine the equation of the parabola using the information given.

Exercise:

Problem: Focus $(4, 0)$ and directrix $x = -4$

Solution:

$y^2 = 16x$

Exercise:

Problem: Focus $(0, -3)$ and directrix $y = 3$

Exercise:

Problem: Focus $(0, 0.5)$ and directrix $y = -0.5$

Solution:

$x^2 = 2y$

Exercise:

Problem: Focus $(2,\ 3)$ and directrix $x = -2$

Exercise:

Problem: Focus $(0, 2)$ and directrix $y = 4$

Solution:

$x^2 = -4\left(y - 3\right)$

Exercise:

Problem: Focus $(-1, 4)$ and directrix $x = 5$

Exercise:

Problem: Focus $(-3, 5)$ and directrix $y = 1$

Solution:

$(x+3)^2 = 8(y-3)$

Exercise:

Problem: Focus $\left(\frac{5}{2}, -4\right)$ and directrix $x = \frac{7}{2}$

For the following exercises, determine the equation of the ellipse using the information given.

Exercise:

Problem:

Endpoints of major axis at $(4, 0), (-4, 0)$ and foci located at $(2, 0), (-2, 0)$

Solution:

$\frac{x^2}{16} + \frac{y^2}{12} = 1$

Exercise:

Problem:

Endpoints of major axis at $(0, 5), (0, -5)$ and foci located at $(0, 3), (0, -3)$

Exercise:

Problem:

Endpoints of major axis at $(0, 2), (0, -2)$ and foci located at $(3, 0), (-3, 0)$

Solution:

$\frac{x^2}{13}+\frac{y^2}{4}=1$

Exercise:

Problem:

Endpoints of major axis at $(-3,3),(7,3)$ and foci located at $(-2,3),(6,3)$

Exercise:

Problem:

Endpoints of major axis at $(-3,5),(-3,-3)$ and foci located at $(-3,3),(-3,-1)$

Solution:

$\frac{(y-1)^2}{16}+\frac{(x+3)^2}{12}=1$

Exercise:

Problem:

Endpoints of major axis at $(0,0),(0,4)$ and foci located at $(5,2),(-5,2)$

Exercise:

Problem: Foci located at $(2,0),\ (-2,0)$ and eccentricity of $\frac{1}{2}$

Solution:

$\frac{x^2}{16}+\frac{y^2}{12}=1$

Exercise:

Problem: Foci located at $(0,-3),\ (0,3)$ and eccentricity of $\frac{3}{4}$

For the following exercises, determine the equation of the hyperbola using the information given.

Exercise:

Problem:

Vertices located at $(5, 0), (-5, 0)$ and foci located at $(6, 0), (-6, 0)$

Solution:

$\frac{x^2}{25} - \frac{y^2}{11} = 1$

Exercise:

Problem:

Vertices located at $(0, 2), (0, -2)$ and foci located at $(0, 3), (0, -3)$

Exercise:

Problem:

Endpoints of the conjugate axis located at $(0, 3), (0, -3)$ and foci located $(4, 0), (-4, 0)$

Solution:

$\frac{x^2}{7} - \frac{y^2}{9} = 1$

Exercise:

Problem: Vertices located at $(0, 1), (6, 1)$ and focus located at $(8, 1)$

Exercise:

Problem:

Vertices located at $(-2, 0), (-2, -4)$ and focus located at $(-2, -8)$

Solution:

$\frac{(y+2)^2}{4} - \frac{(x+2)^2}{32} = 1$

Exercise:

Problem:

Endpoints of the conjugate axis located at $(3, 2), (3, 4)$ and focus located at $(3, 7)$

Exercise:

Problem: Foci located at $(6, -0), (6, 0)$ and eccentricity of 3

Solution:

$\frac{x^2}{4} - \frac{y^2}{32} = 1$

Exercise:

Problem: $(0, 10), (0, -10)$ and eccentricity of 2.5

For the following exercises, consider the following polar equations of conics. Determine the eccentricity and identify the conic.

Exercise:

Problem: $r = \frac{-1}{1+\cos\theta}$

Solution:

$e = 1$, parabola

Exercise:

Problem: $r = \frac{8}{2-\sin\theta}$

Exercise:

Problem: $r = \frac{5}{2+\sin\theta}$

Solution:

$e = \frac{1}{2}$, ellipse

Exercise:

Problem: $r = \frac{5}{-1+2\sin\theta}$

Exercise:

Problem: $r = \frac{3}{2-6\sin\theta}$

Solution:

$e = 3$, hyperbola

Exercise:

Problem: $r = \frac{3}{-4+3\sin\theta}$

For the following exercises, find a polar equation of the conic with focus at the origin and eccentricity and directrix as given.

Exercise:

Problem: Directrix: $x = 4; e = \frac{1}{5}$

Solution:

$r = \frac{4}{5+\cos\theta}$

Exercise:

Problem: Directrix: $x = -4; e = 5$

Exercise:

Problem: Directrix: $y = 2; e = 2$

Solution:

$r = \frac{4}{1+2\sin\theta}$

Exercise:

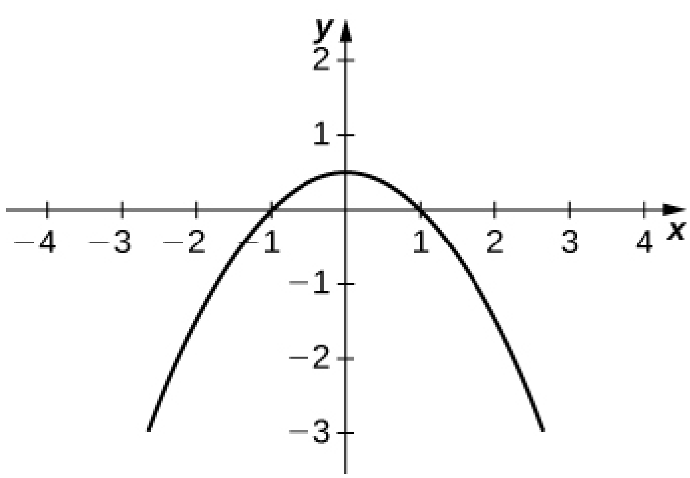
Problem: Directrix: $y = -2; e = \frac{1}{2}$

For the following exercises, sketch the graph of each conic.

Exercise:

Problem: $r = \frac{1}{1+\sin\theta}$

Solution:



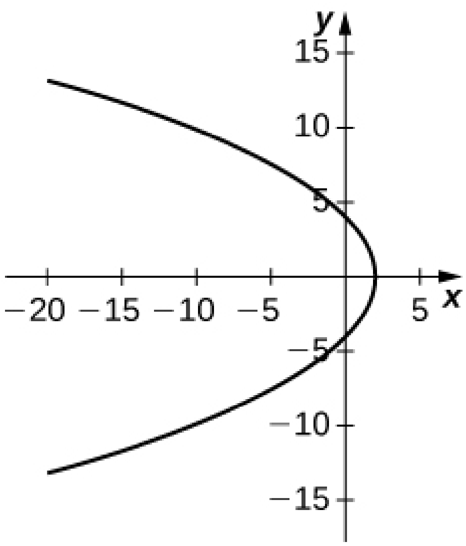
Exercise:

Problem: $r = \frac{1}{1-\cos\theta}$

Exercise:

Problem: $r = \frac{4}{1+\cos\theta}$

Solution:



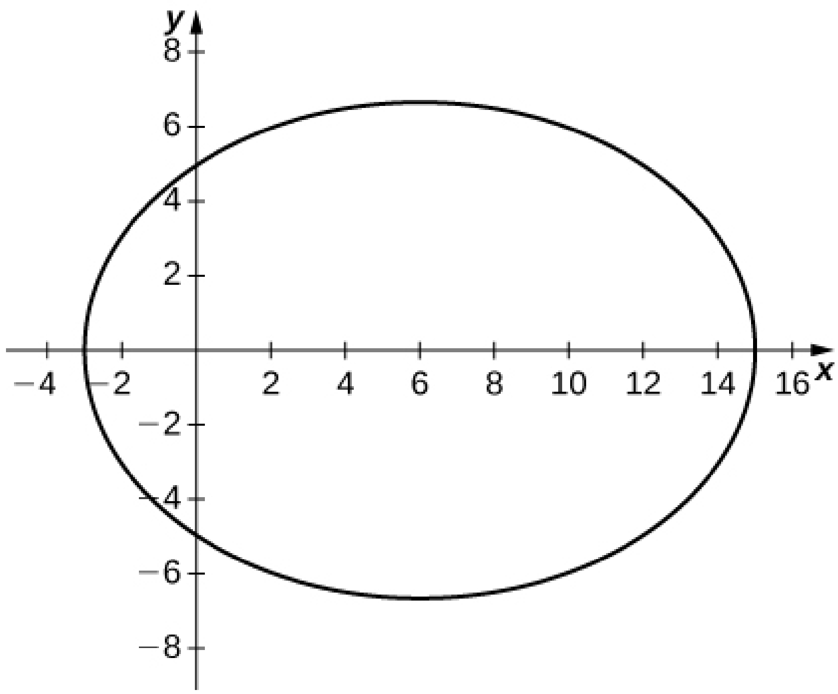
Exercise:

Problem: $r = \frac{10}{5+4\sin\theta}$

Exercise:

Problem: $r = \frac{15}{3-2\cos\theta}$

Solution:



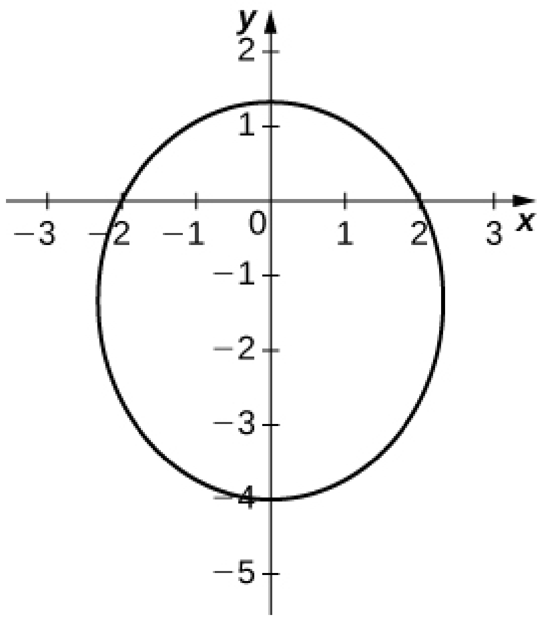
Exercise:

Problem: $r = \frac{32}{3+5\sin\theta}$

Exercise:

Problem: $r(2 + \sin\theta) = 4$

Solution:



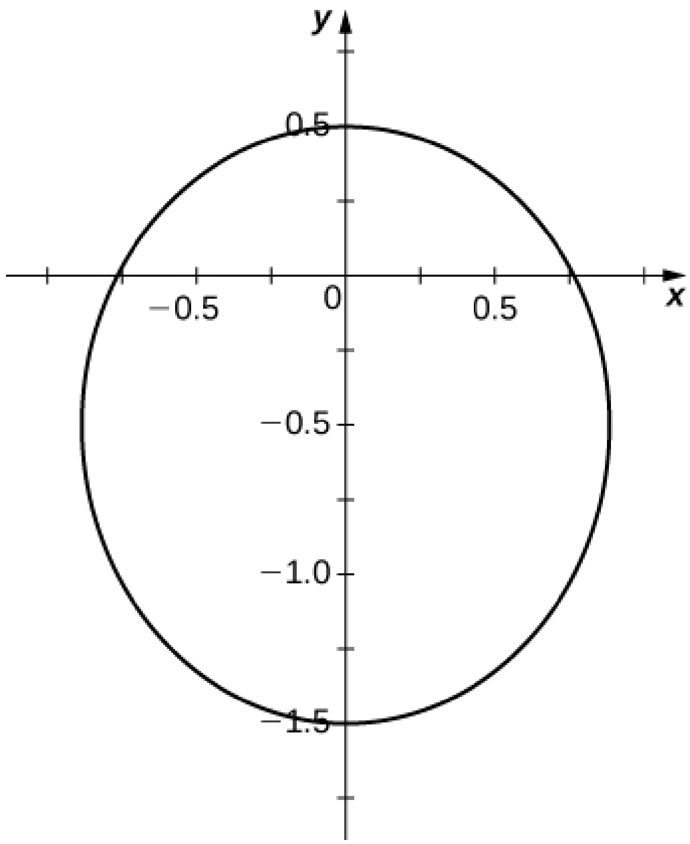
Exercise:

Problem: $r = \frac{3}{2+6\sin\theta}$

Exercise:

Problem: $r = \frac{3}{-4+2\sin\theta}$

Solution:



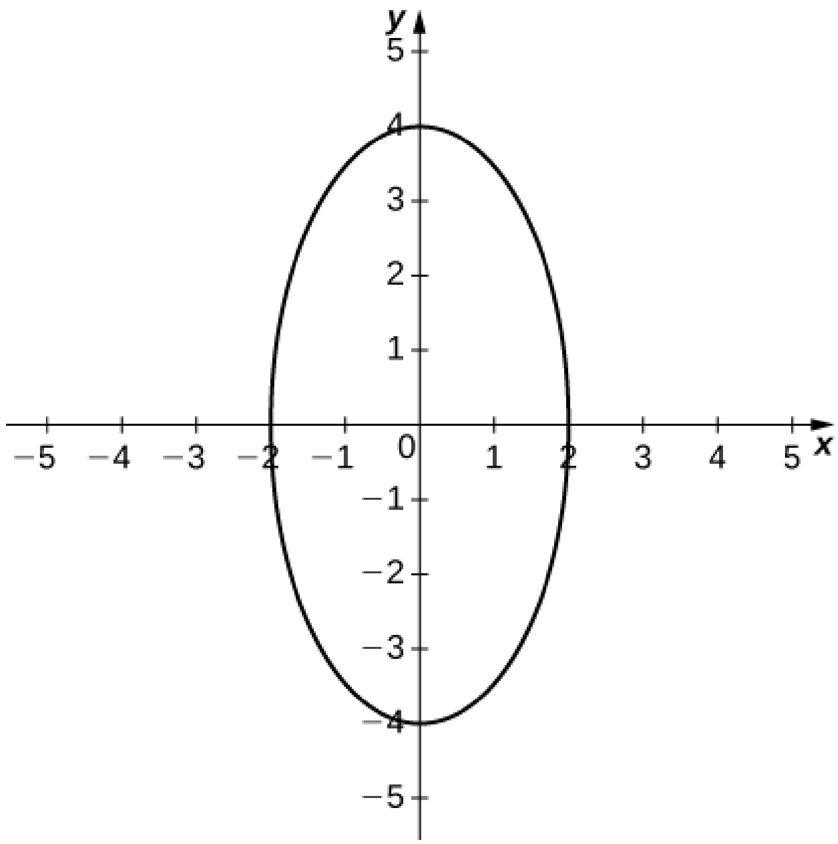
Exercise:

Problem: $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Exercise:

Problem: $\frac{x^2}{4} + \frac{y^2}{16} = 1$

Solution:



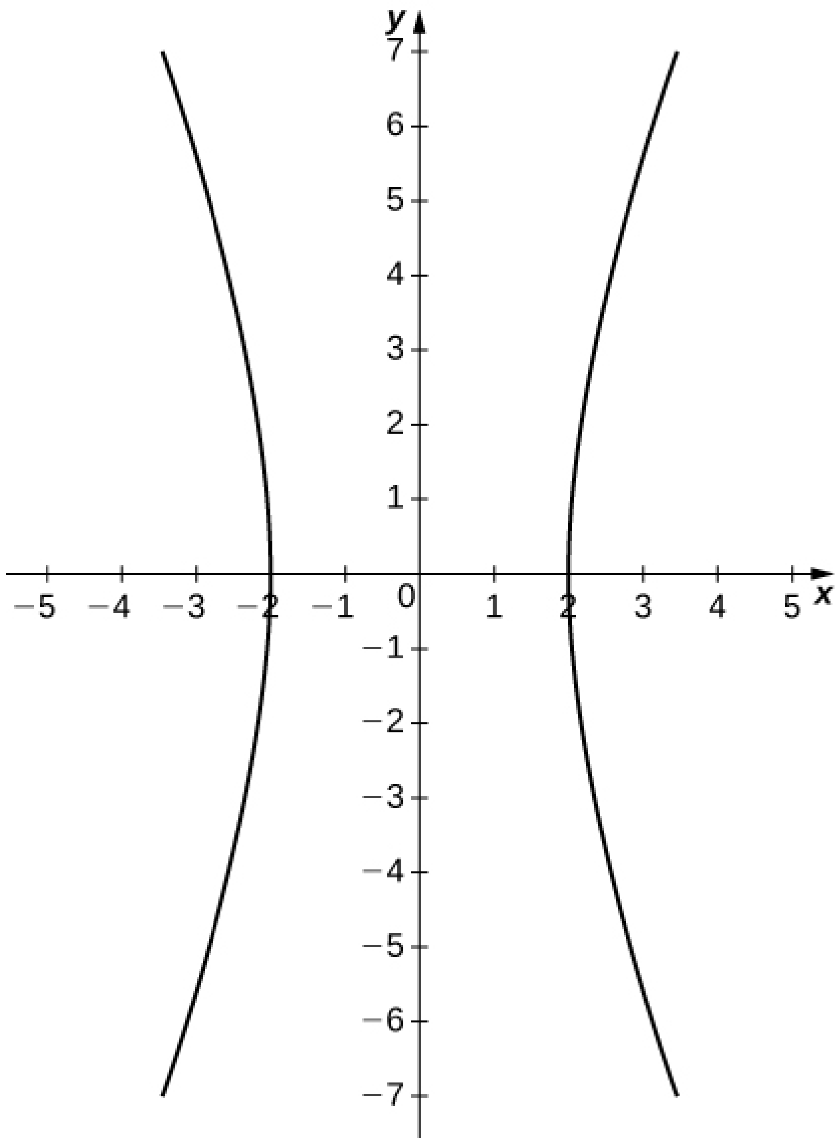
Exercise:

Problem: $4x^2 + 9y^2 = 36$

Exercise:

Problem: $25x^2 - 4y^2 = 100$

Solution:



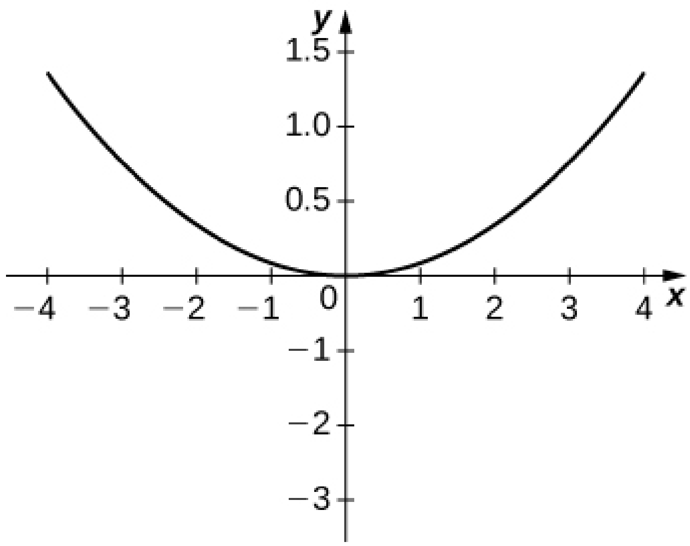
Exercise:

Problem: $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Exercise:

Problem: $x^2 = 12y$

Solution:



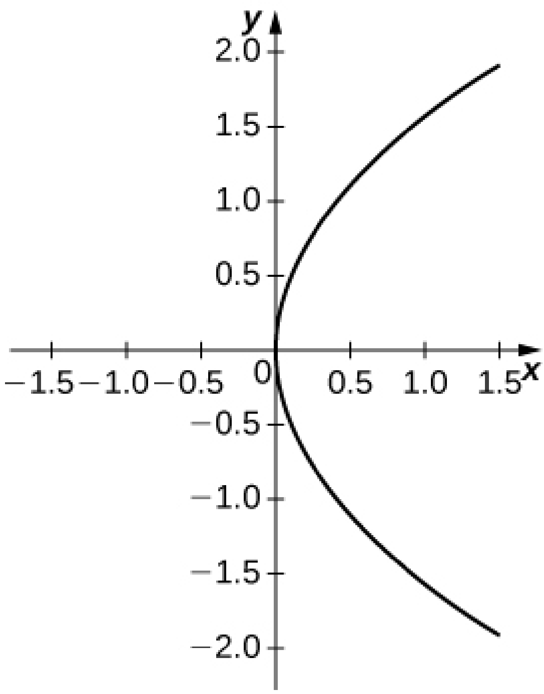
Exercise:

Problem: $y^2 = 20x$

Exercise:

Problem: $12x = 5y^2$

Solution:



For the following equations, determine which of the conic sections is described.

Exercise:

Problem: $xy = 4$

Exercise:

Problem: $x^2 + 4xy - 2y^2 - 6 = 0$

Solution:

Hyperbola

Exercise:

Problem: $x^2 + 2\sqrt{3}xy + 3y^2 - 6 = 0$

Exercise:

Problem: $x^2 - xy + y^2 - 2 = 0$

Solution:

Ellipse

Exercise:

Problem: $34x^2 - 24xy + 41y^2 - 25 = 0$

Exercise:

Problem: $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$

Solution:

Ellipse

Exercise:

Problem:

The mirror in an automobile headlight has a parabolic cross section, with the lightbulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the lightbulb?

Exercise:

Problem:

A satellite dish is shaped like a paraboloid of revolution. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

Solution:

At the point 2.25 feet above the vertex.

Exercise:

Problem:

Consider the satellite dish of the preceding problem. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

Exercise:

Problem:

A searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

Solution:

0.5625 feet

Exercise:

Problem:

Whispering galleries are rooms designed with elliptical ceilings. A person standing at one focus can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet and the foci are located 30 feet from the center, find the height of the ceiling at the center.

Exercise:

Problem:

A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus and the other focus is 80 feet away, what is the length and the height at the center of the gallery?

Solution:

Length is 96 feet and height is approximately 26.53 feet.

For the following exercises, determine the polar equation form of the orbit given the length of the major axis and eccentricity for the orbits of the comets or planets. Distance is given in astronomical units (AU).

Exercise:

Problem:

Halley's Comet: length of major axis = 35.88, eccentricity = 0.967

Exercise:

Problem:

Hale-Bopp Comet: length of major axis = 525.91, eccentricity = 0.995

Solution:

$r = \frac{2.616}{1+0.995\cos\theta}$

Exercise:

Problem: Mars: length of major axis = 3.049, eccentricity = 0.0934

Exercise:

Problem: Jupiter: length of major axis = 10.408, eccentricity = 0.0484

Solution:

$r = \frac{5.192}{1+0.0484\cos\theta}$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem:

The rectangular coordinates of the point $\left(4, \frac{5\pi}{6}\right)$ are $\left(2\sqrt{3}, -2\right)$.

Exercise:

Problem:

The equations $x = \cosh(3t)$, $y = 2\sinh(3t)$ represent a hyperbola.

Solution:

True.

Exercise:

Problem:

The arc length of the spiral given by $r = \frac{\theta}{2}$ for $0 \leq \theta \leq 3\pi$ is $\frac{9}{4}\pi^3$.

Exercise:

Problem:

Given $x = f(t)$ and $y = g(t)$, if $\frac{dx}{dy} = \frac{dy}{dx}$, then $f(t) = g(t) + \text{C}$, where C is a constant.

Solution:

False. Imagine $y = t + 1$, $x = -t + 1$.

For the following exercises, sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

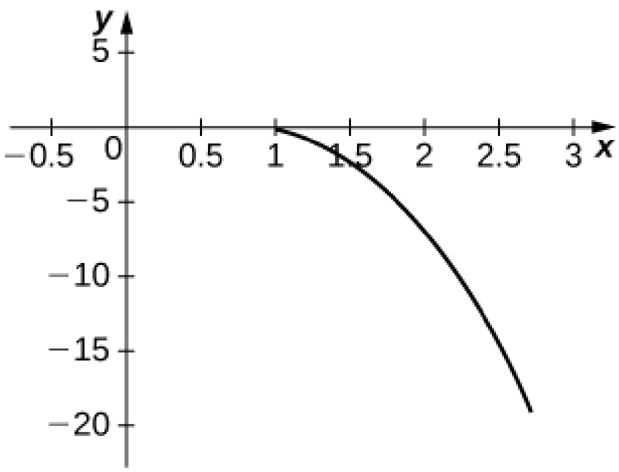
Exercise:

Problem: $x = 1 + t$, $y = t^2 - 1$, $-1 \leq t \leq 1$

Exercise:

Problem: $x = e^t$, $y = 1 - e^{3t}$, $0 \leq t \leq 1$

Solution:



$y = 1 - x^3$

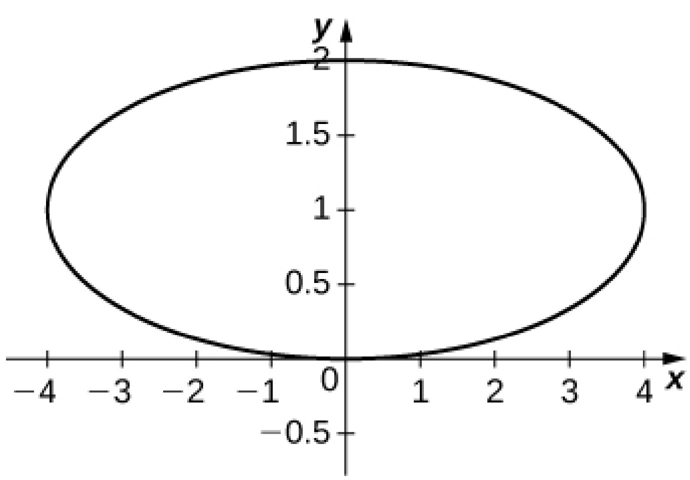
Exercise:

Problem: $x = \sin\theta,\ y = 1 - \csc\theta,\ 0 \le \theta \le 2\pi$

Exercise:

Problem: $x = 4\cos\phi,\ y = 1 - \sin\phi,\ 0 \le \phi \le 2\pi$

Solution:



$\frac{x^2}{16} + (y-1)^2 = 1$

For the following exercises, sketch the polar curve and determine what type of symmetry exists, if any.

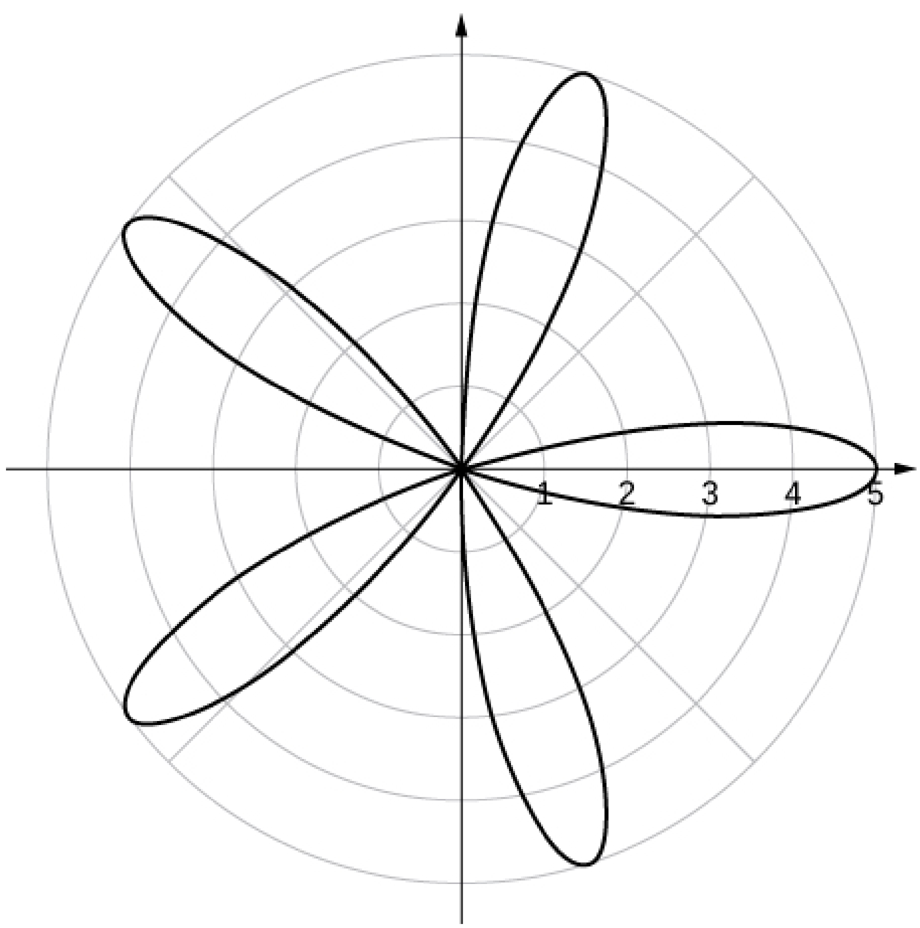
Exercise:

Problem: $r = 4\sin\left(\frac{\theta}{3}\right)$

Exercise:

Problem: $r = 5\cos(5\theta)$

Solution:



Symmetric about polar axis

For the following exercises, find the polar equation for the curve given as a Cartesian equation.

Exercise:

Problem: $x + y = 5$

Exercise:

Problem: $y^2 = 4 + x^2$

Solution:

$r^2 = \frac{4}{\sin^2\theta - \cos^2\theta}$

For the following exercises, find the equation of the tangent line to the given curve. Graph both the function and its tangent line.

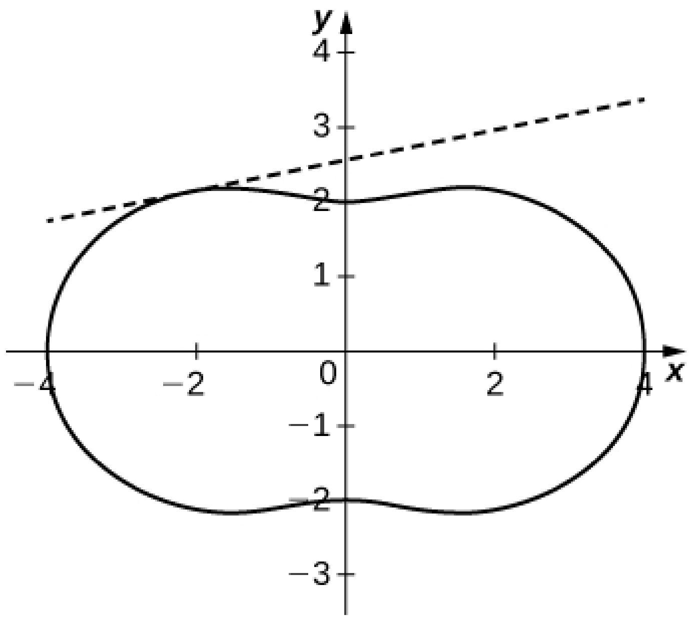
Exercise:

Problem: $x = \ln(t)$, $y = t^2 - 1$, $t = 1$

Exercise:

Problem: $r = 3 + \cos(2\theta)$, $\theta = \frac{3\pi}{4}$

Solution:



$y=\frac{3\sqrt{2}}{2}+\frac{1}{5}\left(x+\frac{3\sqrt{2}}{2}\right)$

Exercise:

Problem: Find $\frac{dy}{dx}$, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ of $y=\left(2+e^{-t}\right)$, $x=1-\sin(t)$

For the following exercises, find the area of the region.

Exercise:

Problem: $x=t^2$, $y=\ln(t)$, $0\le t\le e$

Solution:

$\frac{e^2}{2}$

Exercise:

Problem: $r=1-\sin\theta$ in the first quadrant

For the following exercises, find the arc length of the curve over the given interval.

Exercise:

Problem: $x = 3t + 4$, $y = 9t - 2$, $0 \le t \le 3$

Solution:

$9\sqrt{10}$

Exercise:

Problem: $r = 6 \cos \theta$, $0 \le \theta \le 2\pi$. Check your answer by geometry.

For the following exercises, find the Cartesian equation describing the given shapes.

Exercise:

Problem: A parabola with focus $(2, -5)$ and directrix $x = 6$

Solution:

$(y + 5)^2 = -8x + 32$

Exercise:

Problem:

An ellipse with a major axis length of 10 and foci at $(-7, 2)$ and $(1, 2)$

Exercise:

Problem:

A hyperbola with vertices at $(3, -2)$ and $(-5, -2)$ and foci at $(-2, -6)$ and $(-2, 4)$

Solution:

$$\frac{(y+1)^2}{16} - \frac{(x+2)^2}{9} = 1$$

For the following exercises, determine the eccentricity and identify the conic. Sketch the conic.

Exercise:

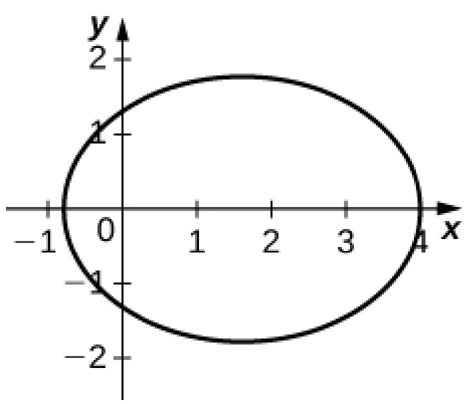
Problem: $r = \frac{6}{1+3\cos(\theta)}$

Exercise:

Problem: $r = \frac{4}{3-2\cos\theta}$

Solution:

$e = \frac{2}{3}$, ellipse



Exercise:

Problem: $r = \frac{7}{5-5\cos\theta}$

Exercise:

Problem:

Determine the Cartesian equation describing the orbit of Pluto, the most eccentric orbit around the Sun. The length of the major axis is 39.26 AU and minor axis is 38.07 AU. What is the eccentricity?

Solution:

$\frac{y^2}{19.03^2} + \frac{x^2}{19.63^2} = 1, e = 0.2447$

Exercise:

Problem:

The C/1980 E1 comet was observed in 1980. Given an eccentricity of 1.057 and a perihelion (point of closest approach to the Sun) of 3.364 AU, find the Cartesian equations describing the comet's trajectory. Are we guaranteed to see this comet again? (*Hint*: Consider the Sun at point $(0, 0)$.)

Glossary

conic section
: a conic section is any curve formed by the intersection of a plane with a cone of two nappes

directrix
: a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

discriminant
: the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving xy, is called a discriminant

focus
: a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

eccentricity
: the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

focal parameter
: the focal parameter is the distance from a focus of a conic section to the nearest directrix

general form
: an equation of a conic section written as a general second-degree equation

major axis
: the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

minor axis
: the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

nappe
: a nappe is one half of a double cone

standard form
: an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

vertex
: a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

Table of Integrals

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, n \neq -1$

2. $\int \frac{du}{u} = \ln|u| + C$

3. $\int e^u\,du = e^u + C$

4. $\int a^u\,du = \frac{a^u}{\ln a} + C$

5. $\int \sin u\,du = -\cos u + C$

6. $\int \cos u\,du = \sin u + C$

7. $\int \sec^2 u\,du = \tan u + C$

8. $\int \csc^2 u\,du = -\cot u + C$

9. $\int \sec u \tan u\,du = \sec u + C$

10. $\int \csc u \cot u\,du = -\csc u + C$

11. $\int \tan u\,du = \ln|\sec u| + C$

12. $\int \cot u \, du = \ln |\sin u| + C$

13. $\int \sec u \, du = \ln |\sec u + \tan u| + C$

14. $\int \csc u \, du = \ln |\csc u - \cot u| + C$

15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$

16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$

17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u \, du = \frac{1}{2} u - \frac{1}{4} \sin 2u + C$

19. $\int \cos^2 u \, du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C$

20. $\int \tan^2 u \, du = \tan u - u + C$

21. $\int \cot^2 u \, du = -\cot u - u + C$

22. $\int \sin^3 u \, du = -\frac{1}{3} \left(2 + \sin^2 u\right) \cos u + C$

23. $\int \cos^3 u \, du = \frac{1}{3} \left(2 + \cos^2 u\right) \sin u + C$

24. $\int \tan^3 u\, du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u\, du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\, du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

29. $\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

30. $\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

31. $\int \cot^n u\, du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du$

32. $\int \sec^n u\, du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$

33. $\int \csc^n u\, du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du$

34. $\int \sin au \sin bu\, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu\, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u \, du = \sin u - u \cos u + C$

38. $\int u \cos u \, du = \cos u + u \sin u + C$

39. $\int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$

40. $\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$

41.
$$\begin{aligned} \int \sin^n u \cos^m u \, du &= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m} \int \sin^{n-2} u \cos^m u \, du \\ &= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m} \int \sin^n u \cos^{m-2} u \, du \end{aligned}$$

Exponential and Logarithmic Integrals

42. $\int u e^{au} \, du = \frac{1}{a^2}(au - 1)e^{au} + C$

43. $\int u^n e^{au} \, du = \frac{1}{a} u^n e^{au} - \frac{n}{a} \int u^{n-1} e^{au} \, du$

44. $\int e^{au} \sin bu \, du = \frac{e^{au}}{a^2+b^2}(a \sin bu - b \cos bu) + C$

45. $\int e^{au} \cos bu \, du = \frac{e^{au}}{a^2+b^2}(a \cos bu + b \sin bu) + C$

46. $\int \ln u \, du = u \ln u - u + C$

47. $\int u^n \ln u \, du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\displaystyle\int \frac{1}{u \ln u}\, du = \ln |\ln u| + C$

Hyperbolic Integrals

49. $\displaystyle\int \sinh u\, du = \cosh u + C$

50. $\displaystyle\int \cosh u\, du = \sinh u + C$

51. $\displaystyle\int \tanh u\, du = \ln \cosh u + C$

52. $\displaystyle\int \coth u\, du = \ln |\sinh u| + C$

53. $\displaystyle\int \operatorname{sech} u\, du = \tan^{-1} |\sinh u| + C$

54. $\displaystyle\int \operatorname{csch} u\, du = \ln \left|\tanh \tfrac{1}{2} u\right| + C$

55. $\displaystyle\int \operatorname{sech}^2 u\, du = \tanh u + C$

56. $\displaystyle\int \operatorname{csch}^2 u\, du = -\coth u + C$

57. $\displaystyle\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

58. $\displaystyle\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\ln\left(1+u^2\right) + C$

62. $\int u\sin^{-1}u\,du = \frac{2u^2-1}{4}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\cos^{-1}u\,du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1}u\,du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$

65. $\int u^n\sin^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1}u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], n \neq -1$

66. $\int u^n\cos^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1}u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], n \neq -1$

67. $\int u^n\tan^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1}u - \int \frac{u^{n+1}\,du}{1+u^2}\right], n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

69.
$\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}\left(a^2+2u^2\right)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int \frac{du}{\left(a^2+u^2\right)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78.
$\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}\left(2u^2-a^2\right)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

79. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln \left| u + \sqrt{u^2 - a^2} \right| + C$$

82. $$\int \frac{u^2\,du}{\sqrt{u^2 - a^2}} = \tfrac{u}{2}\sqrt{u^2 - a^2} + \tfrac{a^2}{2}\ln \left| u + \sqrt{u^2 - a^2} \right| + C$$

83. $$\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$$

84. $$\int \frac{du}{\left(u^2 - a^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $$\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \tfrac{a^2}{2}\sin^{-1}\tfrac{u}{a} + C$$

86. $$\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}\left(2u^2 - a^2\right)\sqrt{a^2 - u^2} + \tfrac{a^4}{8}\sin^{-1}\tfrac{u}{a} + C$$

87. $$\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln \left| \frac{a+\sqrt{a^2-u^2}}{u} \right| + C$$

88. $$\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\tfrac{u}{a} + C$$

89. $$\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{u}\sqrt{a^2 - u^2} + \tfrac{a^2}{2}\sin^{-1}\tfrac{u}{a} + C$$

90. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\tfrac{1}{a}\ln \left| \frac{a+\sqrt{a^2-u^2}}{u} \right| + C$$

91. $$\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\tfrac{1}{a^2u}\sqrt{a^2 - u^2} + C$$

92. $\int \left(a^2-u^2\right)^{3/2} du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

93. $\int \frac{du}{\left(a^2-u^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\int \sqrt{2au-u^2}\, du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

95. $\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

96. $\int u\sqrt{2au-u^2}\, du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

97. $\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

Integrals Involving $a + bu$, $a \neq 0$

98. $\int \frac{u\, du}{a+bu} = \frac{1}{b^2}\left(a+bu-a\ln|a+bu|\right) + C$

99. $\int \frac{u^2\, du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

100. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

101. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

102. $\int \frac{u\, du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

103. $\displaystyle\int \frac{u\,du}{u\,(a+bu)^2} = \tfrac{1}{a(a+bu)} - \tfrac{1}{a^2}\ln\left|\tfrac{a+bu}{u}\right| + C$

104. $\displaystyle\int \frac{u^2\,du}{(a+bu)^2} = \tfrac{1}{b^3}\left(a+bu-\tfrac{a^2}{a+bu}-2a\ln|a+bu|\right)+C$

105. $\displaystyle\int u\sqrt{a+bu}\,du = \tfrac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$

106. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \tfrac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C$

107. $\displaystyle\int \frac{u^2\,du}{\sqrt{a+bu}} = \tfrac{2}{15b^3}\left(8a^2+3b^2u^2-4abu\right)\sqrt{a+bu}+C$

108. $$\int \frac{du}{u\sqrt{a+bu}} = \tfrac{1}{\sqrt{a}}\ln\left|\tfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a>0$$
$$= \tfrac{2}{\sqrt{-a}}\tan-1\sqrt{\tfrac{a+bu}{-a}}+C, \quad \text{if } a<0$$

109. $\displaystyle\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu}+a\int\frac{du}{u\sqrt{a+bu}}$

110. $\displaystyle\int \frac{\sqrt{a+bu}}{u^2}\,du = -\tfrac{\sqrt{a+bu}}{u}+\tfrac{b}{2}\int\frac{du}{u\sqrt{a+bu}}$

111.
$$\int u^n\sqrt{a+bu}\,du = \tfrac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2}-na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

112. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \tfrac{2u^n\sqrt{a+bu}}{b(2n+1)}-\tfrac{2na}{b(2n+1)}\int\frac{u^{n-1}\,du}{\sqrt{a+bu}}$

113. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\tfrac{\sqrt{a+bu}}{a(n-1)u^{n-1}}-\tfrac{b(2n-3)}{2a(n-1)}\int\frac{du}{u^{n-1}\sqrt{a+bu}}$

Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

4. $\frac{d}{dx}(x^n) = nx^{n-1}, \text{ for real numbers } n$

5. $\frac{d}{dx}(cf(x)) = cf'(x)$

6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$

10. $\frac{d}{dx}(\tan x) = \sec^2 x$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x$

12. $\frac{d}{dx}(\cos x) = -\sin x$

13. $\frac{d}{dx}(\cot x) = -\csc^2 x$

14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$

17. $\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}}$

18. $\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$

19. $\frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$

20. $\frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$

22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

23. $\frac{d}{dx}(b^x) = b^x \ln b$

24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$

26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

28. $\frac{d}{dx}(\cosh x) = \sinh x$

29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$

32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} (|x| < 1)$

33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$

34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$

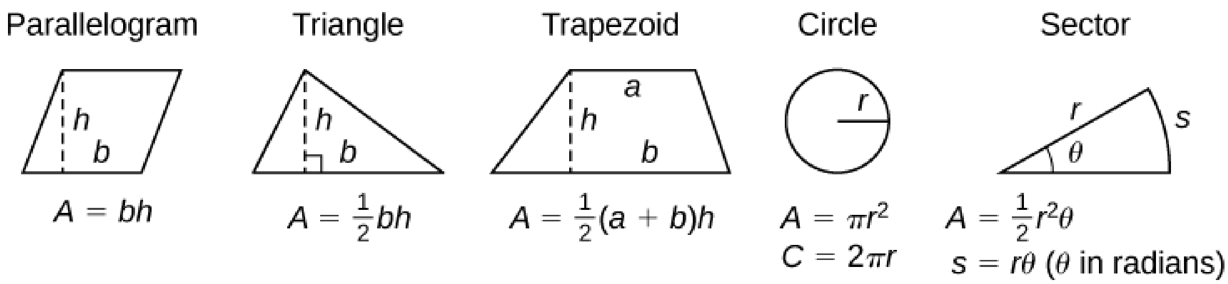
35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| > 1)$

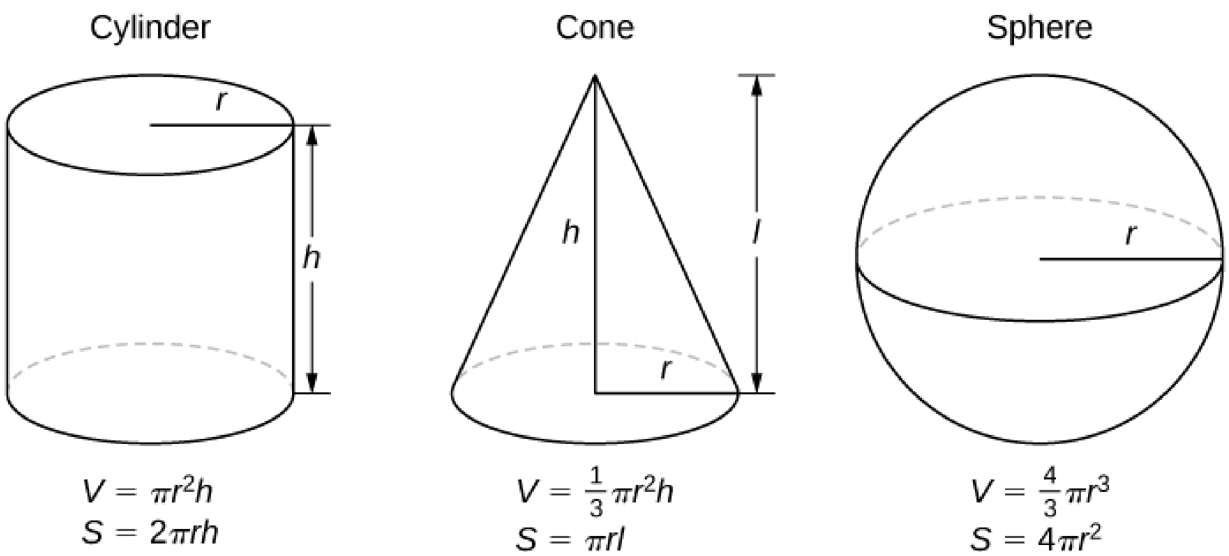
36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{1+x^2}} (x \neq 0)$

Review of Pre-Calculus

Formulas from Geometry

$A = \text{area}, V = \text{Volume}, \text{ and } S = \text{lateral surface area}$





Formulas from Algebra

Laws of Exponents

$$
\begin{aligned}
x^m x^n &= x^{m+n} & \frac{x^m}{x^n} &= x^{m-n} & (x^m)^n &= x^{mn} \\
x^{-n} &= \frac{1}{x^n} & (xy)^n &= x^n y^n & \left(\frac{x}{y}\right)^n &= \frac{x^n}{y^n} \\
x^{1/n} &= \sqrt[n]{x} & \sqrt[n]{xy} &= \sqrt[n]{x}\sqrt[n]{y} & \sqrt[n]{\frac{x}{y}} &= \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \\
x^{m/n} &= \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m
\end{aligned}
$$

Special Factorizations

$$
\begin{aligned}
x^2 - y^2 &= (x+y)(x-y) \\
x^3 + y^3 &= (x+y)\left(x^2 - xy + y^2\right) \\
x^3 - y^3 &= (x-y)\left(x^2 + xy + y^2\right)
\end{aligned}
$$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ca}}{2a}$.

Binomial Theorem

$$(a+b)^n = a^n + \genfrac{}{}{0pt}{}{n}{1} a^{n-1}b + \genfrac{}{}{0pt}{}{n}{2} a^{n-2}b^2 + \cdots + \genfrac{}{}{0pt}{}{n}{n-1} ab^{n-1} + b^n,$$

where $\genfrac{}{}{0pt}{}{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1} = \frac{n!}{k!(n-k)!}$

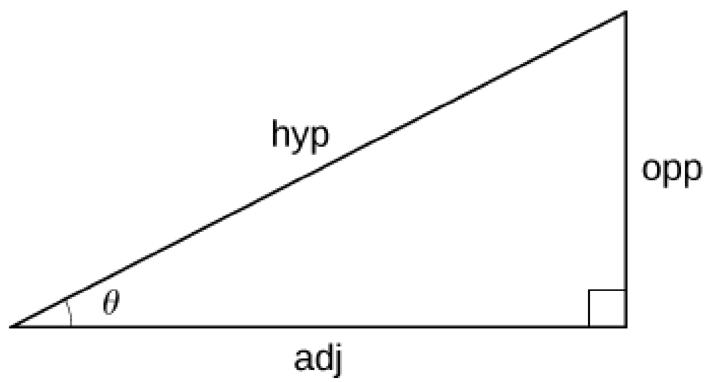
Formulas from Trigonometry

Right-Angle Trigonometry

$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$ $\cot\theta = \frac{\text{adj}}{\text{opp}}$



Trigonometric Functions of Important Angles

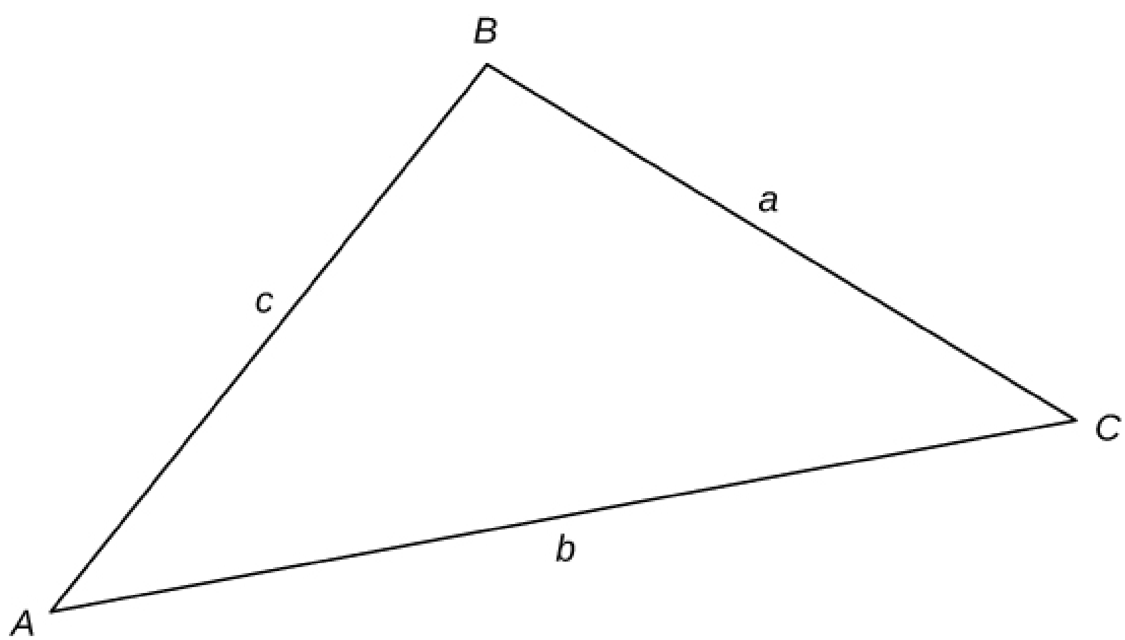
θ	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 & \sin(-\theta) &= -\sin\theta \\ 1 + \tan^2\theta &= \sec^2\theta & \cos(-\theta) &= \cos\theta \\ 1 + \cot^2\theta &= \csc^2\theta & \tan(-\theta) &= -\tan\theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta & \sin(\theta + 2\pi) &= \sin\theta \\ \cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta & \cos(\theta + 2\pi) &= \cos\theta \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot\theta & \tan(\theta + \pi) &= \tan\theta \end{aligned}$$

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Law of Cosines

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

Addition and Subtraction Formulas

$$\begin{aligned} \sin\ (x+y) &= \sin x\cos y + \cos x\sin y \\ \sin\ (x-y) &= \sin x\cos y - \cos x\sin y \\ \cos\ (x+y) &= \cos x\cos y - \sin x\sin y \\ \cos\ (x-y) &= \cos x\cos y + \sin x\sin y \\ \tan\ (x+y) &= \frac{\tan x + \tan y}{1 - \tan x\tan y} \\ \tan (x-y) &= \frac{\tan x - \tan y}{1 + \tan x\tan y} \end{aligned}$$

Double-Angle Formulas

$$\begin{aligned} \sin 2x &= 2\sin x\cos x \\ \cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \\ \tan 2x &= \frac{2\tan x}{1-\tan^2 x} \end{aligned}$$

Half-Angle Formulas

$$\begin{aligned} \sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \end{aligned}$$